Complexity of k-rainbow independent domination and some result on the lexicographic product of graphs

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Joint work with...

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First article on this topic

On k-rainbow independent domination in graphs,

- Tadeja Kraner Šumenjak, Douglas F. Rall and Aleksandra Tepeh.
- Applied Mathematics and Computation, **333**, 2018, 353-361.

Overview

Basics

- Simple and connected graphs

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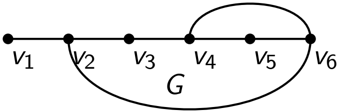

$$N(v_4) = \{v_3, v_5, v_6\},$$
$$N[v_4] = \{v_3, v_4, v_5, v_6\}.$$

Open\closed neighbourhood of a set

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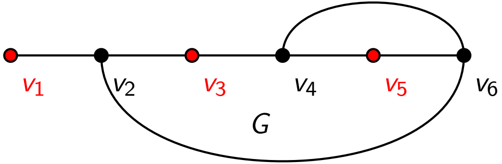

$$A = \{v_1, v_3, v_5\}, N(A) = \{v_2, v_4, v_6\}, N[A] = V(G).$$

Dominating set

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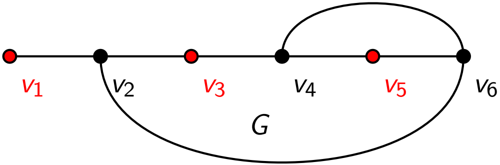

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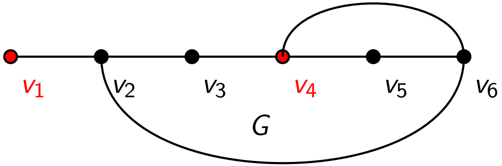
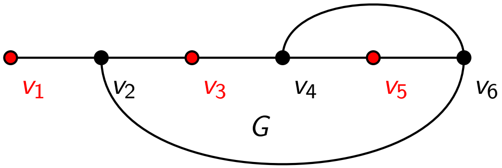


$$\gamma(G) = 2.$$

- The *Domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Independent set

- $\{v_1, v_2, \ldots v_k\} \subseteq V(G)$ is *independent set* of G if $v_i v_j \notin E(G)$, for all $i, j \in \{1, \ldots, k\}$.



Independent dominating set and independent domination number

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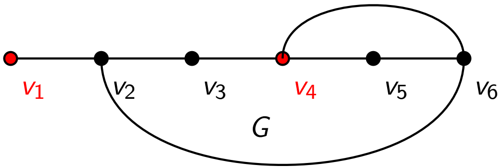

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k-dominating set

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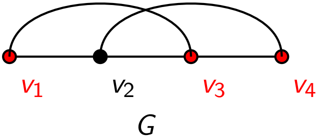

2-dominating set $D = \{v_1, v_3, v_4\}$.

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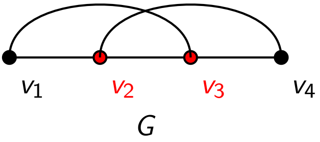

$$D = \{v_2, v_3\}, \gamma_2(G)) = 2.$$

k-rainbow domination

Definition

A function $f\colon V(G) \to 2^{\{1,2,\dots,k\}}$ is called a k-rainbow dominating function (kRDF) of G if for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ it follows

$$\bigcup_{u \in N(v)} f(u) = \{1, 2, \dots, k\}.$$

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Definition

The weight, $w(f)$, of f is defined as

$$w(f) = \sum_{v \in V(G)} |f(v)|.$$

k-rainbow domination number

Definition

Given a graph G, the minimum weight $w(f)$, of f with respect to all k-rainbow dominating functions is called the *k-rainbow domination number* of G, which we denote by $\gamma_{rk}(G)$.

k-rainbow independent domination

Definition

For a function $f : V(G) \to \{0, 1, 2, \dots k\}$ we denote $V_i = \{x \in V(G) \,:\, f(x) = i\}$. A function $f : V(G) \to \{0, 1, \dots, k\}$ is called a *k-rainbow independent dominating function* (*kRiDF*) of G if V_i is independent for $1 \leq i \leq k$, and for every $x \in V_0$ it follows that $N(x) \cap V_i \neq \emptyset$, for every $i \in [k]$.

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Definition

The *k-rainbow independent domination number* of a graph G ($\gamma_{rik}(G)$) is the minimum weight of a kRiDF of G.

Some results

Theorem

Let f be a kRiDF of an arbitrary graph G, then

$$\gamma_k(G) \leq \gamma_{rik}(G).$$

Theorem

Let T be a tree, then for any $k \geq 2$ it holds

$$\gamma_k(T) = \gamma_{rik}(T).$$

k-rainbow independent domination problem

The k-RiDF problem

INSTANCE: A graph G, a positive integer k greater than 2 and a positive integer s.
QUESTION: Does G have a k-RiDF of weight s?

Complexity of k-rainbow independent domination problem

Theorem

For any fixed $k \in \mathbb{Z}^+$, $k \geq 2$, the k-rainbow independent domination problem is NP-complete for bipartite graphs.

Proposition

For any fixed $k \in \mathbb{Z}^+$, $k \geq 2$, the k-rainbow independent domination number of a tree can be computed in linear time.

A sharp lower bound of k-rainbow independent domination number of an arbitrary graph

Corollary

For any graph G,

$$\gamma_{rik}(G) \geq \frac{kn}{\Delta(G) + k}.$$

Sharp bounds on trees

Corollary

Let T be a tree on n vertices and S is the set of all vertices of degree at most $k-1$, then

$$\frac{(k-1)n+1}{k} \leq \gamma_{rik}(T) \leq \frac{n+|S|}{2}.$$

Corollary

Let T be a tree on n vertices with l leaves, then

$$\frac{n+1}{2} \leq \gamma_{ri2}(T) \leq \frac{n+l}{2}.$$

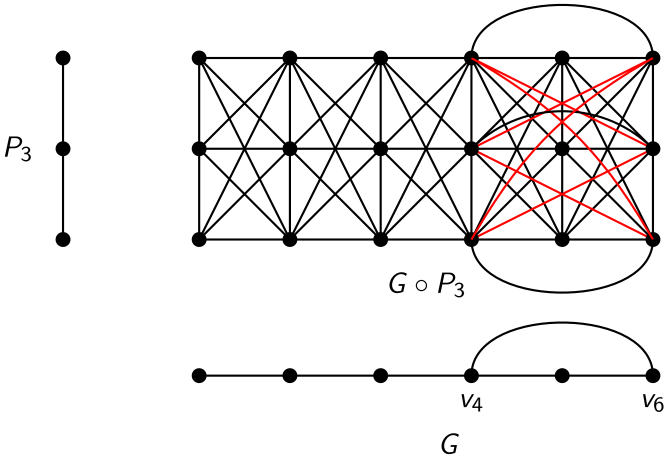
Lexicographic product of a graph

Definition

Lexicographic product $G \circ H$ of graphs G and H is a graph with the vertex set $V(G) \times V(H)$, where vertices (g_1, h_1) and (g_2, h_2) are adjacent if either

- $g_1 g_2 \in E(G)$ or
- $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

Lexicographic product of a graph



Sharp bounds on the lexicographic product

Theorem

For every graph G and every graph H such that $|V(H)| \geq k$,

$$k \cdot i(G) \leq \gamma_{rik}(G \circ H) \leq \gamma_{rik}(H) i(G).$$

2-rainbow independent domination number on the lexicographic product of graphs

Definition

An ordered triple (A, B, C) of pairwise disjoint independent sets $A, B, C \subseteq V(G)$, where also $A \cup B$ and $A \cup C$ are independent, is an *independent dominating triple* of G if

- for every vertex $x \in V(G) \setminus (A \cup B \cup C)$, there exists a vertex $w \in A$ such that $x \in N_G(w)$ or there exist vertices $w_1 \in B$ and $w_2 \in C$ such that $x \in N_G(w_1) \cap N_G(w_2)$,
- for every vertex $x \in B$ there exists a vertex $y \in C$ such that $x \in N_G(y)$ and for every vertex $x \in C$ there exists a vertex $y \in B$ such that $x \in N_G(y)$.

The exact formula

Theorem

Let G be an arbitrary graph and H a non-trivial graph of order n, then

$$\gamma_{ri2}(G \circ H) = \min\{\gamma_{ri2}(H)|A| + i(H)|B \cup C| :$$

$$(A, B, C) \text{ is an independent dominating triple of G}\}.$$

Hvala! Thank you!