Parine

On a Generalization of the Carré Arithmétique de Fermat and on Weighted Sums of the Type $\sum_{k=0}^{n} k^{\ell} a_k^{(r)}$

1. Hypersequences $(a_n^{(r)})_{n\in\mathbb{N}_0}$, $r\in\mathbb{N}_0$, of an arbitrary sequence $(a_n)_{n\in\mathbb{N}_0}$, $a_n^{(0)} := a_n$
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1. Hypersequences $(a_n^{(r)})_{n\in\mathbb{N}_0}$, $r\in\mathbb{N}_0$, of an arbitrary sequence $(a_n)_{n\in\mathbb{N}_0}$, $a_n^{(0)} := a_n$

Let $a_n^{(0)} := a_n$, $n\in\mathbb{N}_0$, be an arbitrary sequence

partial sums of (a_n)

$$a_n^{(1)} := \sum_{k=0}^{n} a_k$$

$$a_n^{(2)} := \sum_{k=0}^{n} a_k^{(1)} = \sum_{k=0}^{n}\left(\sum_{j=0}^{k} a_j\right)$$

$$\begin{aligned}
= {} & a_0 \\
& + a_0 + a_1 \\
& + a_0 + a_1 + a_2 \\
& + \dots \\
& + a_0 + a_1 + a_2 + \dots + a_{n-1} + a_n
\end{aligned}$$

$$= (n+1)a_0 + n\cdot a_1 + (n-1)a_2 + \dots + 2a_{n-1} + 1\cdot a_n$$

$$= \sum_{k=0}^{n}(n+1-k)a_k$$

$$a_n^{(3)} := \sum_{k=0}^{n} a_k^{(2)} = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} (k+1-j) a_j \right)$$

$$\begin{aligned} = \ & a_0 \\ & + 2a_0 + a_1 \\ & + 3a_0 + 2a_1 + a_2 \\ & + \dots \\ & + (n+1)a_0 + na_1 + (n-1)a_2 + \dots + 2a_{n-1} + a_n \end{aligned}$$

$$= (1+2+3+\dots+(n+1))a_0 + (1+2+\dots+n)a_1 + (1+2+\dots+(n-1))a_2 + \dots + (1+2)a_{n-1} + 1 \cdot a_n$$

$$= \binom{n+2}{2} a_0 + \binom{n+1}{2} a_1 + \binom{n}{2} a_2 + \dots + \binom{3}{2} a_{n-1} + \binom{2}{2} a_n$$

or $$a_n^{(3)} = \sum_{k=0}^{n} \binom{n+2-k}{2} a_k$$

. . . .

In general, for $r \in \mathbb{N}$, $n \in \mathbb{N}_0$

$$a_n^{(r)} := \sum_{k=0}^{n} a_k^{(r-1)} \quad , \quad a_n^{(0)} := a_n$$

initial values $a_0^{(r)} := a_0$

we have

Theorem 1 Let $r \in \mathbb{N}$, $n \in \mathbb{N}_0$, then

$$a_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} a_k = \sum_{k=0}^{n} \binom{r+k-1}{k} a_{n-k}$$

Proof By induction on $r \geq 1$.

$r=1$: $\binom{n-k}{0} = 1$ for all $k = 0, 1, \ldots, n$

$$\Rightarrow a_n^{(1)} = \sum_{k=0}^{n} a_k .$$

$r \to r+1$

$$a_n^{(r+1)} = \sum_{k=0}^{n} a_k^{(r)} = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{k+r-1-j}{r-1} a_j \right)$$

by induction assumptions

$$= \binom{r-1}{r-1} a_0$$
$$+ \left[\binom{r}{r-1} a_0 + \binom{r-1}{r-1} a_1 \right]$$
$$+ \left[\binom{r+1}{r-1} a_0 + \binom{r}{r-1} a_1 + \binom{r-1}{r-1} a_2 \right]$$
$$+ \cdots$$
$$+ \left[\binom{n+r-1}{r-1} a_0 + \binom{n+r-2}{r-1} a_1 + \binom{n+r-3}{r-1} a_2 + \cdots + \binom{r-1}{r-1} a_n \right]$$

$$= \left[\binom{r-1}{r-1} + \binom{r}{r-1} + \binom{r+1}{r-1} + \cdots + \binom{n+r-1}{r-1}\right] a_0$$

$$+ \left[\binom{r-1}{r-1} + \binom{r}{r-1} + \cdots + \binom{n+r-2}{r-1}\right] a_1$$

$$+ \left[\binom{r-1}{r-1} + \binom{r}{r-1} + \cdots + \binom{n+r-3}{r-1}\right] a_2$$

$$+ \cdots$$

$$+ \left[\binom{r-1}{r-1} + \binom{r}{r-1}\right] a_{n+1}$$

$$+ \underbrace{\binom{r-1}{r-1}}_{=\binom{r}{r}} a_n$$

$$= \binom{n+r}{r} a_0 + \binom{n+r-1}{r} a_1 + \binom{n+r-2}{r} a_2 + \cdots + \binom{r+1}{r} a_{n-1} + \binom{r}{r} a_n$$

by the formula of Chu-Shih-Chieh (1303) (<u>upper summation</u>)

$$\binom{r-1}{r-1} + \binom{r}{r-1} + \binom{r+1}{r-1} + \cdots + \binom{r+n-1}{r-1} = \binom{r+n}{r}$$

or
$$a_n^{(r+1)} = \sum_{k=0}^{n} \binom{n+(r+1)-1-k}{(r+1)-1} a_k$$

The second equation follows from the fact that if $k \in \{0, 1, .., n\}$, then $n-k \in \{n, n-1, .., 1, 0\}$

$$a_n^{(r)} = \sum_{0 \le k \le n} \binom{n+r-1-k}{r-1} a_k = \sum_{\underbrace{0 \le n-k \le n}_{=:k'}} \binom{r-1+k'}{r-1} a_{n-k'} = \sum_{0 \le k \le n} \binom{r+k-1}{k} a_{n-k} .\ \square$$

By means of Theorem 1 we can construct an $r \times n$-table (array) $A(r,n)$, $r,n \in \mathbb{N}_0$ by setting

$$A(r,n) := a_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} a_k = \sum_{k=0}^{n} \binom{r+k-1}{k} a_{n-k}$$

$$A(0,n) := a_n^{(0)} = a_n$$

$r \backslash n$	0	1	2	3	...
0	a_0	a_1	a_2	a_3	...
1	a_0	a_0+a_1	$a_0+a_1+a_2$	$a_0+a_1+a_2+a_3$	...
2	a_0	$2a_0+a_1$	$3a_0+2a_1+a_2$	$4a_0+3a_1+2a_2+a_3$	...
3	a_0	$3a_0+a_1$	$6a_0+3a_1+a_2$	$10a_0+6a_1+3a_2+a_3$	...
...	...	...	...	...	...

Table 1 The hypersequences $(a_n^{(r)})_{n\in\mathbb{N}_0}$, $0 \le r \le 3$, $0 \le n \le 3$

Theorem 2 For all $r,n \in \mathbb{N}_0$

$$A(r+1,n+1) = A(r,n+1) + A(r+1,n)$$

Proof By Theorem 1

$$A(r+1,n+1) = \sum_{k=0}^{n+1} \binom{n+1+(r+1)-1-k}{(r+1)-1} a_k$$

$$= \sum_{k=0}^{n} \underbrace{\binom{n+r+1-k}{r}}_{=\binom{n+r-k}{r}+\binom{n+r-k}{r-1}} a_k + \binom{r}{r} a_{n+1}$$

(Pascal's rule)

$$= \sum_{k=0}^{n} \underbrace{\binom{n+r-k}{r}}_{=\binom{n+(r+1)-1-k}{(r+1)-1}} a_k + \sum_{k=0}^{n} \underbrace{\binom{n+r-k}{r-1}}_{=\binom{n+1+r-1-k}{r-1}} a_k + \underbrace{\binom{r}{r}}_{=\binom{r-1}{r-1}} a_{n+1}$$

$$= A(r+1,n) + \underbrace{\sum_{k=0}^{n+1} \binom{n+1+r-1-k}{r-1} a_k}_{=A(r,n+1)} . \square$$

Corollary 1 For all $r, n \in \mathbb{N}_0$

$$\sum_{k=0}^{n} A(r,k) = A(r+1,n)$$

Proof By induction on $n \geq 0$

$\underline{n=0}$: $A(r,0) = a_0 = A(r+1,0)$

$\underline{n \to n+1}$:

$$\sum_{k=0}^{n+1} A(r,k) = \underbrace{\sum_{k=0}^{n} A(r,k)}_{=A(r+1,n) \text{ by induction assumptions}} + A(r,n+1) = A(r+1,n+1) . \square$$

1.1 Triangular Array

Shifting column $n \in \mathbb{N}_0$ by n steps downwards we obtain $T(r,n)$

$r \backslash n$	0	1	2	3	...	row sums
0	a_0					a_0
1	a_0	a_1				a_0+a_1
2	a_0	a_0+a_1	a_2			$2a_0+a_1+a_2$
3	a_0	$2a_0+a_1$	$a_0+a_1+a_2$	a_3	⋮	$4a_0+2a_1+a_2+a_3$
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 2 The triangular array $T(r,n)$, $0 \le r \le 3$, $0 \le n \le 3$

<u>Theorem 3</u> For all $r, n \in \mathbb{N}_0$, $r \geq n \geq 0$

i) $$T(r,n) = A(r-n,n) = \sum_{k=0}^{n} \binom{r-1-k}{n-k} a_k$$

$T(r,n)$ satisfies the recurrence relation

ii) $$T(r+1,n+1) = T(r,n+1) + T(r,n)$$

<u>Proof</u> i) $T(r,n) = A(r-n,n)$

ii) $T(r, n+1) + T(r, n) = A(r-(n+1), n+1) + A(r-n, n)$

$= A(r-n, n+1)$ by Theorem 2

$= A(r+1-(n+1), n+1) = T(r+1, n+1)$ by i). $\square$

Row sums: $\sigma_T(r) := \sum_{n=0}^{r} T(r,n)$

<u>Theorem 4</u> For all $r \in \mathbb{N}_0$

i) $\sigma_T(r) = \sum_{k=0}^{r-1} 2^{r-1-k} \cdot a_k + a_r$

ii) $\sigma_T^{(1)}(r) = \sum_{k=0}^{r} 2^{r-k} \cdot a_k = \sigma_T(r+1) - a_{r+1}$

iii) $\sigma_T^{(1)}(r) = 2\sigma_T(r) - a_r$

iv) $\sigma_T(r+1) = 2\sigma_T(r) + a_{r+1} - a_r$, $r \geq 0$, $\sigma_T(0) = a_0$

with the solution

$$\sigma_T(r) = a_0 \cdot 2^r + \sum_{k=0}^{r-1} 2^k \cdot (a_{r-k} - a_{r-1-k})$$

v) For all $r, s \in \mathbb{N}_0$:

$$\sigma_T^{(s)}(r) = 2^s \cdot \sigma_T(r) - \sum_{k=0}^{s-1} 2^{s-1-k} \cdot a_r^{(k)}$$

<u>Proof</u> i) $\sigma_T(r) = \sum_{u=0}^{r} T(r,u) = \sum_{u=0}^{r-1} T(r,u) + \underbrace{T(r,r)}_{=A(0,r)=a_r}$

For $r=0$: $\sigma_T(0) = a_0$

Let $r \geq 1$, then by Theorem 3i)

$$\sigma_T(r) = \sum_{u=0}^{r-1} \left(\sum_{k=0}^{u} \binom{r-1-k}{u-k} a_k \right) + a_r$$

$$= \binom{r-1}{0} a_0$$
$$+ \binom{r-1}{1} a_0 + \binom{r-2}{0} a_1$$
$$+ \binom{r-1}{2} a_0 + \binom{r-2}{1} a_1 + \binom{r-3}{0} a_2$$
$$+ \cdots$$
$$+ \binom{r-1}{r-1} a_0 + \binom{r-2}{r-2} a_1 + \binom{r-3}{r-3} a_2 + \cdots + \binom{0}{0} a_{r-1} + a_r$$

$$= a_r + \left[\binom{r-1}{0} + \binom{r-1}{1} + \binom{r-1}{2} + \cdots + \binom{r-1}{r-1} \right] a_0 +$$
$$\left[\binom{r-2}{0} + \binom{r-2}{1} + \cdots + \binom{r-2}{r-2} \right] a_1 + \cdots + a_{r-1}$$

$$= a_r + 2^{r-1} \cdot a_0 + 2^{r-2} \cdot a_1 + \cdots + 2^0 \cdot a_{r-1}$$

$$= a_r + \sum_{k=0}^{r-1} 2^{r-1-k} \cdot a_k \text{ , since } \sum_{k=0}^{r-j} \binom{r-j}{k} = 2^{r-j}$$

for all $j = 1, 2, \ldots, r$

ii) $\sigma_T^{(1)}(r) = \sum_{\varrho=0}^{r} \sigma_T(\varrho) = \sum_{\cancel{k}\,\varrho=0}^{r} \left(\sum_{k=0}^{\varrho-1} 2^{\varrho-1-k} \cdot a_k + a_\varrho \right)$ by Theorem 4i)

$$= \underbrace{\sum_{\rho=0}^{n} a_\rho} + \sum_{\rho=0}^{n} \left(\sum_{k=0}^{\rho-1} 2^{\rho-1-k} \cdot a_k \right)$$

$$\begin{aligned} = a_n^{(1)} &+ 0 \\ &+ a_0 \\ &+ 2a_0 + a_1 \\ &+ 4a_0 + 2a_1 + a_2 \\ &+ \cdot \; - \; - \; - \; - \\ &+ 2^{n-1} \cdot a_0 + 2^{n-2} \cdot a_1 + 2^{n-3} \cdot a_2 + \cdots + 2^1 \cdot a_{n-2} + 2^0 \cdot a_{n-1} \end{aligned}$$

$$= a_n^{(1)} + \left(1+2+4+\cdots+2^{n-1}\right) a_0 + \left(1+2+\cdots+2^{n-2}\right) a_1$$

$$+ \cdots + (1+2)\, a_{n-2} + 1 \cdot a_{n-1}$$

$$= a_n^{(1)} + (2^n - 1)\, a_0 + (2^{n-1} - 1)\, a_1 + \cdots + (2^2 - 1)\, a_{n-2} + (2^1 - 1)\, a_{n-1}$$

$$= a_n^{(1)} + 2^n \cdot a_0 + 2^{n-1} \cdot a_1 + \cdots + 2^1 a_{n-1} - \Big(\underbrace{a_0 + a_1 + \cdots + a_{n-1}}_{= a_{n-1}^{(1)}} \Big)$$

$$= \underbrace{a_n^{(1)} - a_{n-1}^{(1)}}_{= a_n} + \sum_{k=0}^{n-1} 2^{n-k} \cdot a_k = \sum_{k=0}^{n} 2^{n-k} \cdot a_k .$$

This proves the first equality in ii)

By Theorem 4 i) it follows

$$\sigma_T(n+1) = \underbrace{\sum_{k=0}^{n} 2^{n-k} \cdot a_k}_{\sigma_T^{(1)}(n)} + a_{n+1} . \qquad \square$$

iii) By i) we have

$$2\sigma_T(n) = 2\sum_{k=0}^{n-1} 2^{n-1-k} \cdot a_k + 2a_n$$

$$= \sum_{k=0}^{n-1} 2^{n-k} \cdot a_k + 2a_n = \underbrace{\sum_{k=0}^{n} 2^{n-k} \cdot a_k}_{=\sigma_T^{(1)}(n)} + a_n . \square$$

iv) By ii) and iii) it follows

$$\sigma_T(n+1) - a_{n+1} = \sigma_T^{(1)}(n) = 2\sigma_T(n) - a_n$$

or
$$\sigma_T(n+1) = 2\sigma_T(n) + a_{n+1} - a_n \quad , n \geq 0$$

$$\sigma_T(0) = a_0$$

This recurrence relation of the first order has the solution

$$\sigma_T(n) = a_0 \cdot 2^n + \sum_{k=0}^{n-1} 2^k \cdot (a_{n-k} - a_{n-1-k})$$

$$= \underbrace{\sum_{k=0}^{n} 2^k \cdot a_{n-k}} - \underbrace{\sum_{k=0}^{n-1} 2^k \cdot a_{n-1-k}}$$

$$= \sum_{k=0}^{n} 2^{n-k} \cdot a_k - \sum_{k=0}^{n-1} 2^{n-1-k} \cdot a_k$$

since $k \in \{0,1,..,n\} \iff n-k \in \{0,1,..,n\}$

$$= \sum_{k=0}^{n} 2^{n-k} \cdot a_k - \frac{1}{2}\left(\sum_{k=0}^{n} 2^{n-k} \cdot a_k - a_n\right)$$

$$= \frac{1}{2} \sum_{k=0}^{r} 2^{r-k} \cdot a_k + \frac{a_r}{2}$$

$$= \sum_{k=0}^{r-1} 2^{r-1-k} \cdot a_k + \underbrace{\frac{a_r}{2} + \frac{a_r}{2}}_{= a_r}$$

and this is the formula given in i). $\square$

v) Proof By induction on $s \geq 0$

$s = 0$: $LHS = \sigma_T^{(0)}(r) = \sigma_T(r)$; $RHS = 2^0 \sigma_T(r) - \sum_{k=0}^{-1} 2^{-1-k} \cdot a_r^{(k)}$
$= \sigma_T(r)$ ✓

<u>$s \to s+1$</u> : $\sigma_T^{(s+1)}(r) = \sum_{j=0}^{r} \sigma_T^{(s)}(j) = \sum_{j=0}^{r} \left(2^s \sigma_T(j) - \sum_{k=0}^{s-1} 2^{s-1-k} \cdot a_j^{(k)} \right)$
by induction assumption

$$= 2^s \sum_{j=0}^{r} \sigma_T(j) - \sum_{j=0}^{r} \left(\sum_{k=0}^{s-1} 2^{s-1-k} \cdot a_j^{(k)} \right)$$

$$= 2^s \cdot \sigma_T^{(1)}(r) - \sum_{k=0}^{s-1} 2^{s-1-k} \underbrace{\left(\sum_{j=0}^{r} a_j^{(k)} \right)}_{= a_r^{(k+1)}}$$

$$= 2^s \cdot \left(2\sigma_T(r) - a_r \right) - \sum_{k=0}^{s-1} 2^{s-1-k} \cdot a_r^{(k+1)} \quad \text{by iii)}$$

$$= 2^{s+1} \cdot \sigma_T(r) - 2^s a_r - 2^{s-1} a_r^{(1)} - 2^{s-2} \cdot a_r^{(2)} - \cdots - 2^0 \cdot a_r^{(s)}$$

$$= 2^{s+1} \cdot \sigma_T(r) - \sum_{k=0}^{(s+1)-1} 2^{(s+1)-1-k} \cdot a_r^{(k)}$$ and this is the assertion iv) for $s+1$. $\square$

Corollary Let $a_{r+1} = a_r + a_{r-1}$, $r \geq 1$, then

$$\sigma_T(r+3) = 3\,\sigma_T(r+2) - \sigma_T(r+1) - 2\,\sigma_T(r), \quad r \geq 0$$

$$\sigma_T(0) = a_0, \quad \sigma_T(1) = a_0 + a_1, \quad \sigma_T(2) = 2a_0 + a_1 + a_2$$

with the solution

$$\sigma_T(r) = a_2 \cdot 2^r + (a_0 + a_1 - 2a_2)F_r + (a_0 - a_2)F_{r-1}, \quad r \geq 0$$

$$= (a_0 + a_1)\cdot 2^r - a_0 F_r - a_1 F_{r+1}, \quad r \geq 0$$

where F_r are the Fibonacci numbers ($F_{-1} = 1$).

Proof By repeated use of Theorem 4 iv) we obtain

$$\sigma_T(r+3) = 2\cdot\sigma_T(r+2) + \overbrace{a_{r+3} - a_{r+2}}^{=a_{r+1}}$$

$$= 2\cdot\Big(2\cdot\sigma_T(r+1) + \underbrace{a_{r+2} - a_{r+1}}_{=a_r}\Big) + a_{r+1}$$

$$= 4\,\sigma_T(r+1) + 2a_r + a_{r+1}$$

$$= 4\Big(2\,\sigma_T(r) + \underbrace{a_{r+1} - a_r}_{=a_{r-1}}\Big) + 2a_r + \underbrace{a_{r+1}}_{=a_r + a_{r-1}}$$

$$= 8\,\sigma_T(r) + 4a_{r-1} + 2a_r + a_r + a_{r-1}$$

$$= 8\,\sigma_T(r) + 5a_{r-1} + 3a_r$$

and

$$3\,\sigma_T(r+2) - \sigma_T(r+1) - 2\,\sigma_T(r)$$

$$= 3\Big(2\cdot\sigma_T(r+1) + \underbrace{a_{r+2} - a_{r+1}}_{=a_r}\Big) - \sigma_T(r+1) - 2\,\sigma_T(r)$$

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$$= 6\,\sigma_T(r+1) + 3\,a_r - \sigma_T(r+1) - 2\sigma_T(r)$$
$$= 5\,\sigma_T(r+1) + 3\,a_r - 2\sigma_T(r)$$
$$= 5\Big(2\cdot\sigma_T(r) + \underbrace{a_{r+1} - a_r}_{=a_{r-1}}\Big) + 3\,a_r - 2\sigma_T(r)$$
$$= 8\,\sigma_T(r) + 5\,a_{r-1} + 3\,a_r$$

and this exactly the expression given above.

The solution of this homogeneous recurrence relation of the third order can be obtained by the standard method

$$\sigma_T(r) =: \lambda^r \neq 0$$

We obtain the cubic equation for $\lambda \neq 0$

$$\lambda^3 = 3\lambda^2 - \lambda - 2$$

with the three roots

$$\lambda_1 = 2 \;,\; \lambda_2 = \frac{1+\sqrt{5}}{2} =: \tau \;,\; \lambda_3 = \frac{1-\sqrt{5}}{2} =: \sigma$$

Hence,

$$\sigma_T(r) = \alpha\cdot 2^r + \beta\cdot\tau^r + \gamma\cdot\sigma^r$$

with $\alpha, \beta, \gamma \in \mathbb{C}$.

Inserting the initial values we obtain the linear system in matrix form

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & \tau & \sigma \\ 4 & \tau^2 & \sigma^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a_0 \\ a_0 + a_1 \\ 2a_0 + a_1 + a_2 \end{pmatrix}$$

By Cramer's rule we obtain the solution

$$\alpha = \frac{\begin{vmatrix} a_0 & 1 & 1 \\ a_0+a_1 & \tau & \sigma \\ 2a_0+a_1+a_2 & \tau^2 & \sigma^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & \tau & \sigma \\ 4 & \tau^2 & \sigma^2 \end{vmatrix}}, \quad \beta = \frac{\begin{vmatrix} 1 & a_0 & 1 \\ 2 & a_0+a_1 & \sigma \\ 4 & 2a_0+a_1+a_2 & \sigma^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & \tau & \sigma \\ 4 & \tau^2 & \sigma^2 \end{vmatrix}}, \quad \gamma = \frac{\begin{vmatrix} 1 & 1 & a_0 \\ 2 & \tau & a_0+a_1 \\ 4 & \tau^2 & 2a_0+a_1+a_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & \tau & \sigma \\ 4 & \tau^2 & \sigma^2 \end{vmatrix}}$$

The denominator of the three fractions is a Vandermonde's determinant with the value $\begin{vmatrix} 1 & 1 & 1 \\ 2 & \tau & \sigma \\ 4 & \tau^2 & \sigma^2 \end{vmatrix} = (\tau-2)(\sigma-2)(\sigma-\tau)$

$$= (\underbrace{\tau\sigma}_{=-1} - 2\sigma - 2\tau + 4)(\sigma-\tau) = (3 - 2\underbrace{(\sigma+\tau)}_{=1})(\sigma-\tau) = \underline{\underline{\sigma-\tau}}$$

The numerators of α, β and γ can be calculated by Sarrus's rule. We obtain

$$\alpha = a_2, \quad \beta = \frac{(a_0-a_2)\sigma + (2a_2-a_1-a_0)}{\sigma-\tau}$$

and $$\gamma = \frac{(a_2-a_0)\tau + (a_0+a_1-2a_2)}{\sigma-\tau}$$

Hence,

$$\sigma_T(n) = a_2 \cdot 2^n + \frac{(a_0-a_2)\sigma + (2a_2-a_1-a_0)}{\sigma-\tau} \cdot \tau^n$$
$$+ \frac{(a_2-a_0)\tau + (a_0+a_1-2a_2)}{\sigma-\tau} \cdot \sigma^n$$

or

$$\sigma_T(r) = a_2 \cdot 2^r + (a_0 + a_1 - 2a_2) \cdot \frac{\sigma^r - \tau^r}{\sigma - \tau} + (a_0 - a_2) \cdot \underbrace{\frac{-\tau\sigma^r + \sigma\tau^r}{\sigma - \tau}}_{= \frac{\sigma^{r-1} - \tau^{r-1}}{\sigma - \tau}, \text{ since } \tau\sigma = -1.}$$

$$\sigma_T(r) = a_2 \cdot 2^r + (a_0 + a_1 - 2a_2) \cdot F_r + (a_0 - a_2) F_{r-1} \quad (*)$$

by the Binet's formula for the Fibonacci numbers F_r, $r \geq 0$

or, since $a_2 = a_0 + a_1$

$$\sigma_T(r) = (a_0 + a_1) \cdot 2^r - (a_0 + a_1) F_r - a_1 F_{r-1}$$

$$\sigma_T(r) = (a_0 + a_1) \cdot 2^r - a_0 F_r - a_1 \underbrace{(F_r + F_{r-1})}_{F_{r+1}}$$

o) Note that $(*)$ is also valid for $a_r = \text{const}$, $r \geq 0$, giving the solution $\sigma_T(r) = a_0 \cdot 2^r$

Special cases

a) $a_0 = 0$, $a_1 = 1$, $a_2 = 1$ (Fibonacci sequence)

$$\sigma_T(r) = 2^r - F_{r+1}$$

b) $a_0 = 2$, $a_1 = 1$, $a_2 = 3$ (Lucas sequence)

$$\sigma_T(r) = 3 \cdot 2^r \underbrace{- 2F_r - F_{r+1}}_{\substack{= -F_r - (F_r + F_{r+1}) \\ = -(F_r + F_{r+2})}}$$

$$= 3 \cdot 2^r - L_{r+1}.$$

1.2 Examples

(A000012) in OEIS

E1) Let $a_n = 1$, $n \geq 0$, then

$r \backslash n$	0	1	2	3	4	5	6	7	8	...
0	1	1	1	1	1	1	1	1	1	...
1	1	2	3	4	5	6	7	8	9	...
2	1	3	6	10	15	21	28	36	45	...
3	1	4	10	20	35	56	84	120	165	...
4	1	5	15	35	70	126	210	330	495	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 3

Carré arithmétique de Fermat

$A_{Fe}(r,n)$, $0 \leq r \leq 4$, $0 \leq n \leq 8$

General term

$$A_{Fe}(r,n) = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} \underset{\substack{\uparrow \\ \text{upper summation}}}{=} \binom{n+r}{r}$$

These are the figured numbers / also called the $(n+1)$st regular r-topic numbers $P^{(r)}_{n+1}$,

where $P^{(r)}_n := \binom{n+r-1}{r}$

Combinatorically,

$\binom{n+r-1}{r}$ gives the numbers of combinations of r objects out of n, $0 \leq r \leq n$, with repetition, whereas $\binom{n+r}{r}$ gives the number of shortest routes in the rectangular grid from the lattice point $(0,0)$ to the lattice point (r,n).

The triangular array derived by Table 3 is the famous <u>Pascal's triangle</u> with the general term

$$T_{Fe}(r,n) = \binom{r}{n}, \quad r \geq n \geq 0$$

r\n	0	1	2	3	4	5	6	...	rowsums
0	1								1
1	1	1							2
2	1	2	1						4
3	1	3	3	1					8
4	1	4	6	4	1				16
5	1	5	10	10	5	1			32
6	1	6	15	20	15	6	1		64
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮

Table 4 Pascal's triangle

The sequence of the row sums is by Theorem 4 i)

$$\sigma_{T_{Fe}}(r) = \sum_{k=0}^{r-1} 2^{r-k-1} + 1 = (2^{r-1} + 2^{r-2} + \dots + 2^0) + 1$$

$$= 2^r - 1 + 1 = 2^r$$

This is the geometric sequence defined recursively by

$$a_{r+1} = 2 \cdot a_r, \quad a_0 = 1, \quad r \geq 0$$

but it satisfies also the recurrence relation

$$\begin{cases} a_r = 3a_{r-1} - a_{r-2} - 2a_{r-2}, & r \geq 3 \\ a_0 = 1, \ a_1 = 2, \ a_2 = 4 \end{cases}$$

E2) Let $a_n := \begin{cases} 1, & n = 0 \\ j, & n \geq 1 \end{cases}$, $j \in \mathbb{N}_0$, then

we obtain a generalization of the figured numbers (case $j=1$)

<u>General term</u>

$$A_j(r,n) = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} a_k = \binom{n+r-1}{r-1} +$$

$$+ j \sum_{k=1}^{n} \binom{n+r-1-k}{r-1} = \binom{n+r-1}{r-1} + j \binom{n+r-1}{r}$$

$$= P_{n+1}^{(r-1)} + j P_n^{(r)}.$$

These are the <u>r-dimensional figurate numbers</u>, where

$n \in \mathbb{N}_0$ is the rank

$j \in \mathbb{N}_0$ is the <u>number</u> and

$r \in \mathbb{N}_0$ is the <u>dimension</u> of the figurate numbers

~~General term~~ : $\cancel{A_0(r,n) = \binom{n+r-1}{r-1}}$

r\n	0	1	2	3	4	5	6	...
0	1	j	j	j	j	j	j	...
1	1	$1+j$	$1+2j$	$1+3j$	$1+4j$	$1+5j$	$1+6j$	...
2	1	$2+j$	$3+3j$	$4+6j$	$5+10j$	$6+15j$	$7+21j$	...
3	1	$3+j$	$6+4j$	$10+10j$	$15+20j$	$21+35j$	$28+56j$	...
4	1	$4+j$	$10+5j$	$20+15j$	$35+35j$	$56+70j$	$84+126j$	...
5	1	$5+j$	$15+6j$	$35+21j$	$70+56j$	$126+126j$	$210+252j$	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 5
r-dimensional figurate numbers
$j \in \mathbb{N}_0$

<u>Special cases</u>

For $j=0$: $A_0(r+1, n) = A_{Fe}(r,n)$, $r, n \in \mathbb{N}_0$

$$A_0(r,n) = \binom{n+r-1}{r-1}$$

For $j=1$ we get the figured numbers

$$A_1(r,n) = A_{Fe}(r,n)$$

Special cases

dimension	name	number	first values	sequence
$r=1$	linear group of figurate numbers	$j=0$	$(1,1,1,\ldots)$	
		$j=1$	$(1,2,3,4,5,\ldots)$	natural numbers
		$j=2$	$(1,3,5,7,9,\ldots)$	odd numbers (gnomic) "
$r=2$	polygonal numbers	$j=0$	$(1,2,3,4,5,\ldots)$	natural numbers
		$j=1$	$(1,3,6,10,15,\ldots)$	triangular numbers
		$j=2$	$(1,4,9,16,25,\ldots)$	square numbers
		$j=3$	$(1,5,12,22,35,\ldots)$	pentagonal "
$r=3$	pyramidal numbers	$j=0$	$(1,3,6,10,15,\ldots)$	triangular "
		$j=1$	$(1,4,10,20,35,\ldots)$	tetrahedral "
		$j=2$	$(1,5,14,30,55,\ldots)$	square pyramidal "
		$j=3$	$(1,6,18,40,75,\ldots)$	pentagonal pyramidal "

Some r-dimensional figurate numbers

The triangular array $T_j(r,n)$ has the general term

$$T_j(r,n) = \binom{r-1}{n} + j\binom{r-1}{n-1}$$

and $$\sigma_{T_j}(r) = (r=0) + (r>0)\cdot 2^{r-1}\cdot(1+j)$$

E3) Dirichlet's sequence

Let $a_n = (n=1)$, $n \geq 0$, where (A) is Iverson's convention meaning 1 if the statement A is true and 0 if it is false. It is the characteristic function of 1, (A063524) $(0,1,0,0,\ldots,0,\ldots)$.

The general term is by Theorem 1

$$A_\delta(r,n) = \binom{n+r-2}{r-1}$$

r\n	0	1	2	3	4	5	6	7	...
0	0	1	0	0	0	0	0	0	
1	0	1	1	1	1	1	1	1	
2	0	1	2	3	4	5	6	7	
3	0	1	3	6	10	15	21	28	
4	0	1	4	10	20	35	56	84	
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 6
Dirichlet's table

It is

$$A_{Fe}(r,n) = A_{\delta}(r+1, n+1)$$

and

$$A_0(r,n) = A_{\delta}(r, n+1)$$

The triangular array $T_{\delta}(r,n)$ is

$$T_{\delta}(r,n) = \sum_{k=0}^{n} \binom{r-1-k}{n-k}(k=1) = \binom{r-2}{n-1}, \quad r \geq n \geq 0$$

and the sequence of the row sums is

$$\sigma_{T_{\delta}}(r) = (r=1) + (r \geq 2) \cdot 2^{r-2}$$

r\n	0	1	2	3	4	5	...	row sums
0	0							0
1	0	1						1
2	0	1	0					1
3	0	1	1	0				2
4	0	1	2	1	0			4
5	0	1	3	3	1	0		8
6	0	1	4	6	4	1		16
⋮	⋮	⋮	⋮	⋮	⋮	⋮	...	⋮

E4) Fibonacci sequence (A000045)

Let $a_n := F_n$, $n \geq 0$, defined recursively by
$F_{n+2} = F_{n+1} + F_n$, $n \geq 0$, $F_0 = 0$, $F_1 = 1$

r\n	0	1	2	3	4	5	6	7	...
0	0	1	1	2	3	5	8	13	...
1	0	1	2	4	7	12	20	33	...
2	0	1	3	7	14	26	46	79	...
3	0	1	4	11	25	51	97	176	...
4	0	1	5	16	41	92	189	365	...
5	0	1	6	22	63	155	344	709	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 7
Hyperfibonacci numbers $F_n^{(r)}$

General term

$$A_{Fi}(r,n) = F_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} F_k = \sum_{k=0}^{n} \binom{r+k-1}{k} F_{n-k}$$

The sequence $(F_n^{(r)})_{n \in \mathbb{N}_0}$, $F_n^{(r)} := \sum_{k=0}^{n} F_k^{(r-1)}$, $r \geq 1$, $F_n^{(0)} := F_n$ is known as the hyperfibonacci sequence of the rth generation and has been studied by several authors in the last years.

For example:

- A. Dil and J. Mező
- Ligia L. Cristea, J. Martinjak and J. Urbiha
- J. Martinjak and J. Urbiha
- M. Bahşi, J. Mező, S. Solak

They gave other representations of $F_n^{(r)}$, e.g.

$$F_n^{(r)} = \sum_{k=r}^{\lfloor \frac{n-1+2r}{2} \rfloor} \binom{n+1+2r-k}{k} \quad \text{(Cristea et al.)}$$

which for $r=0$ reduces to the classical formula of

É. Lucas (1876)

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}$$

Another representation:

It is

$$F_n^{(1)} = F_{n+2} - \overbrace{F_2}^{=F_1} \qquad \text{(Lucas, 1876)}$$

$$\Rightarrow F_n^{(2)} = \sum_{k=0}^{n} (F_{k+2} - F_2) = \sum_{k=0}^{n} F_{k+2} - F_2 \sum_{k=0}^{n} 1$$

$$= \sum_{k=0}^{n+2} F_k - \cancel{F_0} - F_1 - F_2(n+1)$$

$$= F_{n+4} - F_2 - F_1 - F_2 \cdot n - F_2$$

$$= F_{n+4} - (F_4 + F_2 n)$$

Similarly,

$$F_n^{(3)} = \sum_{k=0}^{n} F_k^{(2)} = \sum_{k=0}^{n} \left(F_{k+4} - (F_4 + F_2 k) \right)$$

$$= \sum_{k=0}^{n} F_{k+4} - F_4 \sum_{k=0}^{n} 1 - F_2 \sum_{k=0}^{n} k$$

$$= \sum_{k=0}^{n+4} F_k - \cancel{F_0} - F_1 - F_2 - F_3 - F_4 (n+1) - F_2 \binom{n+1}{2}$$

$$= F_{n+6} - F_2 - F_1 - F_2 - F_3 - F_4 (n+1) - F_2 \binom{n+1}{2}$$

or $$F_n^{(3)} = F_{n+6} - \left(F_6 + F_4 n + F_2 \binom{n+1}{2} \right)$$

In general, we have

<u>Theorem 5</u> Let $r, n \in \mathbb{N}_0$, then

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k-1}{k} F_{2(r-k)}$$

Proof By induction on $r \geq 0$.

<u>$r = 0$</u>: $RHS = F_n$, $LHS = F_n^{(0)}$ ✓

<u>$r \to r+1$</u>:

$$F_n^{(r+1)} \underset{F_0^{(r)}=0}{=} \sum_{k=1}^{n} F_k^{(r)} \underset{\text{by induction ass.}}{=} \sum_{k=1}^{n} \left(F_{k+2r} - \sum_{j=0}^{r-1} \binom{k+j-1}{j} F_{2(r-j)} \right)$$

$$= \sum_{k=1}^{n} F_{k+2r} - \sum_{k=1}^{n} \left(\sum_{j=0}^{r-1} \binom{k+j-1}{j} F_{2(r-j)} \right)$$

$$= \sum_{k=1}^{n+2r} F_k - \sum_{k=1}^{2r} F_k - \sum_{j=0}^{r-1} \left(\underbrace{\sum_{k=1}^{n} \binom{k+j-1}{j}}_{=\binom{n+j}{j+1}} \right) F_{2(r-j)}$$

$$= F_{n+2r+2} - \cancel{F_2} - \left(F_{2r+2} - \cancel{F_2} \right) - \sum_{j=0}^{r-1} \binom{n+j}{j+1} F_{2(r-j)}$$

With $j = k-1$, $k = 1, 2, \dots, r$ it is ~~wit~~ $\left(\binom{n+r}{r+1} F_0 = 0 \right)$

$$F_n^{(r+1)} = F_{n+2(r+1)} - \sum_{k=1}^{r} \binom{n+k-1}{k} F_{2(r-k+1)} - \binom{n+0-1}{0} F_{2(r+1-0)}$$

$$= F_{n+2(r+1)} - \sum_{k=0}^{r} \binom{n+k-1}{k} F_{2(r+1-k)}$$

and this is the formula for $r+1$ instead of r. □

Corollary For all $r, n \in \mathbb{N}_0$

$$F_{n+2r} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} F_k + \sum_{k=0}^{r-1} \binom{n+k-1}{k} F_{2(r-k)}$$

$$= \sum_{k=0}^{n} \binom{r+k-1}{k} F_{n-k} + \qquad \text{''} \qquad \text{''}$$

Other representations of $F_n^{(r)}$:

- $F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r-k}{r-1-k}$
- $F_n^{(r)} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1+r-k}{r+k}$

both given by Cristea, Martinjak and Urbiha

Instead of the recurrence relation using the values of two consecutive generations ~~we have the alternative~~

$$F_{n+1}^{(r+1)} = F_{n+1}^{(r)} + F_n^{(r+1)}, \quad n,r \geq 0, \quad F_0^{(r)} = 0, F_1^{(r)} = 1$$

we have the alternative property

<u>Theorem 6</u> (Cristea, Martinjak and Urbiha)

For all $r, n \in \mathbb{N}_0$

$$\begin{cases} F_{n+2}^{(r)} = F_{n+1}^{(r)} + F_n^{(r)} + \binom{n+r}{r-1}, & n \geq 0 \\ F_0^{(r)} = 0, \quad F_1^{(r)} = 1 \end{cases}$$

<u>Proof</u> $F_n^{(r)} = \sum_{k=0}^{n} \binom{r+k-1}{k} F_{n-k}$ ~~satisfies~~ is the solution of the recurrence relation. $\square$

The triangular array T_{F_i} has the general form

$$T_{F_i}(r,n) = \sum_{k=0}^{n} \binom{r-1-k}{n-k} F_k \,, \quad r \geq n \geq 0$$

$$= F_{2r-n} - \sum_{k=0}^{r-n-1} \binom{n+k-1}{k} F_{2(r-n-k)}$$

r\n	0	1	2	3	4	5	6	...	row sums
0	0								0
1	0	1							1
2	0	1	1						2
3	0	1	2	2					5
4	0	1	3	4	3				11
5	0	1	4	7	7	5			24
6	0	1	5	11	14	12	8		51
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 8
Fibonacci's triangle

For the sequence of the row sums we have

Proposition For all $n \in \mathbb{N}_0$

i) $\sigma_{T_{F_i}}(n) = \sum_{k=0}^{n-1} 2^{n-1-k} \cdot F_k + F_n = 2^n - F_{n+1}$ (A027934)

ii) $\sigma^{(1)}_{T_{F_i}}(n) = \sum_{k=0}^{n} 2^{n-k} \cdot F_k = 2^{n+1} - F_{n+3}$ (A008466$(n+1)$)

iii) $a_n := \sigma_{T_{F_i}}(n)$, ~~at~~ $n \geq 0$, satisfies the recurrence relation

$$\begin{cases} a_n = 3a_{n-1} - a_{n-2} - 2a_{n-3}, & n \geq 3 \\ a_0 = 0,\ a_1 = 1,\ a_2 = 2 \end{cases}$$

Proof ii) The second equation ~~is~~ is proved by induction on $n \geq 0$

i) $\sigma_{T_{F_i}}(n) = \underbrace{\sum_{k=0}^{n-1} 2^{n-1-k} \cdot F_k}_{=2^n - F_{n+2}} + F_n = 2^n - F_{n+1}$.

iii) It follows by using i). ▯

E5) Lucas sequence (A000032)

Let $a_n := L_n$, $n \geq 0$, defined recursively by

$$L_{n+2} = L_{n+1} + L_n, \quad n \geq 0, \quad L_0 = 2, \; L_1 = 1$$

r\n	0	1	2	3	4	5	6	7	...
0	2	1	3	4	7	11	18	29	...
1	2	3	6	10	17	28	46	75	...
2	2	5	11	21	38	66	112	187	...
3	2	7	18	39	77	143	255	442	...
4	2	9	27	66	143	286	541	983	...
5	2	11	38	104	247	533	1074	2057	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 9
Hyperlucas numbers $L_n^{(r)}$

General term

$$A_{L_n}(r,n) = L_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} L_k = \sum_{k=0}^{n} \binom{r+k-1}{k} L_{n-k}$$

The sequence $\left(L_n^{(r)}\right)_{n \in \mathbb{N}_0}$, $L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)}$, $r \geq 1$, $L_n^{(0)} := L_n$ is known as the hyperlucas sequence of the rth generation and has been studied by several authors in the last years.

Other representation of $L_n^{(r)}$

It is $\quad L_n^{(1)} = L_{n+2} - L_1 \quad$ (Lucas, 1876)

$$\Rightarrow L_n^{(2)} = \sum_{k=0}^{n} (L_{k+2} - L_1) = \sum_{k=0}^{n} L_{k+2} - L_1 \sum_{k=0}^{n} 1$$

$$= \sum_{k=0}^{n+2} L_k - \underbrace{L_0 - L_1}_{-L_2} - L_1(n+1)$$

$$= L_{n+4} - L_1 - L_2 - L_1(n+1)$$

or $\quad L_n^{(2)} = L_{n+4} - \big(L_3 + L_1(n+1)\big)$

Similarly,

$$L_n^{(3)} = \sum_{k=0}^{n} L_k^{(2)} = \sum_{k=0}^{n} \Big(L_{k+4} - \big(L_3 + L_1(k+1)\big)\Big)$$

$$= \sum_{k=0}^{n+4} L_k - L_0 - L_1 - L_2 - L_3 - L_3 \sum_{k=0}^{n} 1 - L_1 \sum_{k=0}^{n} (k+1)$$

$$= L_{n+6} - L_1 - L_0 - L_1 - L_2 - L_3 - L_3(n+1) - L_1 \sum_{k=1}^{n+1} k$$

$$= L_{n+6} - \left(L_5 + L_3(n+1) + L_1 \binom{n+2}{2}\right)$$

In general, we have

Theorem 7 Let $r, n \in \mathbb{N}_0$, then

$$L_n^{(r)} = L_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k}{k} L_{2(r-k)-1}$$

Proof By induction on $r \geq 0$

$\underline{r=0}$ $\quad RHS = L_n \quad , LHS = L_n^{(0)}$

$\underline{r \to r+1}$

$$L_n^{(r+1)} = \sum_{k=0}^{n} L_k^{(r)} = \sum_{k=0}^{n} \left(L_{k+2r} - \sum_{j=0}^{r-1} \binom{k+j}{j} L_{2(r-j)-1} \right)$$

by induction assumptions

$$= \sum_{k=0}^{n} L_{k+2r} - \sum_{j=0}^{r-1} \underbrace{\left(\sum_{k=0}^{n} \binom{k+j}{j} \right)}_{= \binom{j+n+1}{j+1}} \cdot L_{2(r-j)-1}$$

$$= \sum_{k=0}^{n+2r} L_k - \sum_{k=0}^{2r-1} L_k - \sum_{j=0}^{r-1} \binom{j+n+1}{j+1} L_{2(r-j)-1}$$

$$= L_{n+2r+2} - \cancel{L_1} - \left(L_{2r+1} - \cancel{L_1} \right) - \sum_{k=1}^{r} \binom{n+k}{k} L_{2(r-k+1)-1}$$

with $j = k-1$, $k = 1, 2, \dots, r$

$$= L_{n+2(r+1)} - \cancel{L_{2r+1}} - \sum_{k=0}^{r} \binom{n+k}{k} L_{2(r-k+1)-1} + \cancel{\binom{n}{0} L_{2(r+1)-1}}$$

$$= L_{n+2(r+1)} - \sum_{k=0}^{(r+1)-1} \binom{n+k}{k} L_{2(r+1-k)-1}$$

and this is the assertion for $r+1$ instead of r. $\square$

<u>Corollary</u> For all $r, n \in \mathbb{N}_0$

$$L_{n+2r} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} L_k + \sum_{k=0}^{r-1} \binom{n+k}{k} L_{2(r-k)-1}$$

$$= \sum_{k=0}^{n} \binom{r+k-1}{k} L_{n-k} + \sum_{k=0}^{r-1} \binom{n+k}{k} L_{2(r-k)-1}$$

Instead of the recurrence relation using the values of two consecutive generations

$$L_{n+1}^{(r+1)} = L_{n+1}^{(r)} + L_n^{(r+1)}, \quad n, r \geq 0$$

with $L_0^{(r)} = 2$, $L_1^{(r)} = 2r+1$

we have the alternative property

Theorem 8 For all $r \in \mathbb{N}_0$ it is

$$L^{(r)}_{n+2} = L^{(r)}_{n+1} + L^{(r)}_{n} + \frac{n+2r}{n+2}\binom{n+r}{r-1}, \quad n \geq 0$$

$$L^{(r)}_0 = 2, \quad L^{(r)}_1 = 2r+1$$

Proof $L^{(r)}_n = \sum_{k=0}^{n} \binom{r+k-1}{k} L_{n-k}$ is the solution of the recurrence relation. ∎

The triangular array $\tilde{T}_{Lu}$ has the general term

$$T_{Lu}(r,n) = \sum_{k=0}^{n} \binom{r-1-k}{n-k} L_k, \quad r \geq n \geq 0$$

$$= L_{2r-n} - \sum_{k=0}^{r-n-1} \binom{n+k}{k} L_{2(r-n-k)-1}$$

Table 10
Lucas's triangle

r\n	0	1	2	3	4	5	6		rowsums
0	2								2
1	2	1							3
2	2	3	3						8
3	2	5	6	4					17
4	2	7	11	10	7				37
5	2	9	18	21	17	11			78
6	2	11	27	48	38	28	18		163
	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮

<u>Proposition</u> For all $n \in \mathbb{N}_0$

i) $\sigma_{T_{Lu}}(n) = \sum_{k=0}^{n-1} 2^{n-1-k} \cdot L_k + L_n = 3 \cdot 2^n - L_{n+1}$

ii) $\sigma^{(1)}_{T_{Lu}}(n) = \sum_{k=0}^{n} 2^{n-k} \cdot L_k = 3 \cdot 2^{n+1} - L_{n+3}$

iii) $a_n := \sigma_{T_{Lu}}(n)$, $n \geq 0$, satisfies the recurrence relation

$$\begin{cases} a_n = 3a_{n-1} - a_{n-2} - 2a_{n-3}, & n \geq 3 \\ a_0 = 2,\ a_1 = 3,\ a_2 = 8 \end{cases}$$

<u>Remark</u> Neither i) ~~nor~~ ii) are available in Sloane-Plouffe Encyclopedia of Integer Sequences EOIS

<u>Proof</u> ii) the second equation is proved by induction on $n \geq 0$

i) $\sigma_{T_{Lu}}(n) = \underbrace{\sum_{k=0}^{n-1} 2^{n-1-k} \cdot L_k}_{= 3 \cdot 2^n - L_{n+2}} + L_n$ by ii) for $n-1$ instead of n

$= 3 \cdot 2^n - L_{n+1} - \cancel{L_n} + \cancel{L_n}$.

iii) It follows by using i) □

2. Weighted Sums of the Type $\sum_{k=0}^{n} k^\ell a_k^{(r)}, r \geq 0$

Let $t_\ell(n) := \sum_{k=0}^{n} k^\ell a_k \quad , \ell \geq 0$

or, more generally, for $r \geq 0$

$$t_\ell^{(r)}(n) := \sum_{k=0}^{n} k^\ell a_k^{(r)} \quad , \ell \geq 0$$

We shall show that these sums can be expressed in terms of $a_n^{(r+m+1)}$, $m = 0, 1, \ldots, \ell$.

Obviously, $t_\ell(n) = t_\ell^{(0)}(n)$.

It is

$$\boxed{t_0(n) = \sum_{k=0}^{n} \underbrace{k^0}_{=1} a_k = a_n^{(1)}}$$

$$a_n^{(2)} = \sum_{k=0}^{n} (n+1-k) a_k = (n+1) \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} k a_k$$

or

$$\boxed{t_1(n) = \sum_{k=0}^{n} k a_k = (n+1) a_n^{(1)} - a_n^{(2)}} \qquad (*)$$

Similarly, from $a_n^{(3)} = \sum_{k=0}^{n} \binom{n+2-k}{2} a_k$

$$\binom{n+2-k}{2} = \frac{(n+2-k)(n+1-k)}{2} = \frac{(n+2)(n+1)-(2n+3)k+k^2}{2}$$

$$\Rightarrow \quad a_n^{(3)} = \frac{(n+2)(n+1)}{2}\sum_{k=0}^{n} a_k - \frac{2n+3}{2}\sum_{k=0}^{n} k a_k + \frac{1}{2}\sum_{k=0}^{n} k^2 a_k$$

or, solving for $t_2(n) = \underbrace{\sum_{k=0}^{n} k^2 a_k}$ and using $(*)$

$$\boxed{t_2(n) = \sum_{k=0}^{n} k^2 a_k = (n+1)^2 a_n^{(1)} - (2n+3)\,a_n^{(2)} + 2a_n^{(3)}}$$

Similarly, for $r=4$

$$\boxed{t_3(n) = \sum_{k=0}^{n} k^3 a_k = (n+1)^3 a_n^{(1)} - (3n^2+9n+7)\,a_n^{(2)} + 6(n+2)\,a_n^{(3)} - 6a_n^{(4)}}$$

From these formulas we can see some patterns for $\ell = 0,1,2,\ldots$

0) $t_\ell(n)$ contains $\ell+1$ terms
1) the coefficients in $t_\ell(n)$ alternate in signs
2) " " of $a_n^{(1)}$ in $t_\ell(n)$ is $(n+1)^\ell$
3) " " of $a_n^{(\ell+1)}$ in $t_\ell(n)$ is $(-1)^\ell \cdot \ell!$
4) The sum of all coefficients of $t_\ell(n)$ is equal to n^ℓ.

The last formulas can be written in matrix form

$$\begin{pmatrix} t_0(n) \\ t_1(n) \\ t_2(n) \\ t_3(n) \\ \vdots \end{pmatrix} = \underbrace{\left(\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & \vdots \\ n+1 & -1 & 0 & 0 & \vdots \\ (n+1)^2 & -(2n+3) & 2 & 0 & \vdots \\ (n+1)^3 & -(3n^2+9n+7) & 6(n+2) & -6 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}\right)}_{=:\,A(n)} \cdot \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \\ a_n^{(3)} \\ a_n^{(4)} \\ \vdots \end{pmatrix}$$

with the infinite lower triangular matrix

$$A(n) := \left(a_{\ell,m}(n) \right)_{\ell,m=0,1,\ldots}$$

Observed patterns can be written for $\ell = 0,1,2,..$, as

i) $a_{\ell,0}(n) = (n+1)^{\ell}$

ii) $a_{\ell,\ell}(n) = (-1)^{\ell} \cdot \ell!$ (independent of n!)

iii) $\displaystyle\sum_{m=0}^{\ell} a_{\ell,m}(n) = n^{\ell}$

Moreover, for $p_\ell(n) := -a_{\ell,1}(n)$ we have

$$p_{\ell+1}(n) = (n+1)\, p_\ell(n) + (n+2)^{\ell}, \quad \ell \geq 0, \quad p_0(n) = 0$$

with the solution $p_\ell(n) = (n+2)^{\ell} - (n+1)^{\ell}$, that is

$$a_{\ell,1}(n) = (n+1)^{\ell} - (n+2)^{\ell}. \qquad \square$$

<u>Theorem 9</u> Let $n, j \in \mathbb{N}_0$, then

$$j!\, a_n^{(j+1)} = (-1)^j \sum_{k=0}^{j} (-1)^k \alpha_k^{(j)}(n) \cdot t_{j-k}(n)$$

$$= \sum_{k=0}^{j} (-1)^k \cdot \alpha_{j-k}^{(j)}(n) \cdot t_k(n)$$

where
$$\alpha_0^{(j)}(n) := 1$$
$$\alpha_1^{(j)}(n) := x_1(n) + x_2(n) + \cdots + x_j(n)$$
$$\alpha_2^{(j)}(n) := x_1(n) \cdot x_2(n) + x_1(n) \cdot x_3(n) + \cdots + x_{j-1}(n) \cdot x_j(n)$$
$$\cdots\cdots$$
$$\alpha_j^{(j)}(n) := x_1(n) \cdot x_2(n) \cdots x_j(n)$$

$x_1(n) := n+1$, $x_2(n) := n+2, \ldots, x_j(n) := n+j$, $j = 1, 2, \ldots$

or, in general,

$$\alpha_k^{(j)}(n) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le j} x_{i_1}(n) \cdot x_{i_2}(n) \cdots x_{i_k}(n)$$

are the <u>elementary symmetric functions</u>.

Proof It is by Theorem 1 (with $j+1$ instead of r)

$$a_n^{(j+1)} = \sum_{k=0}^{n} \binom{n+j-k}{j} a_k$$

By definition

$$\binom{n+j-k}{j} = \frac{(n+j-k)\cdot(n+j-1-k)\cdot\ldots\cdot(n+1-k)}{j!}$$

$$= \frac{(-1)^j}{j!} \left(k - x_1(n)\right)\cdot\left(k - x_2(n)\right)\cdot\ldots\cdot\left(k - x_j(n)\right)$$

By Vieta's theorem it follows

$$\binom{n+j-k}{j} = \frac{(-1)^j}{j!}\Big(\underbrace{\alpha_0^{(j)}(n)}_{=1} k^j - \alpha_1^{(j)}(n) k^{j-1} + \alpha_2^{(j)}(n) k^{j-2} + \ldots$$
$$\ldots + (-1)^{j-1}\alpha_{j-1}^{(j)}(n)\, k + (-1)^j \alpha_j^{(j)}(n)\Big)$$

$$\Rightarrow a_n^{(j+1)} = \sum_{k=0}^{n} \binom{n+j-k}{j} a_k$$

$$= \frac{(-1)^j}{j!}\left\{ \sum_{k=0}^{n} k^j a_k - \alpha_1^{(j)} \sum_{k=0}^{n} k^{j-1}\cdot a_k + \ldots \right.$$
$$\left. \ldots + (-1)^{j-1}\alpha_{j-1}^{(j)}(n) \sum_{k=0}^{n} k\, a_k + (-1)^j \alpha_j^{(j)}(n) \sum_{k=0}^{n} a_k \right\}$$

$$= \frac{(-1)^j}{j!}\Big(t_j(u) - \alpha_1^{(j)} t_{j-1}(u) + \dots + (-1)^{j-1} \alpha_{j-1}^{(j)}(u)\, t_1(u) + (-1)^j \alpha_j^{(j)}(u) \cdot t_0(u) \Big)$$

The assertion follows by multiplication with $j!$.
The second equation follows immediately by noting that $k \in \{0,1,\dots,j\} \Leftrightarrow j-k \in \{0,1,\dots,j\}$

$$\Rightarrow \quad j!\, a_u^{(j+1)} = \sum_{k=0}^{j} (-1)^k \alpha_{j-k}^{(j)}(u) \cdot t_k(u) . \qquad \square$$

Theorem 9 in matrix form

$$\begin{pmatrix} 0!\, a_u^{(1)} \\ 1!\, a_u^{(2)} \\ 2!\, a_u^{(3)} \\ 3!\, a_u^{(4)} \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_0^{(0)}(u) & 0 & 0 & 0 & \cdots \\ \alpha_1^{(1)}(u) & -\alpha_0^{(1)}(u) & 0 & 0 & \cdots \\ \alpha_2^{(2)}(u) & -\alpha_1^{(2)}(u) & \alpha_0^{(2)}(u) & 0 & \cdots \\ \alpha_3^{(3)}(u) & -\alpha_2^{(3)}(u) & \alpha_1^{(3)}(u) & -\alpha_0^{(3)}(u) & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}}_{=:\, B(u) \text{ lower triangular matrix}} \cdot \begin{pmatrix} t_0(u) \\ t_1(u) \\ t_2(u) \\ t_3(u) \\ \vdots \end{pmatrix}$$

or

$$\begin{pmatrix} 0! & & & \\ & 1! & & \\ & & 2! & \\ & & & \ddots \end{pmatrix} \cdot \begin{pmatrix} a_u^{(1)} \\ a_u^{(2)} \\ a_u^{(3)} \\ \vdots \end{pmatrix} = B(u) \cdot A(u) \cdot \begin{pmatrix} a_u^{(1)} \\ a_u^{(2)} \\ a_u^{(3)} \\ \vdots \end{pmatrix}$$

that is

$$\begin{pmatrix} 0! & & & 0 \\ & 1! & & \\ & & 2! & \\ 0 & & & \ddots \end{pmatrix} = B(u) \cdot A(u)$$

- Some properties of the matrix

$$\underline{B(u) := \left(b_{jk}(u) \right)}_{j,k=0,1,\ldots}$$

0) $$b_{jk}(u) := (-1)^k \alpha_{j-k}^{(j)}(u) \quad , j,k = 0,1,2,\ldots$$

1) By definition $\alpha_0^{(j)}(u) = 1$, $j = 0,1,2,\ldots$,
that is the main diagonal is given by
$$b_{jj}(u) = (-1)^j \quad , j = 0,1,2,\ldots$$

2) $$\alpha_1^{(j)}(u) = ju + \binom{j+1}{2} \quad , j = 0,1,2,\ldots ,$$
that is $$b_{j,j-1}(u) = (-1)^{j-1}\left(ju + \binom{j+1}{2} \right) , j = 1,2,\ldots$$

3) $$\alpha_j^{(j)}(u) = \frac{(u+j)!}{u!} \quad , j = 0,1,2,\ldots$$
that is $$b_{j0}(u) = \alpha_j^{(j)}(u) = \frac{(u+j)!}{u!}$$

4) Let $k \geq 0$ and $j \geq k$, then $\left(\alpha_{-1}^{(j)}(u) := 0 \right)$

$$\alpha_k^{(j+1)}(u) = \alpha_k^{(j)}(u) + \alpha_{k-1}^{(j)}(u) \cdot (u+j+1)$$
$$\alpha_k^{(k)}(u) = \frac{(u+k)!}{u!}$$

5) (Rowsums)

For all $u, j \in \mathbb{N}_0$

$$\sum_{k=0}^{j} (-1)^k \alpha_{j-k}^{(j)}(n) = j! \binom{n+j-1}{j} = \frac{(n+j-1)!}{(n-1)!} = n^{\overline{j}}$$

$$= \sum_{k=0}^{j} \left[{j \atop k}\right] n^k ,$$

where $n^{\overline{j}} := n(n+1)(n+2)\cdots(n+j-1)$ rising factorial powers of n

and $\left[{j \atop k}\right]$ are the Stirling numbers of the first kind counting the numbers of ways to arrange j objects into k cycles, $0 \le k \le j$.

Proof By Example E3 (Dirichlet's sequence) it is

$$\underbrace{\sum_{k=0}^{n} a_k}_{=t_0(n)} = \underbrace{\sum_{k=0}^{n} k a_k}_{=t_1(n)} = \cdots = \underbrace{\sum_{k=0}^{n} k^j a_k}_{t_j(n)} = 1, \; j \ge 0$$

Hence, by Theorem 9

$$\underbrace{j!\, a_n^{(j+1)}}_{=\binom{n+j-1}{j}} = \sum_{k=0}^{j} (-1)^k \cdot \alpha_{j-k}^{(j)}(n) \cdot \underbrace{t_k(n)}_{=1}$$

$$\Longrightarrow \sum_{k=0}^{j} (-1)^k \alpha_{j-k}^{(j)}(n) = j! \binom{n+j-1}{j} = n(n+1)\cdots(n+j-1) = n^{\overline{j}}. \; \square$$

6) (Alternate Row Sums)

For all $n, j \in \mathbb{N}_0$

$$\sum_{k=0}^{j} \alpha_k^{(j)}(n) = \frac{(n+j+1)!}{(n+1)!}$$

Proof Let $f_j := \sum_{k=0}^{j} \alpha_k^{(j)}(n)$, then by 4)

$$f_j = \sum_{k=0}^{j} \alpha_k^{(j-1)}(n) + (n+j) \sum_{k=0}^{j} \alpha_{k-1}^{(j-1)}(n)$$

$$= \underbrace{\alpha_j^{(j+1)}(n)}_{=0} + \sum_{k=0}^{j-1} \alpha_k^{(j-1)}(n) + (n+j) \underbrace{\sum_{k=1}^{j} \alpha_{k-1}^{(j-1)}(n)}_{= \sum_{k=0}^{j-1} \alpha_k^{(j-1)}(n)}, \text{ since } \alpha_{-1}^{(j-1)}(n) = 0$$

or $f_j = f_{j-1} + (n+j) f_{j-1} = (n+j+1) f_{j-1}$, $f_0 = \alpha_0^{(0)}(n) = 1$

Solution: $f_j = \sum_{k=0}^{j} \alpha_k^{(j)}(n) = (n+j+1)(n+j) \cdots (n+2)(n+1)$

$$= \frac{(n+j+1)!}{(n+1)!}$$

□

Matrix $B(n)$

$$
\left(\begin{array}{c|c|c|c|c}
1 & 0 & 0 & 0 & \cdots \\
n+1 & -1 & 0 & 0 & \vdots \\
(n+1)(n+2) & -(2n+3) & 1 & 0 & \vdots \\
(n+1)(n+2)(n+3) & -(3n^2+12n+11) & 3n+6 & -1 & \vdots \\
(n+1)(n+2)(n+3)(n+4) & -(4n^3+30n^2+70n+50) & 6n^2+30n+35 & -(4n+10) & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)
$$