Domino and square tilings of Euclidean and hyperbolic boards

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Tilings and Fibonacci numbers

Tilings of a $(1 \times n)$-board with squares and dominoes

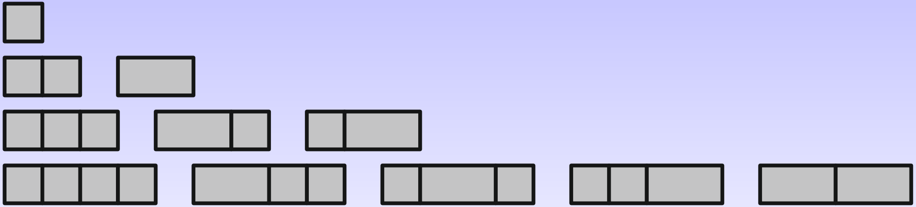
(1×8)-board square domino

$n = 1,\ f_1 = 1$:

$n = 2,\ f_2 = 2$:

$n = 3,\ f_3 = 3$:

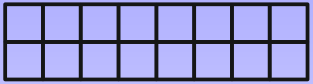
$n = 4,\ f_4 = 5$:



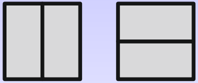
generally, $f_n = f_{n-1} + f_{n-2}$:

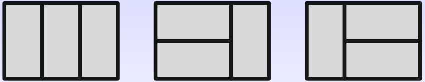
Tilings and Fibonacci numbers

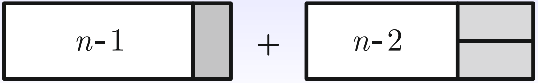
Tilings of a $(2 \times n)$-board with dominoes

(2×8)-board : 

$n = 1,\ f_1 = 1$: 

$n = 2,\ f_2 = 2$: 

$n = 3,\ f_3 = 3$: 

generally, $f_n = f_{n-1} + f_{n-2}$: 

Tilings of a $(k \times n)$-board with dominoes

$(2 \times n)$-board: $f_n = f_{n-1} + f_{n-2}$ (OEIS: A000045)

$(3 \times n)$-board: $f_n = 4f_{n-2} - f_{n-4}$ (OEIS: A001835)

$(4 \times n)$-board: $f_n = f_{n-1} + 5f_{n-2} + f_{n-3} - f_{n-4}$ (OEIS: A005178)

$(5 \times n)$-board: $f_n = 15f_{n-2} - 32f_{n-4} + 15f_{n-6} - f_{n-8}$ (OEIS: A003775)

Tilings with colored squares and dominoes

Tiling a $(1 \times n)$-board with colored squares and dominoes

Let u_n be the generalized Fibonacci number, where

$$u_n = au_{n-1} + bu_{n-2}, \quad (n \geq 2) \tag{1}$$

with initial values $u_0 = 1$, $u_1 = a$, is interpreted as the number of ways to tile a $(1 \times n)$-board using a colors of squares and b colors of dominoes.

For example, (1×8)-board  .

If $a = 1$, $b = 1$, then $u_n = f_n$.

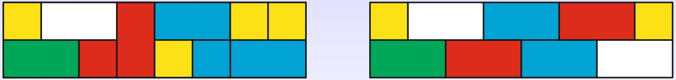
Tilings of a $(2 \times n)$-board with colored squares and dominoes

Let R_n be the number of tilings of $(2 \times n)$-board using a colors of squares and b colors of dominoes. Katz and Stenson (2009) showed the recurrence rule

$$R_n = (a^2 + 2b)R_{n-1} + a^2 b\, R_{n-2} - b^3 R_{n-3}, \quad (n \geq 3) \tag{2}$$

with initial values $R_0 = 1$, $R_1 = a^2 + b$ and $R_2 = a^4 + 4a^2 b + 2b^2$.

For example, (2×8)-board and $a = 3$, $b = 4$, breakable and unbreakable



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If $a = 1$, $b = 1$, then $R_n = 3R_{n-1} + R_{n-2} - R_{n-3}$.

A $(2 \times n)$-board on the hyperbolic mosaic with Schäfli symbol $\{4, 5\}$

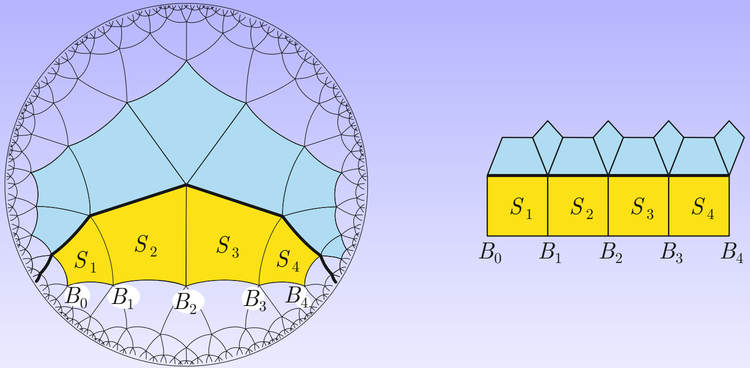


Figure: (2×4)-board on hyperbolic mosaic $\{4, 5\}$

A $(2 \times n)$-board on the hyperbolic mosaic with Schäfli symbol $\{4, q\}$

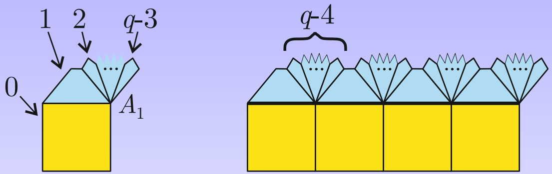


Figure: (2×1)-board and (2×4)-board on hyperbolic mosaic $\{4, q\}$ $(q \geq 5)$

Tiling exercise of the hyperbolic (2×1)-board on the mosaic $\{4, q\}$ $(q \geq 5)$ is the same as the tiling exercise of the Euclidean $(1 \times (q-2))$-board. So $R_1 = u_{q-2}$ and $r_1 = f_{q-2}$.

Tilings with colored squares and dominoes of a hyperbolic $(2 \times n)$-board

Subboards of a board

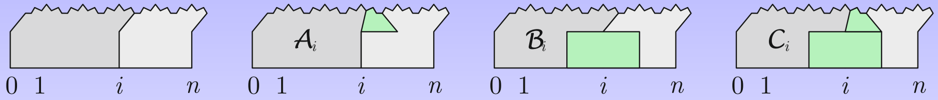


Figure: Breakable and unbreakable tilings in position i when $q = 7$

Let A_n, B_n, C_n and R_n be the number of the tiling on $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$ subboards and all the board, respectively.

Tilings with colored squares and dominoes of a hyperbolic $(2 \times n)$-board

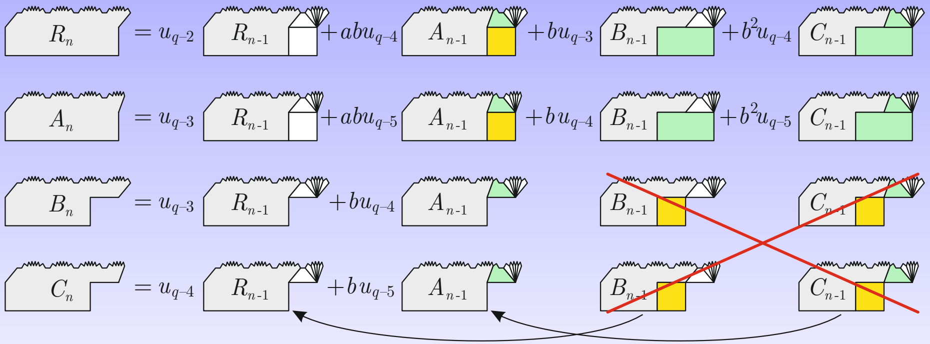


Figure: Base of recurrence connections of the subboards

The recurrence equations for $n \geq 1$ satisfy the system

$$\begin{array}{rcl} R_n & = & u_{q-2}R_{n-1} + ab\, u_{q-4}A_{n-1} + b\, u_{q-3}B_{n-1} + b^2\, u_{q-4}C_{n-1} \\ A_n & = & u_{q-3}R_{n-1} + ab\, u_{q-5}A_{n-1} + b\, u_{q-4}B_{n-1} + b^2\, u_{q-5}C_{n-1} \\ B_n & = & u_{q-3}R_{n-1} + b\, u_{q-4}A_{n-1} \\ C_n & = & u_{q-4}R_{n-1} + b\, u_{q-5}A_{n-1} \end{array} . \qquad (3)$$

$R_0 = 1$, $R_1 = u_{q-2}$, $R_2 = u_{q-2}^2 + abu_{q-4}u_{q-3} + bu_{q-3}^2 + b^2u_{q-4}^2$, $R_3 = (u_{q-2}^2 + 2abu_{q-4}u_{q-3} + 2bu_{q-3}^2 + 2b^2u_{q-4}^2)u_{q-2} + b^2(u_{q-3}u_{q-4} + (a^2+b)u_{q-4}u_{q-5} + au_{q-4}^2)u_{q-3} + ab^3u_{q-4}^2u_{q-5}$.

The matrix of the coefficients of (3) is

$$\mathbf{M} = \begin{pmatrix} u_{q-2} & ab\, u_{q-4} & b\, u_{q-3} & b^2\, u_{q-4} \\ u_{q-3} & ab\, u_{q-5} & b\, u_{q-4} & b^2\, u_{q-5} \\ u_{q-3} & b\, u_{q-4} & 0 & 0 \\ u_{q-4} & b\, u_{q-5} & 0 & 0 \end{pmatrix} .$$

Theorem

Assume $q \geq 4$. The sequence $\{R_n\}_{n=0}^{\infty}$ can be described by the fourth order linear homogeneous recurrence relation

$$R_n = \alpha_q R_{n-1} + \beta_q R_{n-2} + \gamma_q R_{n-3} - b^{2(q-2)} R_{n-4}, \quad (n \geq 4)$$

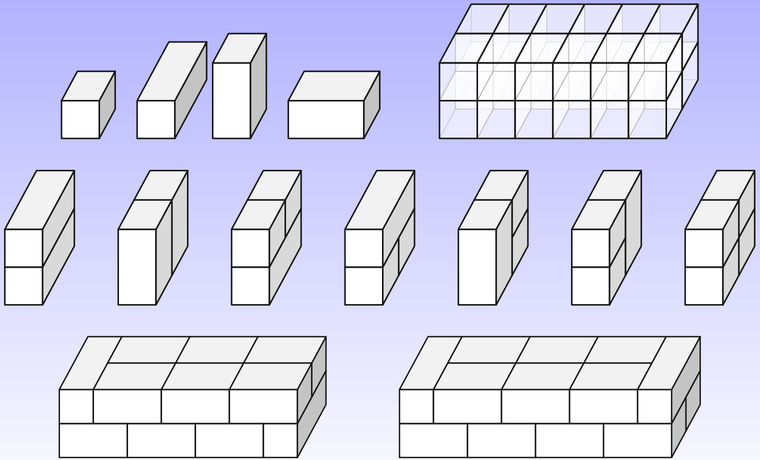
where

$$\begin{aligned} \alpha_{q+2} &= a\alpha_{q+1} + b\alpha_q, \\ \beta_{q+3} &= (a^2+b)\beta_{q+2} + b(a^2+b)\beta_{q+1} - b^3\beta_q, \\ \gamma_{q+2} &= -ab\gamma_{q+1} + b^3\gamma_q \end{aligned}$$

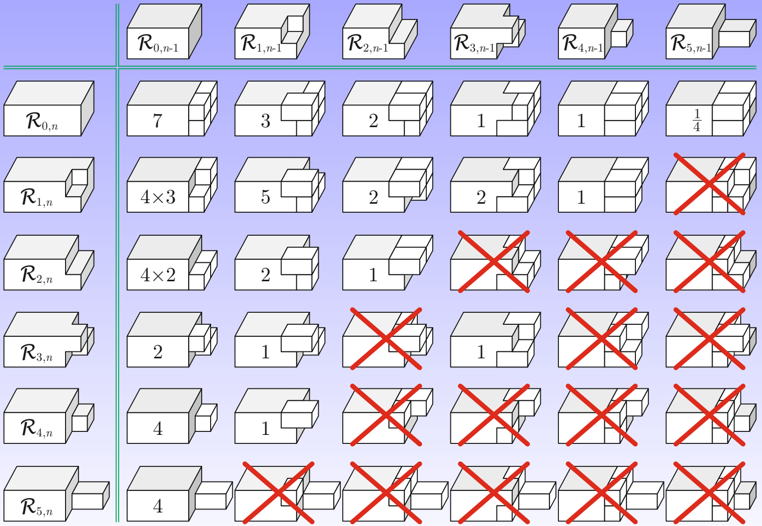
with initial values

$$\begin{aligned} &\alpha_4 = a^2+b,\ \alpha_5 = a(a^2+3b), \\ &\beta_4 = 2b(a^2+b),\ \beta_5 = b(a^2+b)(a^2+2b),\ \beta_6 = b(a^6+6a^4b+10a^2b^2+2b^3), \\ &\gamma_4 = b^2(a^2-b),\ \gamma_5 = -ab^3(a^2+b). \end{aligned}$$

Cubes, bricks, $(2 \times 2 \times n)$-board, tilings when $n = 1$, unbreakable tilings



Tilings with colored cubes and bricks in the Euclidean space



Theorem

The sequence $\{R_n\}_{n=0}^{\infty}$ *can be described by the sixth order linear homogeneous recurrence relation*

$$R_n = \sum_{i=1}^{6} \alpha_i R_{n-i} \quad (n \geq 6), \tag{4}$$

where

$$\begin{aligned}
\alpha_1 &= a^4 + 7a^2 b + 6b^2, \\
\alpha_2 &= b(a^6 + 6a^4 b + 6a^2 b^2 - 7b^3), \\
\alpha_3 &= -2b^3(a^6 + 5a^4 b + 13a^2 b^2 + 4b^3), \\
\alpha_4 &= b^5(a^6 + 2a^4 b + 6a^2 b^2 + 9b^3), \\
\alpha_5 &= b^8(a^4 - a^2 b + 2b^2), \\
\alpha_6 &= -b^{12},
\end{aligned}$$

and the initial values R_n $(n = 0, \ldots, 5)$ *are* $1,\ a^4 + 4a^2 b + 2b^2,$
$a^8 + 12a^6 b + 42a^4 b^2 + 44a^2 b^3 + 9b^4,$

Connections between the tilings and the unbreakable tilings

Theorem

If $n \geq 1$ *and* $m \geq 1$*, then*

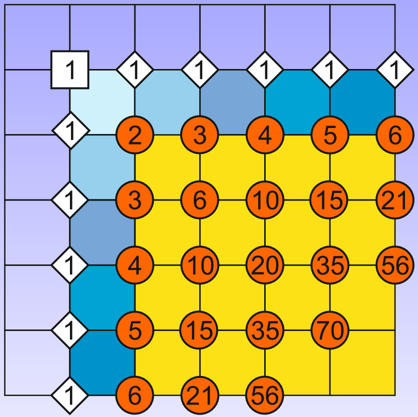
$$R_n = \sum_{i=0}^{n-1} R_i \widetilde{R}_{n-i}.$$

$$R_{n+m} = R_n R_m + \sum_{i=1}^{n} \sum_{j=1}^{m} R_{n-i} R_{m-j} \widetilde{R}_{i+j}.$$

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \widetilde{R}_i \cdot \sum_{j=0}^{n-i} R_j.$$

For the proof of these identities we just use the tilings.

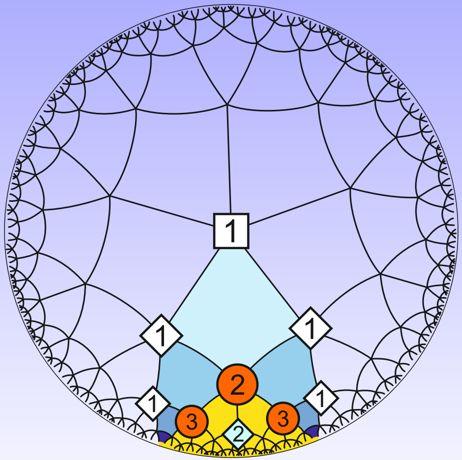
Classical Pascal's triangle



Value of a vertex is the number of the shortest paths from the vertex to the base vertex along the edges.
The layers are the vertices with same distance from base vertex.

Hyperbolic Pascal triangle $\{4, 5\}$

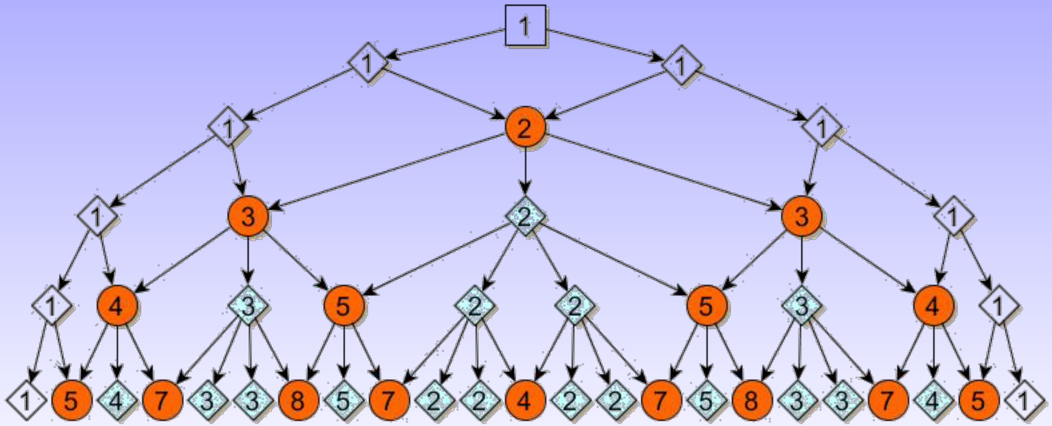
Extension in hyperbolic plane



Vertices are “Type A”, “Type B” and wingers.

Hyperbolic Pascal triangle $\{4, 5\}$

In a graph form



Vertices are "Type A", "Type B" and wingers.

Hyperbolic Pascal triangle $\{4, q\}$

Number of elements in a row

$$s_n = (q-1)s_{n-1} - (q-1)s_{n-2} + s_{n-3} \qquad (n \geq 4).$$

Sum of values in a row

$$\hat{s}_n = q\hat{s}_{n-1} - (q+1)\hat{s}_{n-2} + 2\hat{s}_{n-3} \qquad (n \geq 4).$$

Alternating sum of values a row

If $q \geq 5$ and odd

$$\widetilde{s}_n = \begin{cases} 0, & \text{if} \quad n = 3t+1, \quad n \geq 1, \\ (-2)^t(q-5)^{t-1} + 2, & \text{if} \quad n = 3t-1, \quad n \geq n_1, \\ 2(-2)^t(q-5)^{t-1} + 2, & \text{if} \quad n = 3t, \quad n \geq n_2. \end{cases}$$

If $q \geq 6$ and even

$$\widetilde{s}_n = \begin{cases} 0, & \text{if} \quad n = 2t+1, \quad n \geq 1, \\ -2(5-q)^{t-1} + 2, & \text{if} \quad n = 2t, \quad n \geq 2. \end{cases}$$

References

https://www.researchgate.net/profile/Laszlo_Nemeth3

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- L. Németh, On the hyperbolic Pascal pyramid, *Beitr Algebra Geom.*, **57** (2016) 913–927.
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Thank you for your attention.

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