Maximum External Wiener Index of Graphs

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Overview

- **Introduction**:

Overview

- **Introduction**:
  - the Wiener index,

Overview

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  - the Wiener index,
  - the internal & the external Wiener index,

Overview

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Overview

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  - the Wiener index,
  - the internal & the external Wiener index,
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- **An overview of our results**:

Overview

- **Introduction**:
  - the Wiener index,
  - the internal & the external Wiener index,
  - motivation.
- **An overview of our results**:
  - auxiliary definitions,

Overview

- **Introduction**:
  - the Wiener index,
  - the internal & the external Wiener index,
  - motivation.
- **An overview of our results**:
  - auxiliary definitions,
  - proof of the main theorem,

Overview

- **Introduction**:
  - the Wiener index,
  - the internal & the external Wiener index,
  - motivation.
- **An overview of our results**:
  - auxiliary definitions,
  - proof of the main theorem,
  - open problems.

Overview

- **Introduction**:
  - the Wiener index,
  - the internal & the external Wiener index,
  - motivation.
- **An overview of our results**:
  - auxiliary definitions,
  - proof of the main theorem,
  - open problems.
- **References**

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

The Wiener index

Definition

$$\boxed{\mathrm{W}(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v)}$$

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

The Wiener index

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- [Wiener, 1947]

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

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- [Wiener, 1947]
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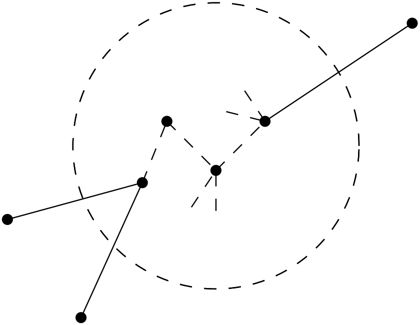
Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

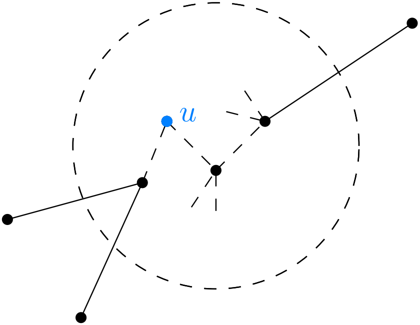
The internal & the external Wiener index



$$\mathrm{W}(G) = \sum_{\substack{u,v\in V(G)\\ \min\{d(u),d(v)\}\geq 2}} d(u,v) \quad + \sum_{\substack{u,v\in V(G)\\ \min\{d(u),d(v)\}=1}} d(u,v)$$

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

The internal & the external Wiener index



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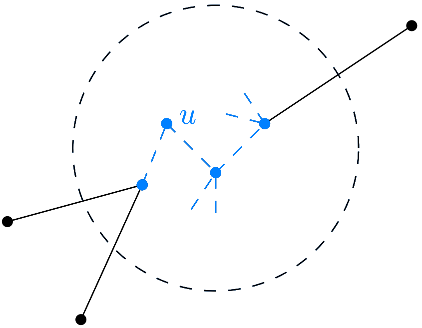

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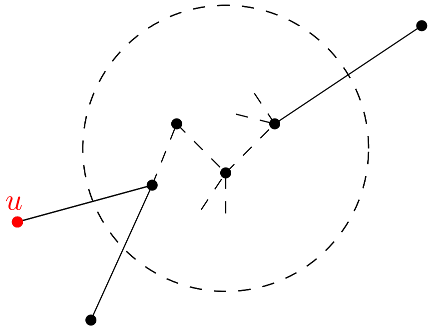

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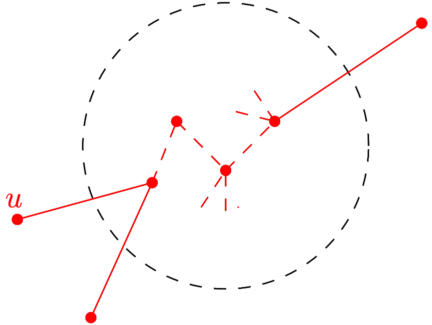
$\mathrm{W}_{\mathrm{ex}}(G)$

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

The internal & the external Wiener index



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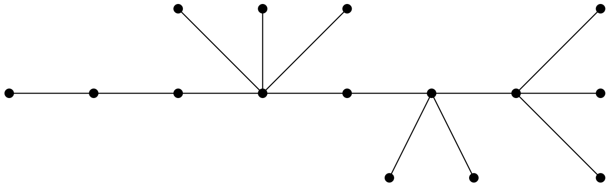
Motivation

Definition

A **caterpillar** is a tree with a central path in which vertices located outside this path are directly connected to it by an edge.



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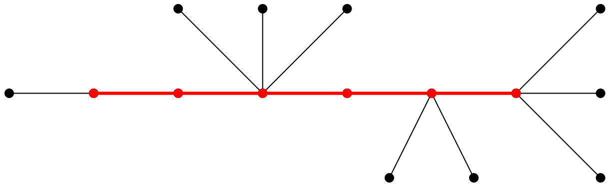

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Motivation

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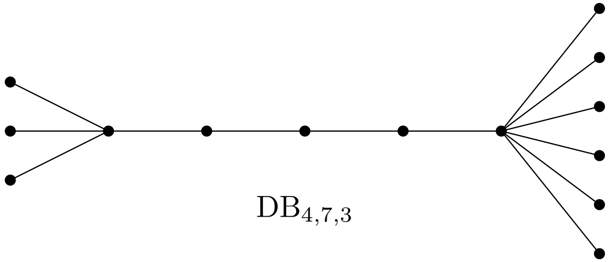
A **double broom** $\mathrm{DB}_{a,b,c}$ is a caterpillar obtained from P_{c+2} by attaching $a-1$ pendant vertices to one of its endpoints and $b-1$ pendant vertices to its other endpoint. A double broom is **balanced** if $|a-b| \leq 1$.



$\mathrm{DB}_{4,7,3}$

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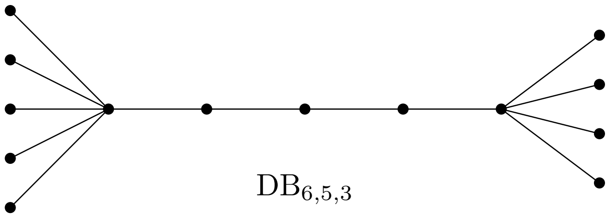

$\mathrm{DB}_{6,5,3}$

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Motivation

Conjecture [Gutman et al., 2016]

Among all trees of a fixed order, double brooms have the greatest W_{ex}.

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The double broom that attains the maximal W_{ex} is the balanced double broom $\mathrm{DB}_{a,b,c}$.

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The double broom that attains the maximal W_{ex} is the balanced double broom $\mathrm{DB}_{a,b,c}$.

Theorem [main result]

The graphs on n vertices with the maximum W_{ex} are balanced double brooms $\mathrm{DB}_{a,b,c}$ with suitably chosen a, b and c, $n = a + b + c$.

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Outline of the proof

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Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

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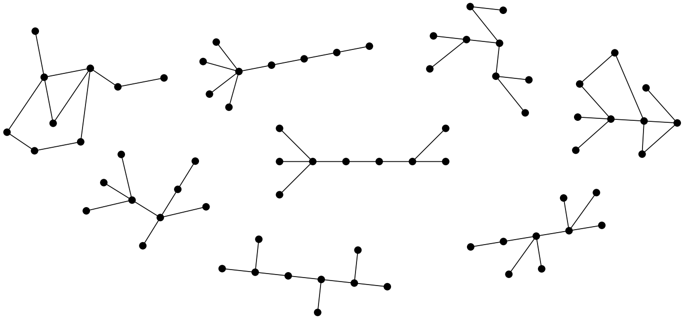

Graphs

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

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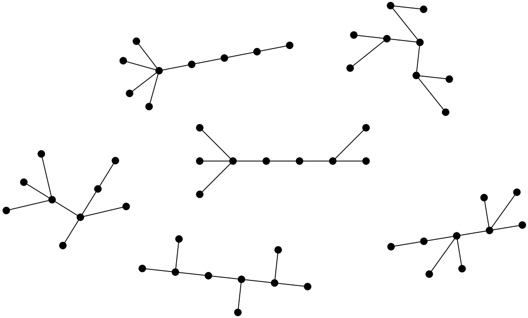

Graphs $\rightarrow$ trees

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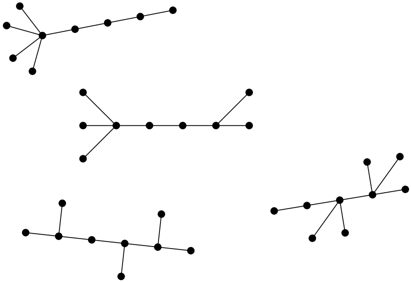

Graphs $\rightarrow$ trees $\rightarrow$ caterpillars

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

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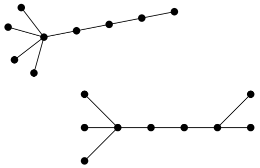

Graphs $\rightarrow$ trees $\rightarrow$ caterpillars $\rightarrow$ $\mathrm{DB}_{a,b,c}$

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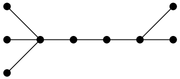

Graphs $\rightarrow$ trees $\rightarrow$ caterpillars $\rightarrow$ $\mathrm{DB}_{a,b,c}$ $\rightarrow$ balanced $\mathrm{DB}_{a,b,c}$

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Auxiliary definitions

Definition

A vertex is a **branching point** if at least three of its neighbours are non-leaf vertices.

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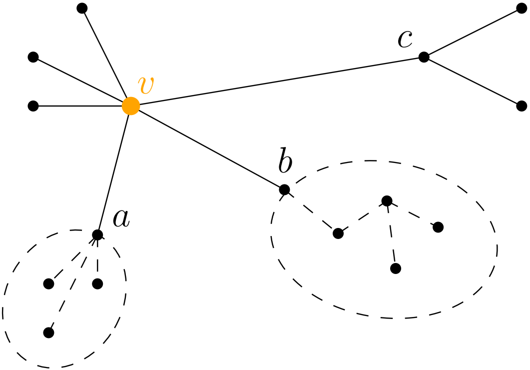
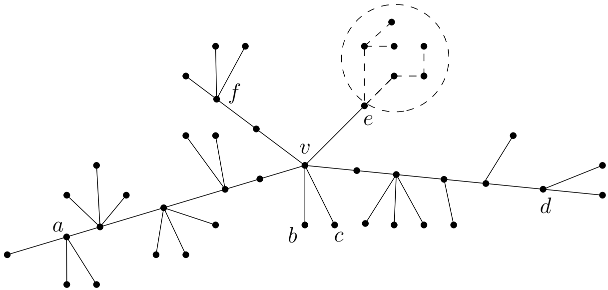


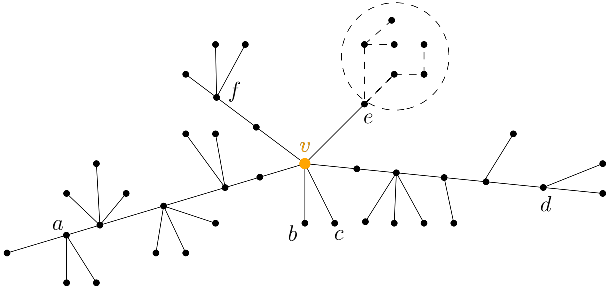
Auxiliary definitions



Auxiliary definitions

Definition

A branching point v is **peripheral** if all connected components of $G - v$, except at most one, are caterpillars with an endpoint adjacent to v.

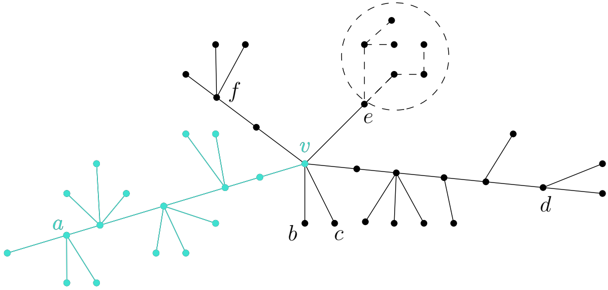


Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Auxiliary definitions

Definition

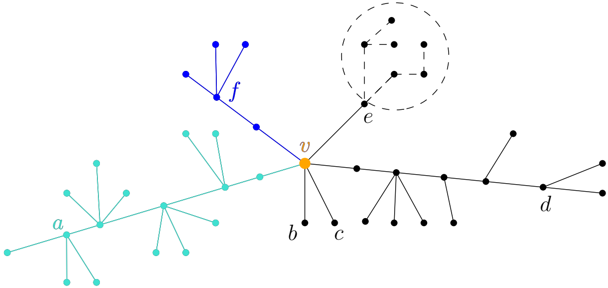
A (peripheral) branching point together with such a caterpillar component is called a **brush**.



Auxiliary definitions

Definition

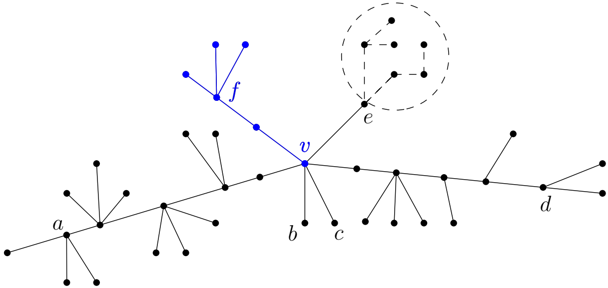
Two brushes are **adjacent** if they share the same attachment point.



Auxiliary definitions

Definition

A **broom** is a brush in which every vertex of the central path, except for the attachment point and (possibly) the other endpoint of this path, has degree two.

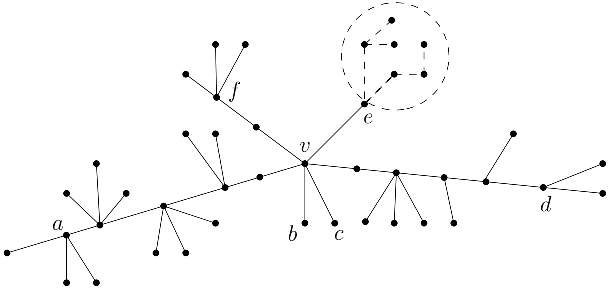


Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Auxiliary definitions

Observation

Every pair of brushes either has disjoint vertex sets or shares precisely one vertex, the attaching vertex. Additionally, $d_B(v) = 1$ holds for every brush B and its attaching vertex v.



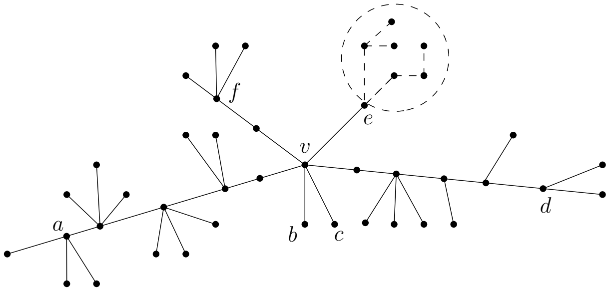
Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Auxiliary definitions

Definition

The **internal degree** $d_i(v)$ of v is the number of its non-leaf neighbours. The **branching sum** of G:

$$\mathrm{BS}(G) = \sum_{\substack{v \text{ is a branching} \\ \text{point in } G}} d_i(v).$$



Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Key steps

Theorem [main result]

The graphs on n vertices with the maximum W_{ex} are balanced double brooms $\mathrm{DB}_{a,b,c}$ with suitably chosen a, b and c, $n = a + b + c$.

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Claim 1

The maximum W_{ex} is attained by a tree.

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Key steps

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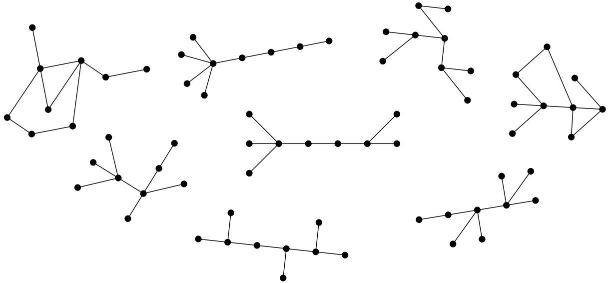

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Key steps

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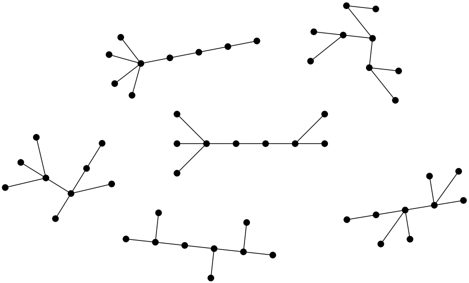


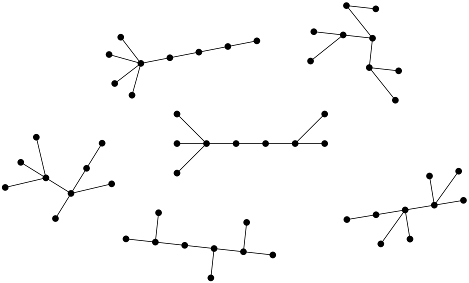
Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Key steps

Assume $\mathrm{BS}(G) > 0$.

Claim 2

Let B be a brush of the extremal graph. Then there exist two other non-trivial brushes that are adjacent to each other.



Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Key steps

Claim 3

All brushes of the extremal graph except for at most one are brooms.

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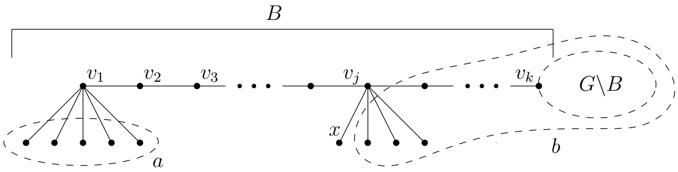
Lemma

Let B be a non-trivial brush. If $b > a$, moving a pendant vertex x from v_j to v_1 strictly increases $\mathrm{W_{ex}}$. If b $<$ a, this move strictly decreases $\mathrm{W_{ex}}$, and if b $=$ a, $\mathrm{W_{ex}}$ remains the same.



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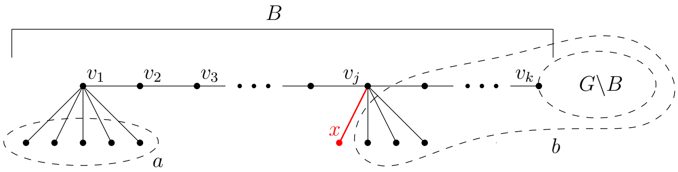

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Key steps

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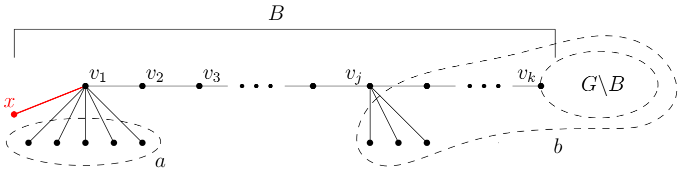
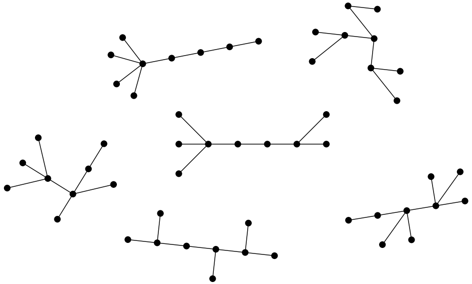


Key steps

The extremal graph is a tree with at most one non-broom brush; for every brush there exist two brooms that are adjacent to each other.

Claim 4

The maximum external Wiener index is attained by a caterpillar tree.



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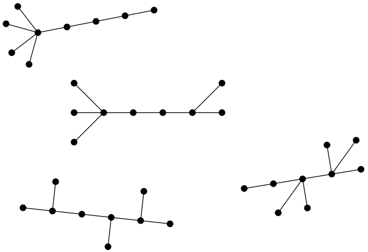

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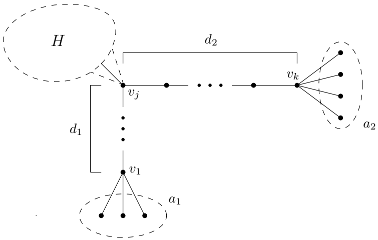
The maximum external Wiener index is attained by a caterpillar tree.

The Sliding Lemma

If G consists of a double broom D with a subgraph H attached to v_j, then the maximum W_{ex} is attained for $j = 1$ or $j = k$.



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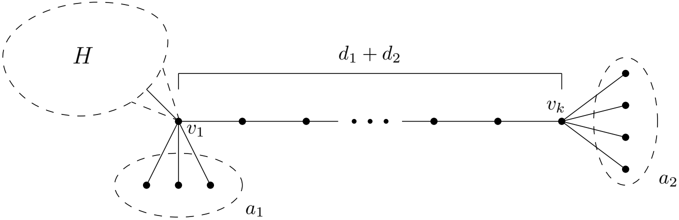
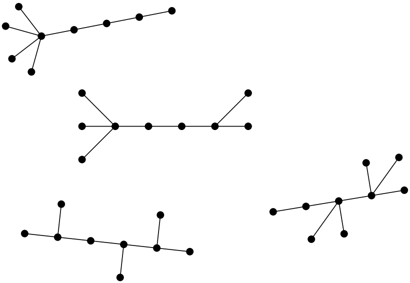


Key steps

Claim 5

The maximum external Wiener index is attained by a double broom.



Key steps

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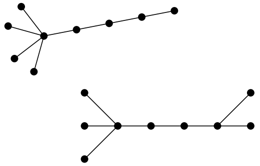

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Key steps

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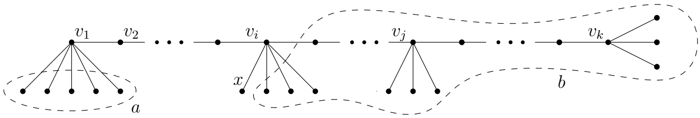

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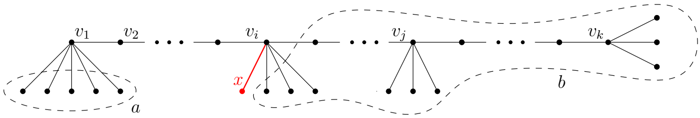

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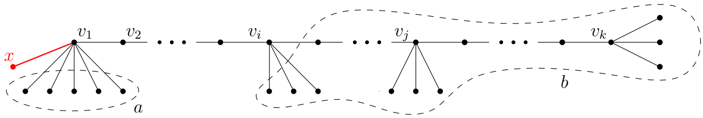

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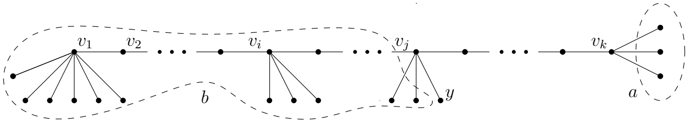

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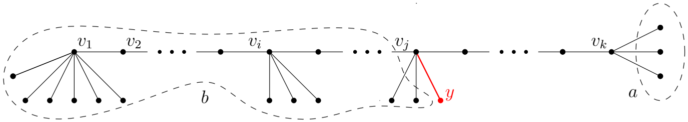

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Key steps

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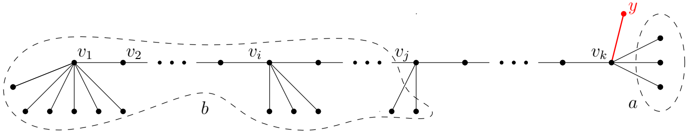

Open problems

[Stevanović, 2008]

Problem

Determine the graphs of a given order n and maximum degree Δ that attain the maximum value of the external Wiener index.

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

Open problems

[Stevanović, 2008]

Problem

Determine the graphs of a given order n and maximum degree Δ that attain the maximum value of the external Wiener index.

[Plesník, 1984]

Problem

Determine the graphs of a given order n and diameter that attain the maximum value of the external Wiener index.

Introduction
The Wiener index
The internal & the external Wiener index
Motivation
An overview of our results
Main theorem
Auxiliary definitions
Proof of the main theorem
Open problems
References

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