Graphs preserving total distance upon vertex removal

28.09.2018

Snježana Majstorović

Wiener index of a graph

⧫ Wiener index or total distance of a connected graph G is the sum of distances between all (unordered) pairs of vertices in G:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = \frac{1}{2} \sum_{v \in V(G)} t_G(v).$$

• first appearence: 1947. by H. Wiener, chemistry: the approximation formula for the boiling point of paraffin (correlations between the boiling points of paraffins and the structure of their molecules.)
$\rightarrow$ Wiener index $=$ topological index in chemistry

• first math paper: 1976. by R.C. Entringer, D. E. Jackson and D. A. Snyder
$\rightarrow$ Wiener index $=$ graph invariant in mathematics

• nowadays: sociometry, social networks
$\rightarrow$ Wiener index $\sim$ centrality measure

Wiener index of a graph

♦ Wiener index - very popular subject of study

LOTS OF CONJECTURES...

• **Conjecture 1**: Let G be a graph with diameter $d > 2$ and order $2d + 1$. Then $W(G) \leq W(C_{2d+1})$.

• **Conjecture 2**: Among all regular graphs on n vertices, the maximum Wiener index is attained by a graph with the maximum possible diameter.

• **Conjecture 3**: In the class of graphs G on n vertices, $W(L(G))$ attains maximum for some dumbbell graph.

• **Conjecture 4**: For every integer $g \geq 3$, there exist infinitely many graphs G of girth g satisfying $W(G) = W(L(G))$.

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Wiener index of a graph

... AND OPEN PROBLEMS

• **Problem 1**: What is the maximum Wiener index among graphs of order n and diameter d ?

• **Problem 2**: Find the maximum Wiener index among unicyclic graphs with n vertices and matching number i, $3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

• **Problem 3**: Do the extremal graphs for the maximum Wiener index in the classes of k-connected and k-edge-connected graphs coincide?

[M. Knor, R. Škrekovski and A. Tepeh: Mathematical aspects on Wiener index, Ars Math. Contemp. **11** (2016) 327–352.]

• **Problem 4**: [Šoltes, 1991.] Find all graphs G for which $W(G) = W(G - v)$ $\forall v \in V(G)$. One such graph is C_{11}.

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Relaxed version of Šoltes's problem

◆ **Problem**: Find all graphs G for which

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Solution: (We show that) there are infinitely many graphs with this property!!!

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Solutions of relaxed version of Šoltes's problem

◆ A. A. Dobrynin, R. Entringer and I. Gutman, Wiener index of trees: theory and application, *Acta Appl. Math.* **66** (2001) 211–249

→ several examples of G for which $G - v = T$ (T - tree) and $W(G) = W(T)$
→ for G as a join of T and some vertex v_0: $W(G) = W(T)$ if and only if T is:

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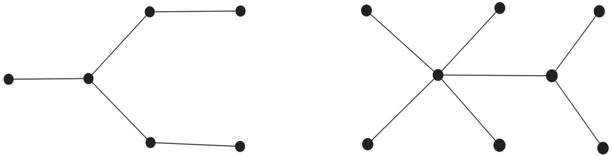

Solutions of relaxed version of Šoltes's problem

• What are the conditions on the particular vertex v_0 that we wish to remove from G so that $W(G) = W(G - v_0)$?

Answer

Let G be a graph and v_0 its vertex so that $W(G) = W(G - v_0)$. Then:
(i) $d(v_0) \neq 1$, $W(G) = W(G - v_0) + t_{G-v_0}(u) + |V(G - v_0)|$, $v_0u \in E(G)$
(ii) v_0 belongs to at least one cycle,
(iii) if $d(v_0) = 2$, then v_0 is not a vertex of triangle,
(iv) if $d(v_0) = 2$, then v_0 is not a vertex of chordless 4-cycle.

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Results for unicyclic graphs

◆ K. Knor, S. Majstorović, R. Škrekovski, Graphs whose Wiener index does not change when a specific vertex is deleted, *Discrete Appl. Math.* **238** (2018) 126–132.

Theorem 1 [R. Škrekovski, M. Knor, S.M.]

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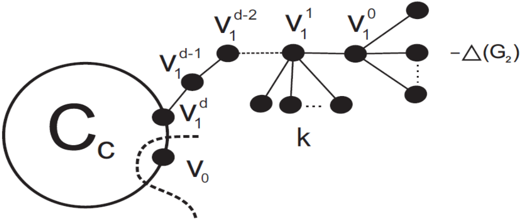

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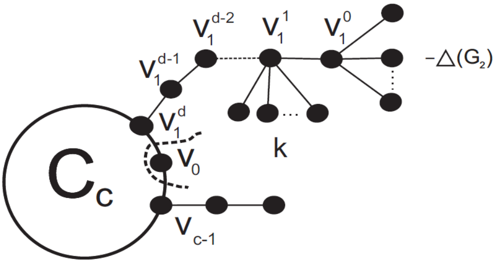
$\boxed{c = \{5, 6, 7\}}$



Results for unicyclic graphs

Theorem 2 [R. Škrekovski, M. Knor, S.M.]

Let $c \in \{3, 4\}$. Then there is no unicyclic graph G with a cycle of length c satisfying $W(G) = W(G - v_0)$.

Results for unicyclic graphs

Theorem 3 [R. Škrekovski, M. Knor, S.M.]

A unicyclic graph G on n vertices for which $W(G) = W(G - v_0)$ exists if and only if $n \geq 9$.

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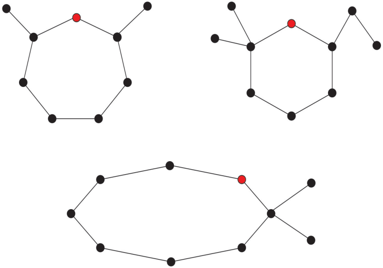

Induced subgraphs

Lemma 1 [R. Škrekovski, M. Knor, S.M.]

Let C_c be a cycle of even length, $c = 2a$, such that a is a square ($C_c = v_0 v_1 v_2 \cdots v_n v_0$). Moreover, let G_m be a graph with a vertex u for which $t_{G_m}(u) = \frac{a}{3}[a^2 - 6a + 2]$. Let H be obtained from G_m and C_c by identifying u with v_i, where $i = a - \sqrt{a}$. Then $W(H) = W(H - v_0)$.

Lemma 2 [R. Škrekovski, M. Knor, S.M.]

Let C_c be a cycle of odd length, $c = 2a + 1$, such that $4a + 1$ is a square ($C_c = v_0 v_1 v_2 \cdots v_n v_0$). Moreover, let G_m be a graph with a vertex u for which $t_{G_m}(u) = \frac{a}{6}[2a^2 - 9a - 5]$. Let H be obtained from G_m and C_c by identifying u with v_i, where $i = \frac{1}{2}(2a + 1 - \sqrt{4a + 1})$. Then $W(H) = W(H - v_0)$.

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Induced subgraphs

Basic tool:

Theorem (O. E. Polansky & D. Bonchev, 1986)

Let G_u and G_v be two graphs with n_u and n_v vertices, respectively, and let $u \in V(G_u)$, $v \in V(G_v)$.

(a) If G arises from G_u and G_v by connecting u and v by an edge, then

$$W(G) = W(G_u) + W(G_v) + n_u t_{G_v}(v) + n_v t_{G_u}(u) + n_u n_v.$$

(b) If G arises from G_u and G_v by identifying u and v, then

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Induced subgraphs

Theorem 4 [R. Škrekovski, M. Knor, S.M.]

Let G be an arbitrary graph. Then there are infinitely many connected graphs H, containing G as an induced subgraph, and such that $W(H) = W(H - v_0)$ for some vertex $v_0 \in V(H) \setminus V(G)$.

Further studies

- If G is unicyclic and $W(G) = W(G - v_0)$, then $d(v_0) = 2$.

Question: are there graphs G for which $W(G) = W(G - v_0)$ for $d(v_0) > 2$?

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Theorem 5 [R. Škrekovski, M. Knor, S.M.]

For every $k \geq 3$ there exist infinitely many graphs G with vertex v such that $d_G(v) = k$ and $W(G) = W(G - v)$.

Idea of the proof: For each k we show the existence of a graph G_1 with a vertex v such that $d_{G_1}(v) = k$ and $W(G_1) = W(G_1 - v)$. Then we construct an infinite class of graphs by attaching to G_1 a new graph G_2.

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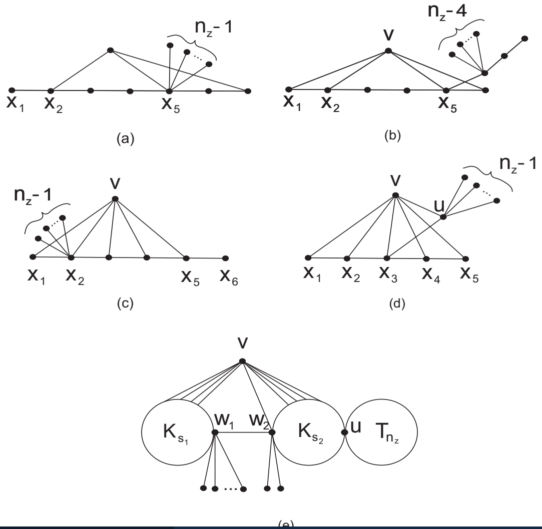


Further studies

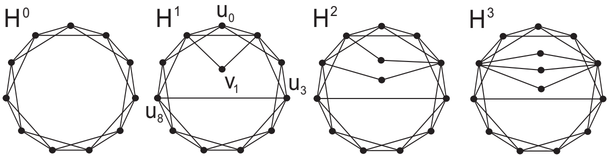
High degree vertices ($d(v)$ is a function of $|V(G)|$)....

Theorem 6 [R. Škrekovski, M. Knor, S.M.]

Let $n \geq 7$. Then there exists an n-vertex graph G with vertex v such that $d_G(v) = n - 2$ and $W(G) = W(G - v)$.

Theorem 7 [R. Škrekovski, M. Knor, S.M.]

Let $n \geq 7$. Then there exists an n-vertex graph G with vertex v such that $d_G(v) = n - 1$ and $W(G) = W(G - v)$.



Further studies

Theorem 8 [R. Škrekovski, M. Knor, S.M.]

If G is an n-vertex graph for which $\delta(G) \geq n/2$, then $W(G) \neq W(G - v)$ for every $v \in V(G)$.

Some open problems

Well.... Šoltes's problem is still open!

• Are there k-regular connected graphs G other than C_{11} for which the equality $W(G) = W(G - v)$ holds for at least one vertex $v \in V(G)$?

• For a given r, find (infinitely many) graphs G for which

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End of slides

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