On some open problems in minimum coloring games

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3. show some progress on them.

Minimum coloring games – what is it?

game theory

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game theory $+$

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game theory + combinatorics

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or more precisely:

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A game theory addendum

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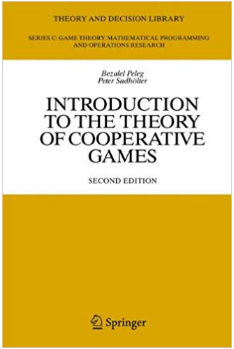
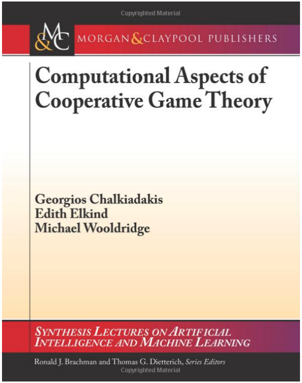
Cooperative game theory is a very rich theory.





Intro to cooperative games I: Definition

Classical cooperative game

Classical cooperative game is an ordered pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function. We further assume $v(\emptyset) = 0$.

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The set N is called *grand coalition* and the set $\emptyset$ is called *null coalition*.

Intro to cooperative games II: An example

Consider a game (N, v) with player set $N = \{1, 2, 3\}$ and characteristic function v defined as:

C	$v(C)$
$\emptyset$	0
$\{1\}$	1
$\{2\}$	2
$\{3\}$	3
$\{1,2\}$	4
$\{1,3\}$	4
$\{2,3\}$	5
$\{1,2,3\}$	6

Intro to cooperative games III: Core

One of the most important concepts in cooperative game theory is *core*. It is considered as a set of stability points – equilibria – in cooperative games.

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Core

The core of a cooperative game (N, v) is the set

$$C((N,v)) = \left\{ x \in \mathbb{R}^{|N|};\ \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \leq v(S), \forall S \subseteq N \right\}.$$

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We can interpret the value of a coalition as a cost that must be paid for this group. Then, the core consist of vectors where the cost of grand coalition must be fully paid and every other coalition does not need to pay more than it was set.

The main course of study is to determine if the core of a given game is empty and if not how we can describe it.

Intro to cooperative games IV: An example of core

Consider the previous game:

C	$v(C)$
$\emptyset$	0
$\{1\}$	1
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Intro to cooperative games IV: An example of core

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The vector $x = (1,2,3)$ is in the core. The vector $y = (3,1,2)$ is not.

Definition of minimum coloring game

Definition

We can associate a cooperative game (V, w_G) with every graph $G = (V, E)$ such that

$$w_G(S) = \chi(G[S]), \forall S \subseteq V.$$

We call such game the *minimum coloring game* for G.

History and examples

Origin

Minimum coloring games were introduced by Deng, Ibaraki, and Nagamochi in 90's.

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Example I: A complete graph

Take a complete graph K_7. For every coalition of vertices S, its value $w_G(S)$ is equal to $|S|$.

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Minimum coloring games were introduced by Deng, Ibaraki, and Nagamochi in 90's.

Example I: A complete graph

Take a complete graph K_7. For every coalition of vertices S, its value $w_G(S)$ is equal to $|S|$.

Example II: A path graph

Take a path graph. For every nonempty coalition of vertices S, its value $w_G(S)$ is equal to 2 if the induced graph contains an edge and is equal to 1 otherwise.

Core emptiness

As I said, the main problem is to determine if the core of a given game is empty or not.

PROBLEM:	Core Emptiness – COREEMPTINESS
INPUT:	A graph $G = (V, E)$.
QUESTION:	Is $C((V, w_G))$ nonempty?

The problem is NP-complete in general and polynomial for perfect graphs, for example.

Largeness

Definition

The core $C((N, w))$ of a cooperative game (N, w) is said to be *large* if for every $y \in \mathbb{R}^N$ satisfying that $y(S) \leq w(S), \forall S \subseteq N$, there exists $x \in C((N, w))$ such that $y \leq x$.

PROBLEM:	Core Largeness – CORELARGENESS
INPUT:	A graph $G = (V, E)$.
QUESTION:	Is $C((V, w_G))$ large?

The problem is NP-complete for general graphs.

Exactness

Definition

The core of a cooperative game (N, w) is said to be *exact* if for every $S \subseteq N$ there exist a vector $x \in C((N, w))$ such that $x(S) = w(S)$.

PROBLEM:	Core Exactness – COREEXACTNESS
INPUT:	A graph $G = (V, E)$.
QUESTION:	Is $C((V, w_G))$ exact?

Again, the problem is NP-complete for general graphs.

Open problems

Okamoto proposed to study complexity of these three problems on the class of *outerplanar graphs* and *maximal planar graphs*.

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- an outerplanar graph is a graph that has a planar drawing for which all vertices belong to the outer face of the drawing
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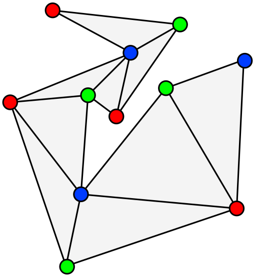
- an outerplanar graph is a graph that has a planar drawing for which all vertices belong to the outer face of the drawing
- maximal planar graphs: planar graphs in which every face is bounded by exactly three edges

The following is an example of outerplanar graph, K_4 is an example of maximal planar graph.



My results for outerplanars

Theorem, B. 2018+

The core of a give outerplanar graph is nonempty if and only if the graph does not contain any induced odd cycle other than triangle.

Theorem, B. 2018+

The problems CoreEmptiness, CoreLargeness and CoreExactness are polynomial on outerplanar graphs.

Edge coloring

In edge coloring we assign colors to edges so that every pair of adjacent edges has different colors. The least number of colors we can use to properly color all the edges of a given graph is called the *chromatic index*. It is denoted by χ'.

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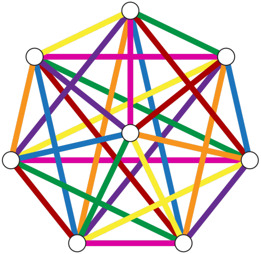

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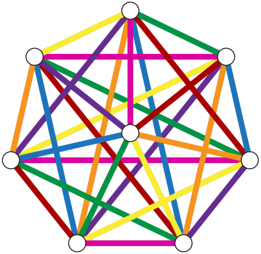

Vizing theorem

By a well-known theorem of Vizing, the chromatic index of G is either $\Delta(G)$ or $\Delta(G)+1$. When $\chi'(G)=\Delta(G)$, G is said to be of class 1; otherwise, it is said to be of class 2.

Definition of edge minimum coloring game

Definition

We can associate a cooperative game $(E, w_G^{\chi'})$ with every graph $G = (V, E)$ such that

$$w_G^{\chi'}(S) = \chi'(G[S]), \forall S \subseteq E.$$

We call such game the *edge minimum coloring game* for G.

Core of edge minimum coloring game on path

Observation

The chromatic index of every path is 2. Simply alternate the two colors.

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We can identify every pair adjacent edges with a characteristic vector of these edges. All the possible convex combinations of such vectors are exactly the core of edge minimum coloring game on path.

Core of edge minimum coloring game on complete graph

Proposition

The core of a complete graph K_n can be described as

$$C((V, w_{\chi'})) = \mathrm{conv}\{x^{(v)} | v \in V\},$$

where

$$x^{(v)}(e) := \begin{cases} 1 & \text{if } v \in e, \\ 0 & \text{if } v \notin e. \end{cases}$$

Core of edge coloring game on cycle

So far, we had edge coloring games with nonempty core.

Proposition

Cycle graphs always have empty core.

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We see a pattern now...

Characterization of core for edge minimum coloring games

Theorem, B. 2018+

An edge minimum coloring game on graph G has an nonempty core if and only if the graph G has the chromatic index $\chi'(G)$ equal to $\Delta(G)$. In other words, graphs with empty core are exactly the graphs of class 2.

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Thank you for attention and for the conference!

Hvala!