Relating Brunn-Minkowski and Rogers-Shephard inequalities with the asymmetry measure of Minkowski

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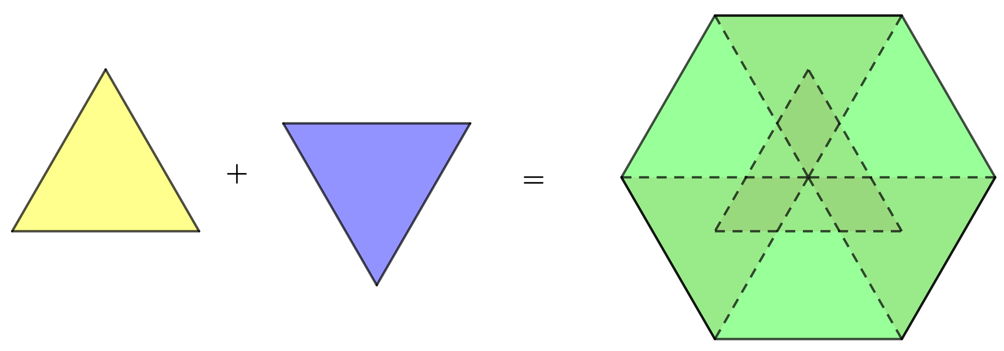
Abstract

In this work we propose to improve Brunn-Minkowski and Rogers-Shephard inequality in terms of the asymmetry measure of Minkowski. We do a first step by computing some bounds via stability results of those inequalities.

Definitions and properties

- Let $\mathcal{K}^n$ be the *set of full-dimensional compact and convex sets* in $\mathbb{R}^n$.
- A *simplex* Δ in $\mathbb{R}^n$ is the convex hull of $n+1$ affinely independent points.
- Let the *Minkowski sum* of K and L be defined by

$$K + L := \{x + y \in \mathbb{R}^n \,|\, x \in K,\ y \in L\}.$$



- Let $\mathrm{vol}(K)$ be the *n-dimensional volume* (or *Lebesgue measure*) of K.
- Let the *Minkowski measure of asymmetry* of K be defined by

$$s(K) := \inf\{\lambda \geq 1 \,|\, -K \subset x + \lambda \cdot K, \text{ for some } x \in \mathbb{R}^n\}.$$

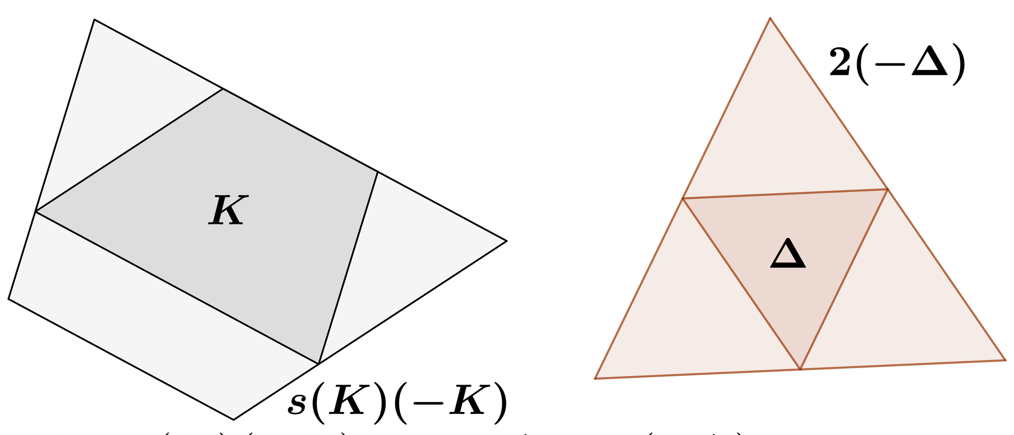


Fig. 2: $K \subset s(K)(-K)$ and $\Delta \subset 2(-\Delta)$ for a triangle Δ

Lemma: Let $K \in \mathcal{K}^n$. Then $1 \leq s(K) \leq n$. Moreover, $s(K) = 1$ iff $K = x - K$, $x \in \mathbb{R}^n$ and $s(K) = n$ iff K is a simplex.

References

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Volume and Minkowski addition

- The *Brunn-Minkowski inequality* (BM) (cf. [4,5]) states for $K, L \in \mathcal{K}^n$ that

$$\mathrm{vol}(K+L)^{\frac{1}{n}} \geq \mathrm{vol}(K)^{\frac{1}{n}} + \mathrm{vol}(L)^{\frac{1}{n}}.$$

Moreover, equality holds iff $L = x + \lambda \cdot K$, for some $x \in \mathbb{R}^n$ and $\lambda > 0$.

- The *Rogers-Shephard inequality* (RS) (cf. [6]) states for $K, L \in \mathcal{K}^n$ that

$$\mathrm{vol}(K+L)\mathrm{vol}(K \cap (-L)) \leq \binom{2n}{n}\mathrm{vol}(K)\mathrm{vol}(L).$$

Moreover, equality holds iff $L = -K$ is a simplex (cf. [3]).

- Letting $L = -K$, then (BM) and (RS) summarizes as

$$2^n \leq \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \leq \binom{2n}{n}.$$

Moreover, $=$ on LHS iff K=x-K, $x \in \mathcal{K}^n$, resp. on RHS iff K is a simplex.

> QUESTION: Let $K \in \mathcal{K}^n$ and $s \in [1, n]$ s.t. $s = s(K)$.
> What are the smallest $C(s) > 0$ and largest $c(s) > 0$ s.t.
>
> $$c(s) \leq \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \leq C(s)\,?$$

References

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Stability of Brunn-Minkowski and Rogers-Shephard

- A *stability version of Brunn-Minkowski inequality* (cf. [8,9]) states for $K \in \mathcal{K}^n$ that

$$\frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \leq 2^n \left(1 + \frac{A(K)^2}{14n^2 4^{n-1}}\right)^n,$$

where $A(K) = \inf_{x \in \mathbb{R}^n} \frac{\mathrm{vol}((K \setminus (x-K)) \cup ((x-K) \setminus K))}{\mathrm{vol}(K)}$.

- A *stability version of Rogers-Shephard inequality* (cf. [7]) states for $K \in \mathcal{K}^n$ that

$$1 - n(d_{BM}(K, \Delta) - 1) \leq \binom{2n}{n}^{-1} \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \leq 1 - \frac{d_{BM}(K, \Delta) - 1}{n^{50n^2}},$$

where the *Banach-Mazur distance* between K and a simplex Δ is defined by

$$d_{BM}(K, \Delta) = \inf_{x, y \in \mathbb{R}^n,\, M \in \mathbb{R}^{n \times n}} \{\lambda \geq 1 \,|\, \Delta \subset x + M(K) \subset y + \lambda\Delta\}.$$

References

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First answers the question

Theorem 1: Let $K \in \mathcal{K}^n$ and let $s = s(K)$. Then

$$c(s) \geq \begin{cases} 2^n \left(1 + \frac{1}{n\,4^{n-1}} \left(\frac{(s-1)^n vol_{n-1}(\mathbb{B}_2^{n-1})}{2^{n-1} n^{2n} vol_n(\mathbb{B}_2^n)}\right)^2\right)^n & \text{if } 1 < s < n, \\ \binom{2n}{n}(1 - 4n^2(n-s)) & \text{if } n - \frac{1}{4n} < s < n \end{cases}$$

and

$$C(s) \leq \begin{cases} (1+s)^n & \text{if } 1 < s < n, \\ \binom{2n}{n}\left(1 - \frac{n-s}{n^{1+50n^2}}\right) & \text{if } n - \frac{1}{4n} < s < n. \end{cases}$$

Remark: The 1. (resp. 2.) upper and lower bounds are specially good when $s(K) \approx 1$ (resp. $s(K) \approx n$).

The *diagram* $f : [1, n] \to \left[2^n, \binom{2n}{n}\right]$ is defined by $f(K) := \left(s(K), \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)}\right)$.

Theorem 2: $f(\mathcal{K}^n)$ is simply connected, contains $(1, 2^n)$ and $(n, \binom{2n}{n})$.

References

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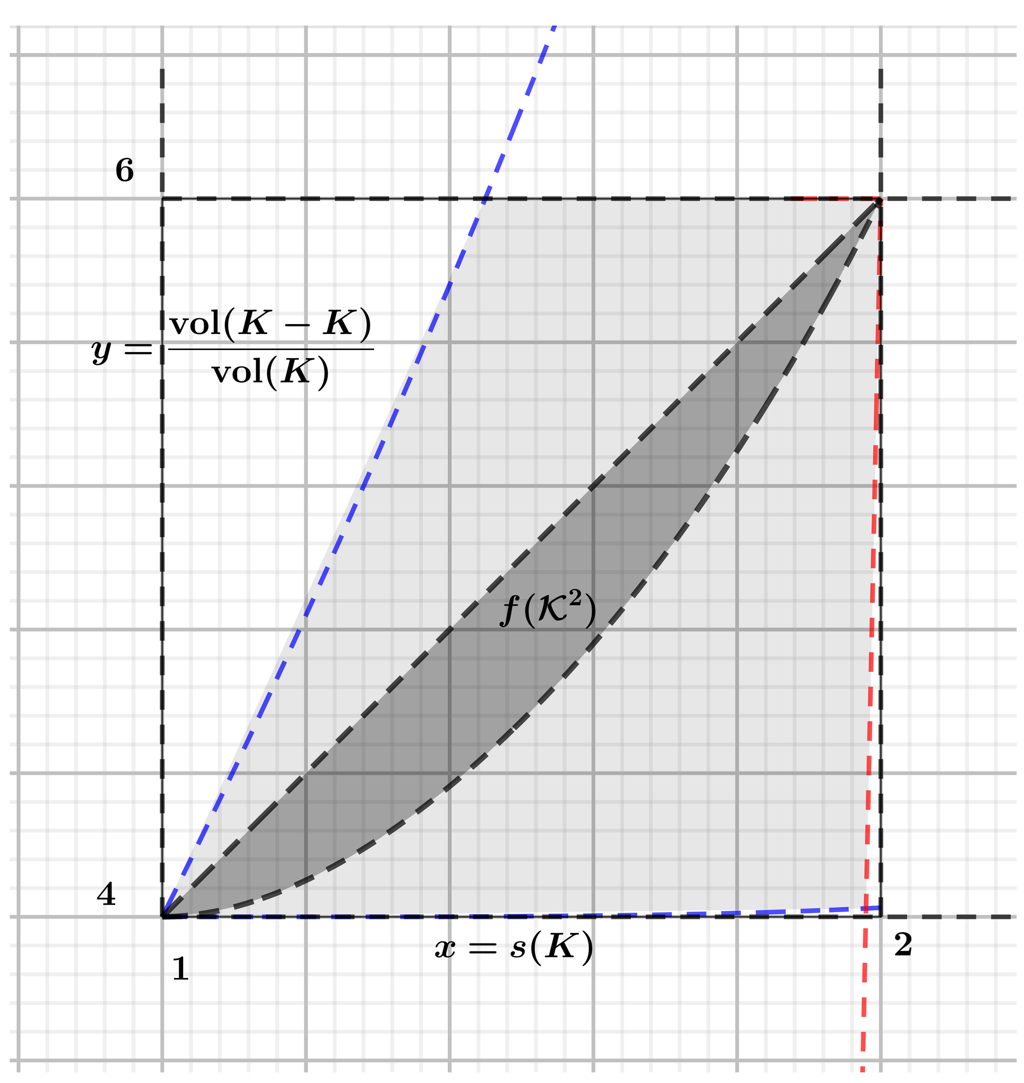
Stability results in the planar case



Let Δ be a regular simplex with center 0,
$K_s := \Delta \cap (s(-\Delta))$, $C_s := \mathrm{conv}(\Delta \cup (s(-\Delta)))$.

- $f(\mathcal{K}^2)$ contains the dark grey area, is contained in the light grey one.
- The lower boundary of the dark grey area is given by $f(K_s) = \left(s, \frac{2(s+1)^2}{2s-(s-1)^2}\right)$.
- The upper boundary of the dark grey area is given by $f(C_s) = (s, 2(s+1))$.
- The blue and red dashed lines are given by Theorem 1.

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