Periodic Parallelogram Polyominoes

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Parallelogram Polyominoes

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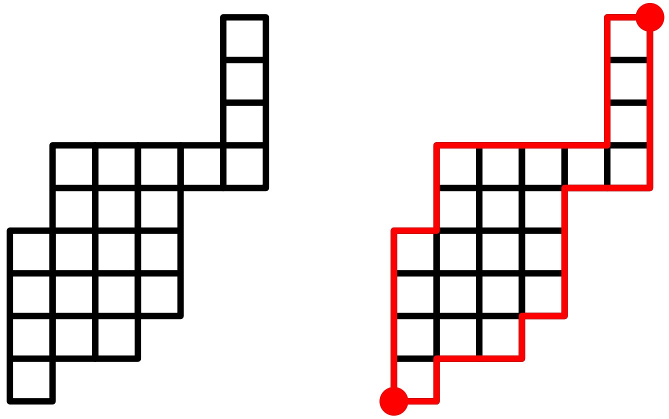
A parallelogram polyomino (PP) is a connected polyomino whose boundary is the union of two North-East paths.



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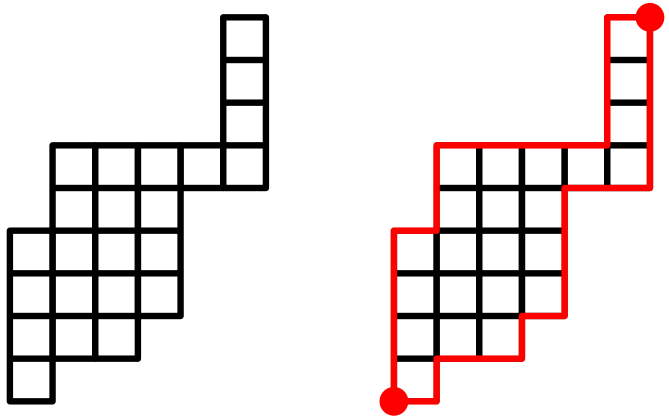

It is well-known that the number of PPs with half-perimeter $n+1$ is the Catalan number $Cat_n = \frac{1}{n+1}\binom{2n}{n}$.

These are recurring objects in combinatorics, and also occur as simple models in statiscal physics.

Parallelogram Polyominoes

We want to count polyominoes by: number of rows (height), number of columns (width), and total number of unit squares (area)

$$PP(x,y,q) = \sum_{p \in PP} x^{\mathsf{height(p)}} y^{\mathsf{width(p)}} q^{\mathsf{area(p)}}$$

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Introduce the series

$$N(x,y,q) = \sum_{n \geq 0} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q;q)_n (xq;q)_n} \qquad \hat{N}(x,y,q) = \sum_{n \geq 1} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q;q)_{n-1} (xq;q)_n}$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$.

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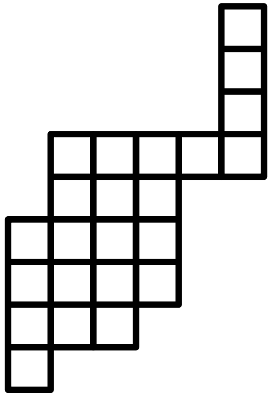
Theorem[Bousquet-Mélou, Viennot '92]

$$PP(x, y, q) = -x \frac{\hat{N}(x, y, q)}{N(x, y, q)}$$

Encoding of polyominoes

Let p be a PP with n columns. Let $b_1, \ldots, b_n$ be the number of cells of each column from left to right.
For $i = 2, \ldots, n$, let a_i be the number of common rows between the $i-1$th and ith column.



$$b_1, \ldots, b_6 = 4, 5, 5, 4, 1, 4$$

$$a_2, \ldots, a_6 = 3, 5, 4, 1, 1$$

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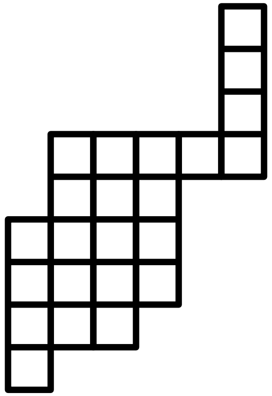

$$b_1, \ldots, b_6 = 4, 5, 5, 4, 1, 4$$

$$a_2, \ldots, a_6 = 3, 5, 4, 1, 1$$

By convention set $a_1 = 1$. So one can encode a PP of width n by a sequence

$$1 = a_1 \leq b_1 \geq a_2 \leq b_2 \cdots \geq a_n \leq b_n.$$

Heaps

Fix a set of *pieces* $P = \{a, b, c, \ldots\}$ for which we impose a certain number of commutations $ab = ba$, $bd = db$, etc... We then call heap of pieces any word on the alphabet P up to the allowed commutations.

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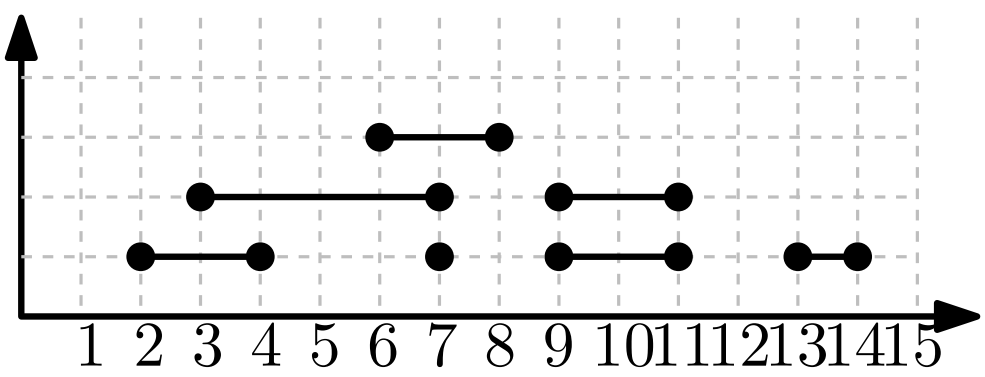
[9,11][7][9,11][2,4][13,14][3,7][6,8]

Such a heap can then be represented by letting the pieces drop one by one when reading the word.



PPs and heaps

Recall that to a PP we associated sequences a_i and b_i.

Bousquet-Mélou and Viennot showed if one stacks the segments $[a_n, b_n], \ldots, [a_1, b_1]$, this constitutes a bijection between PPs and semi-pyramids, i.e. heaps with a unique maximal element, which is of the form $[1, b]$.

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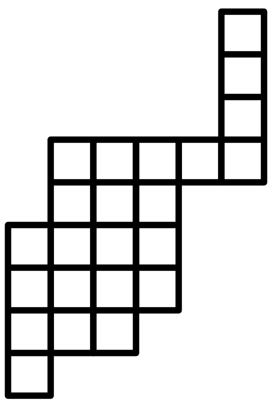

$$a_1, \ldots, a_6 = 1, 3, 5, 4, 1, 1$$
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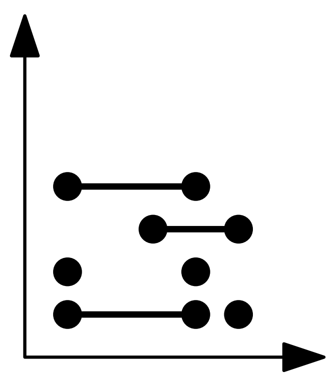

Enumeration of heaps

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Furthermore, when multiplied by this series, cancellations occur in the series counting semi-pyramids: what is left is a series counting certain trivial heaps, which can be expressed as $\hat{N}(x, y, q)$, and that completes the proof.

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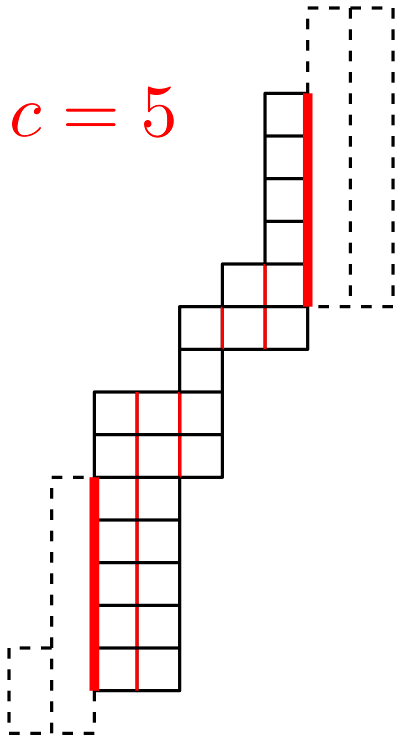
We want to adapt this proof to the case of periodic parallelogram polyominoes.

Periodic PP

A periodic PP (PPP) is a PP that one "extends periodically": this can be encoded by an integer c not larger than the height of the first or last columns, which represents the overlap between these columns in the periodic extension.



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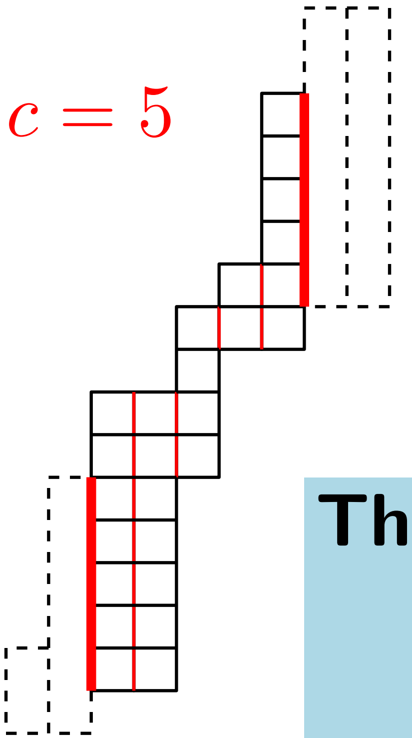


$$PPP(x,y,q) := \sum_{p \in PPP} x^{\text{height'(p)}} y^{\text{width(p)}} q^{\text{area(p)}}$$

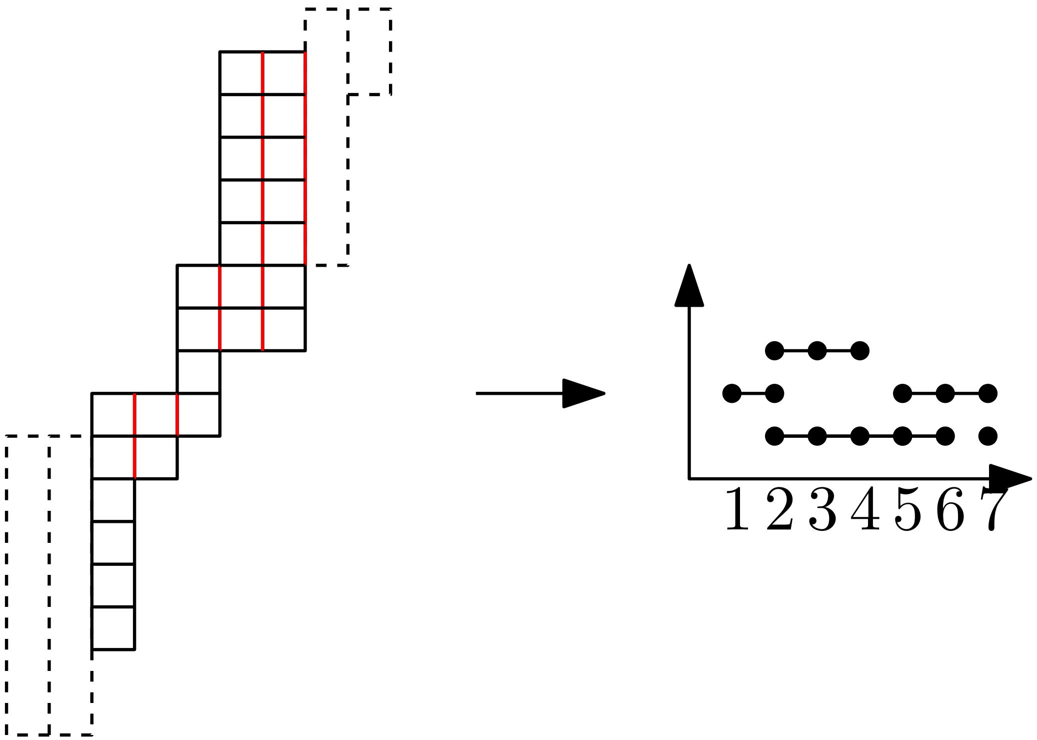
Theorem[Biagioli, Jouhet, N. ’17]

$$PPP(x,y,q) = -y \frac{\partial_y N(x,y,q)}{N(x,y,q)}$$

Proof

If one defines $a_1 = c$, one gets a sequence

$$a_1 \leq b_1 \geq a_2 \leq b_2 \cdots \geq a_n \leq b_n \geq a_1.$$



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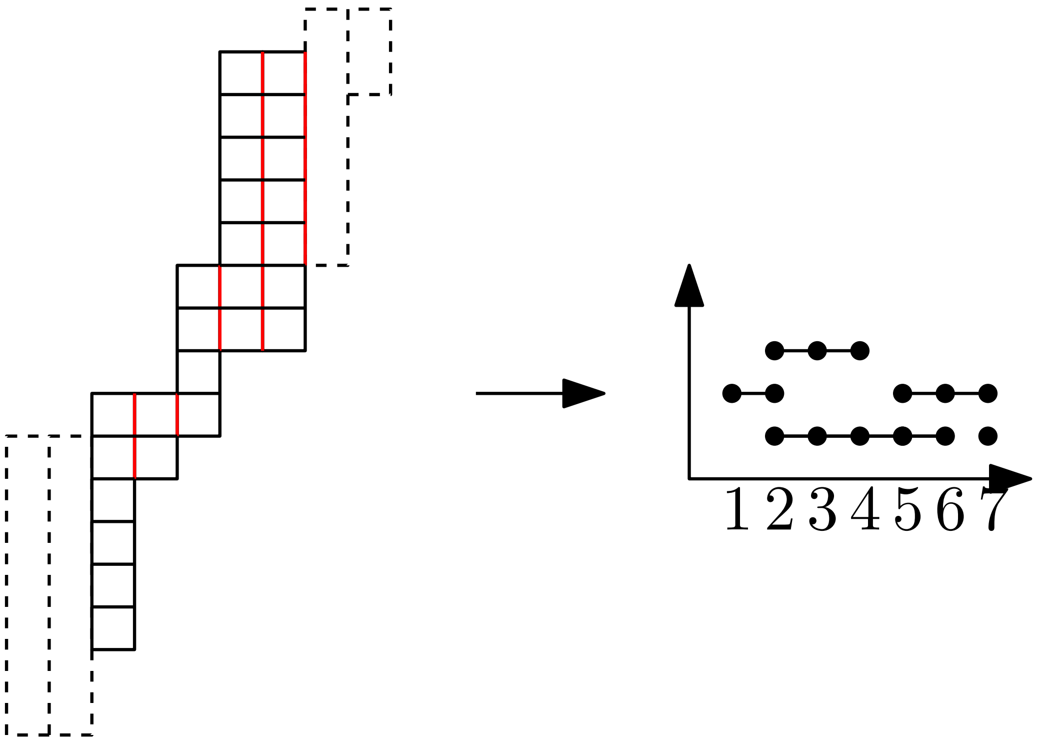

By stacking the segments $[a_i, b_i]$, one shows that PPPs are in bijection with heaps of segments with a special condition between minima and maxima (special heaps).

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THANK YOU FOR YOUR ATTENTION