Some results on unique-maximum coloring of plane graphs

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2nd CroCoDays, Zagreb

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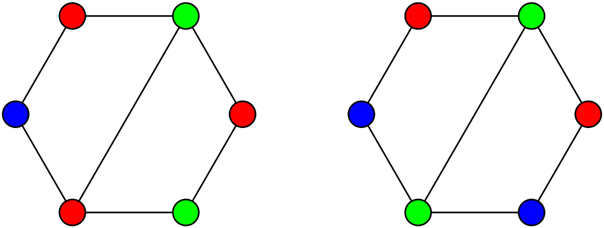
Joint work with V. Andova, B. Lidickeý, B. Lužar, K. Messerschmidt

Graph Coloring

A *(proper) coloring* of a graph G is a mapping $\varphi : V(G) \to \mathcal{C}$ such that for every $uv \in E(G) : \varphi(u) \neq \varphi(v)$.



G is *k-colorable* if there is a (proper) coloring of G with $|\mathcal{C}| = k$.

Minimum k such that G is k-colorable is denote by $\chi(G)$.

Here we color with $\{1, 2, \ldots, k\}$ instead of arbitrary $\mathcal{C}$.

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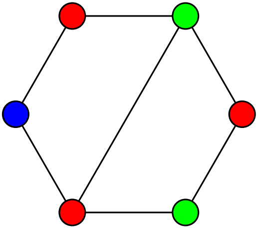

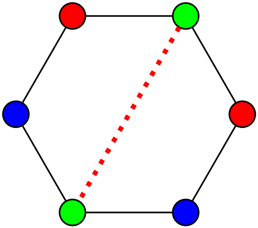

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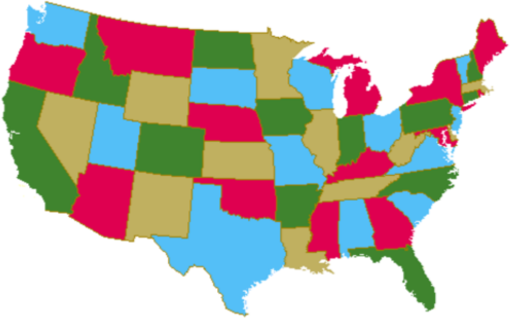

Figure: States in colors

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If G is a plane graph, then there is a proper coloring of the vertices of G by colors in $\{1, 2, 3, 4\}$ such that every face contains a unique vertex colored with the maximal color appearing on that face.

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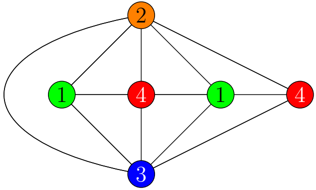

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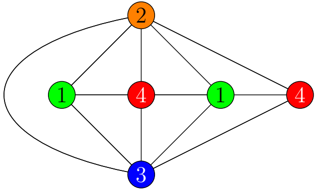

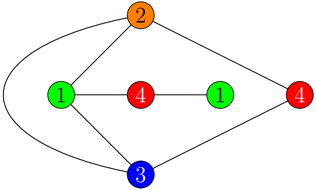

Fabrici and Göring

A proper coloring of a graph G embedded on some surface, where

(1) *colors are natural numbers, and*

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Example: $\chi_{\mathrm{fum}}(K_4) = \chi(K_4)$ and $\chi_{\mathrm{fum}}(Q_3) \neq \chi(Q_3)$.

Unique-maximum coloring

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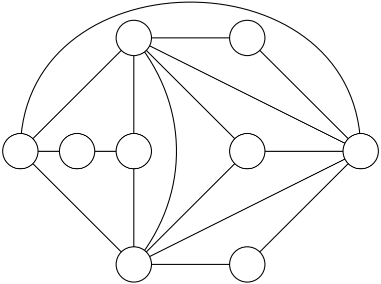

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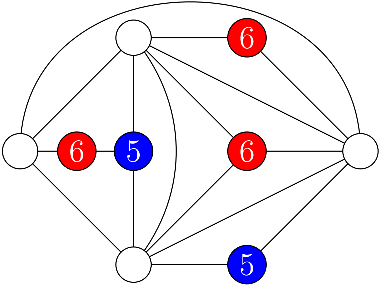
If G is a plane graph, then $\chi_{\mathrm{fum}}(G) \leq 6$.

Color *some* vertices of G by colors 5 and 6 such that each face contains unique 6 or (no 6 and unique 5).



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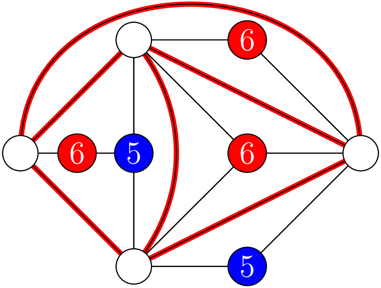

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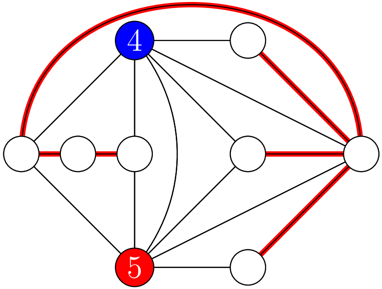

Color rest by 4-color theorem with $\{1, 2, 3, 4\}$.
Wendland: Make *the rest* triangle-free and use Grötzsch's theorem.
Just $\{4, 5\} \cup \{1, 2, 3\}$ colors needed in total.

Our Results

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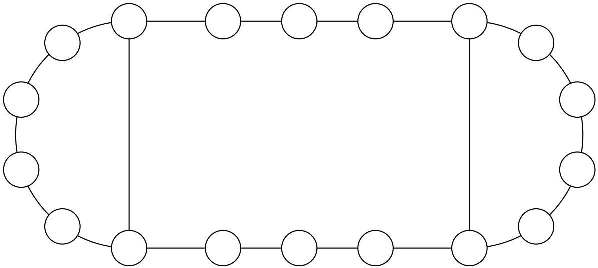
Theorem (Andova, Lidický, Lužar, Š.)

If G is an outerplane graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Both results are tight.

Tight Example

For the following graph G, $\chi_{\text{fum}}(G) > 3$.
Suppose for contradiction $\chi_{\text{fum}}(G) = 3$:



Notice G is subcubic, bipartite, 2-connected, and outerplane.

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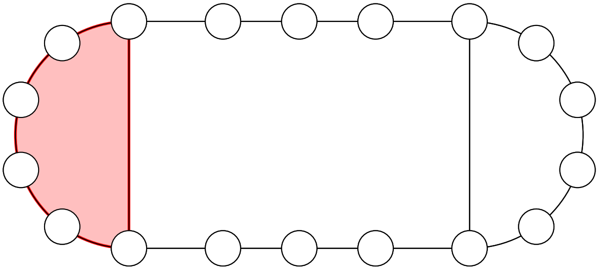

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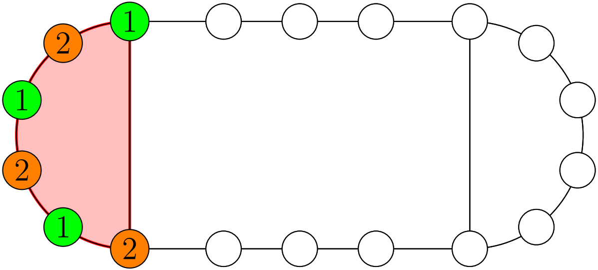

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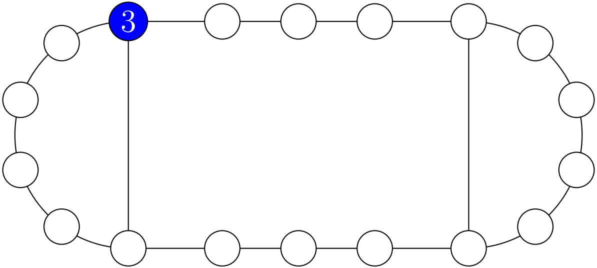

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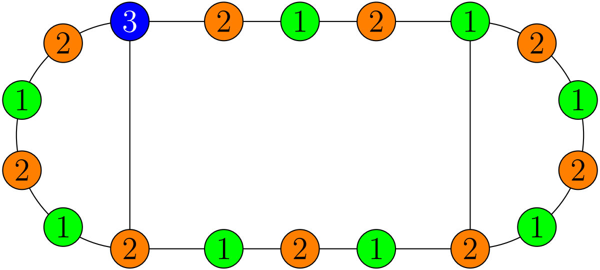

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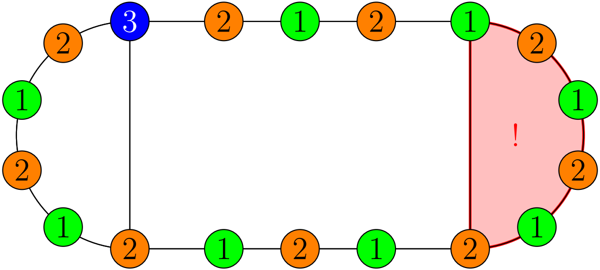
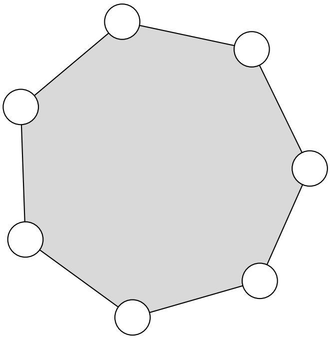


Notice G is subcubic, bipartite, 2-connected, and outerplane.
Also, G can have arbitrarily large girth.

Proof Idea

If G is a plane subcubic graph, then $\chi_{\text{fum}}(G) \leq 4$.

Use precoloring extension method (Thomassen's 5-list coloring)



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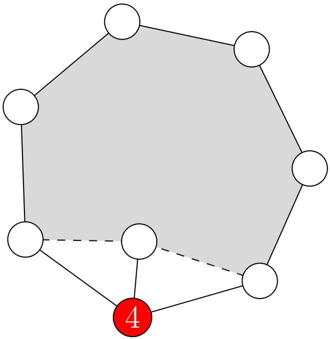

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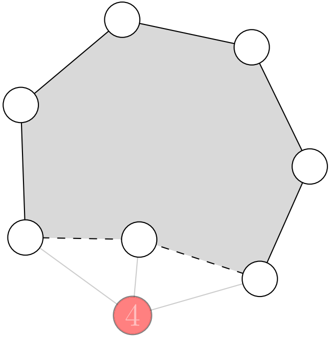

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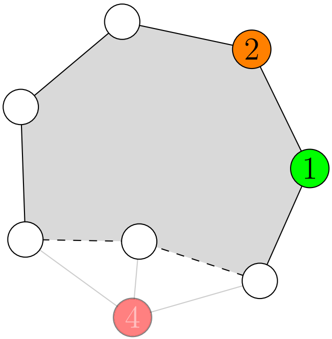

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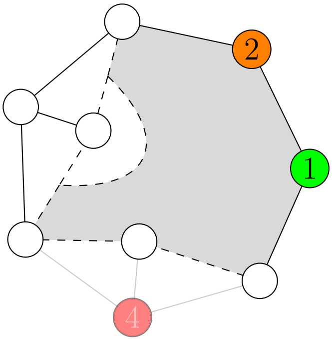

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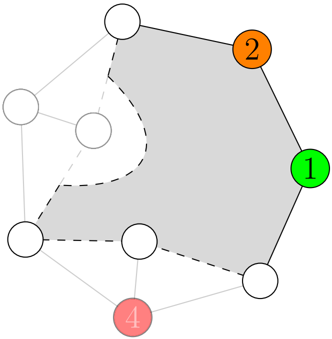

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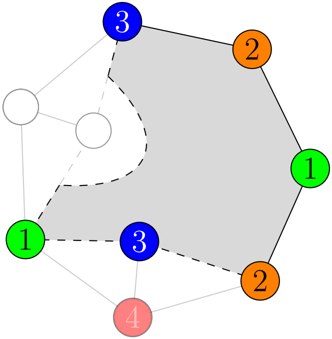

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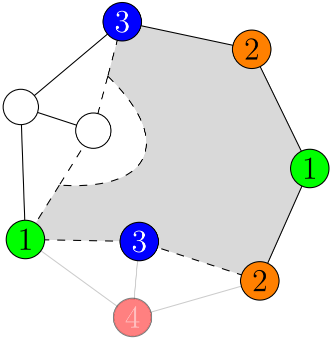

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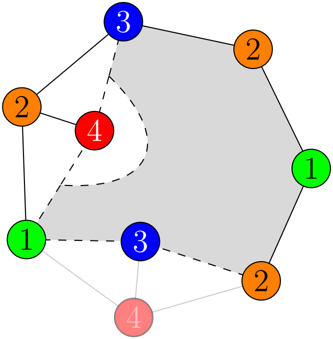

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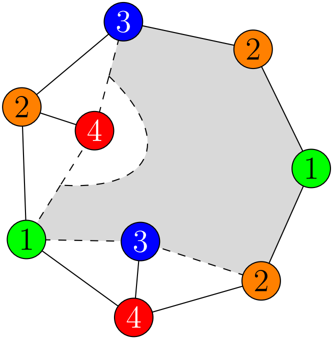

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Theorem (Lidický, Messerschmidt, Š.)

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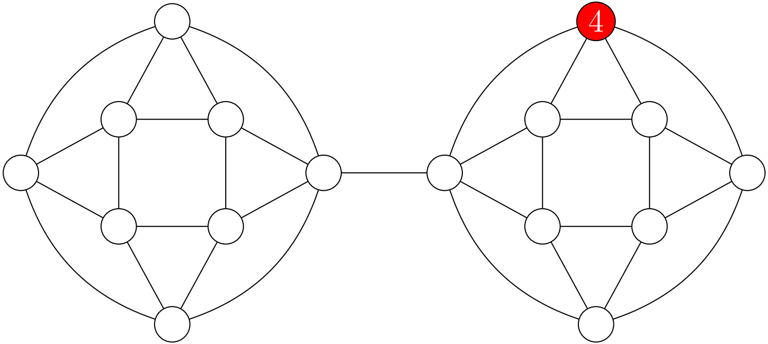

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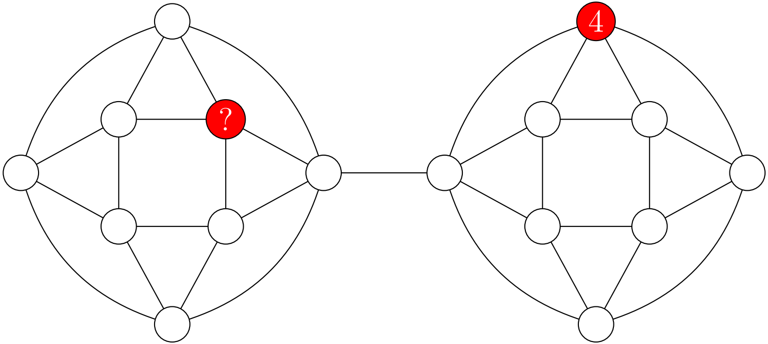

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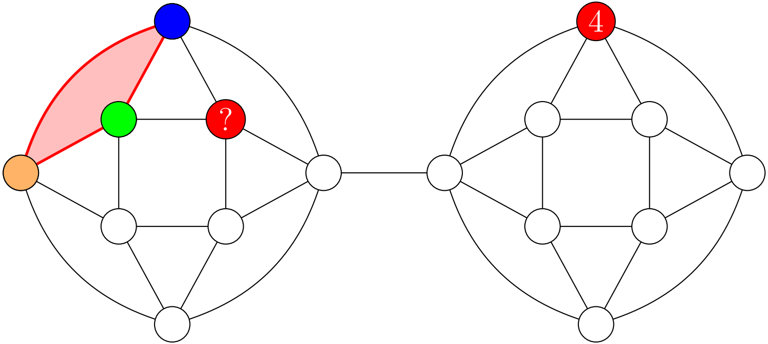

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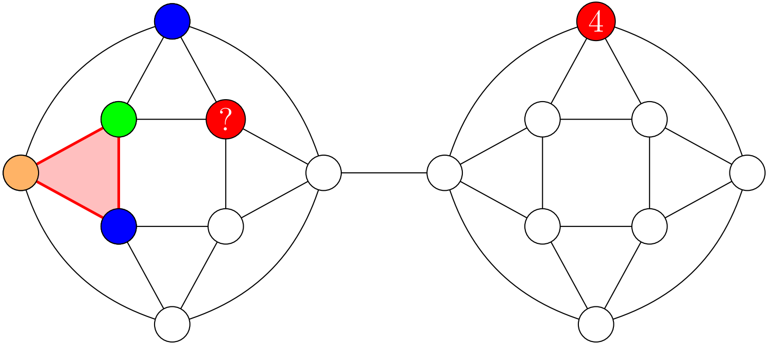

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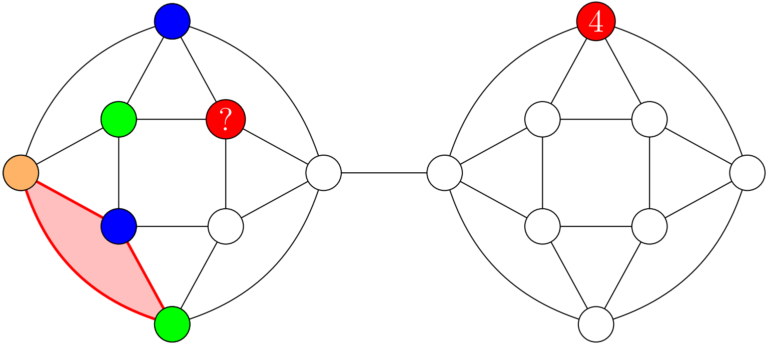

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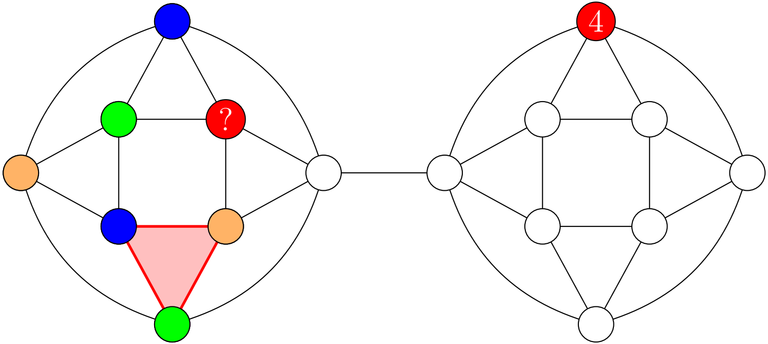

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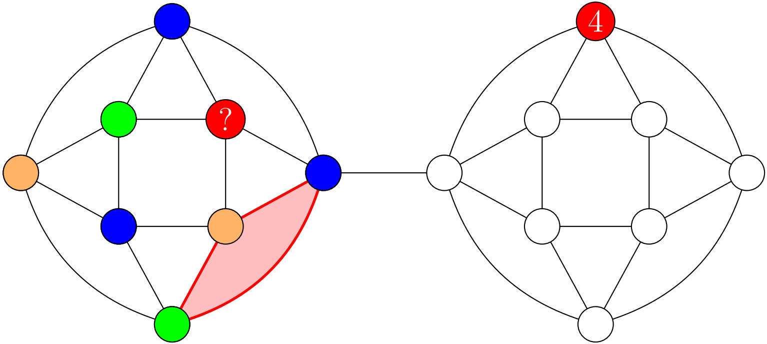

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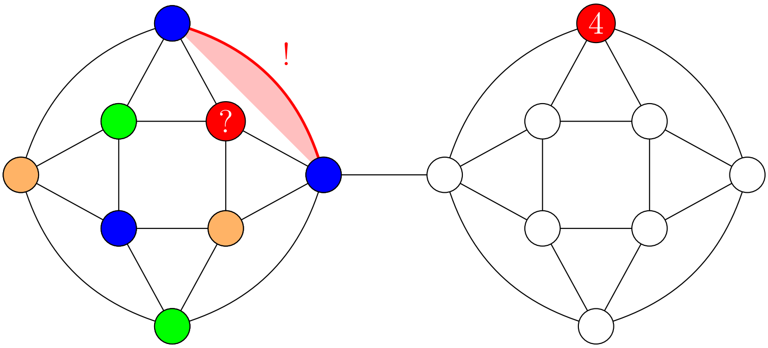

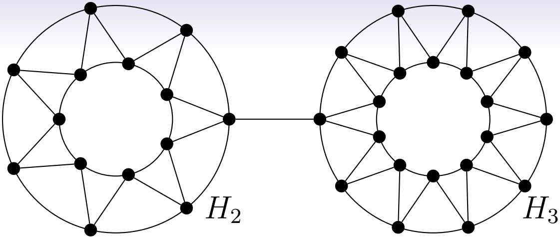
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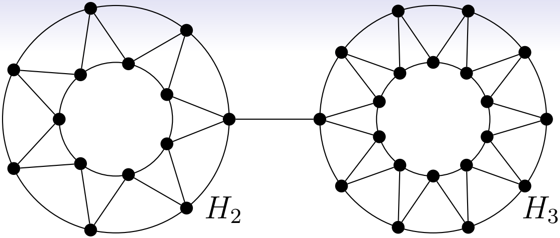
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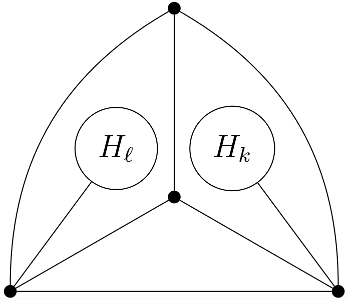
H_ℓ
H_k

SOME
VARIATIONS
&
DIRECTIONS

1. Graphs on surfaces

For a surface Σ, we define the facial unique-maximum chromatic number of Σ,

$$\chi_{\mathrm{fum}}(\Sigma) = \max_{G \hookrightarrow \Sigma} \chi_{\mathrm{fum}}(G),$$

as the maximum of $\chi_{\mathrm{fum}}(G)$ over all graphs G embedded into Σ.

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Problem

Determine $\chi_{\mathrm{fum}}(\Sigma)$ for surfaces Σ of higher genus.

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Conjecture

If G is a plane graph, then $\chi_{2\text{-fum}}(G) \leq 4$.

2. Weakening the uniqueness condition

We can study a variation of this coloring, where

(1) *colors are natural numbers, and*

(2') *every face has at most k vertices colored with its maximal color.*

Let call this coloring by a facial k-unique-maximum coloring or k-FUM-coloring for short. And, denote this chromatic number by $\chi_{k\text{-fum}}(G)$.

Conjecture

If G is a plane graph, then $\chi_{2\text{-fum}}(G) \leq 4$.

Conjecture

There exists large k such that for every plane graph G, it holds

$$\chi_{k\text{-fum}}(G) \leq 4.$$

3. The case $\Delta = 4$

We now introduce a variation of Fabrici and Göring's Conjecture with maximum degree and connectivity conditions added.

Theorem (Andova, Lidický, Lužar, Š.)
If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Conjecture
If G is a connected plane graph with maximum degree 4, then $\chi_{\mathrm{fum}}(G) \leq 4$.

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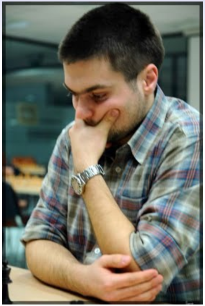
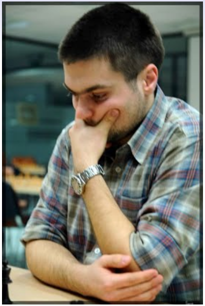
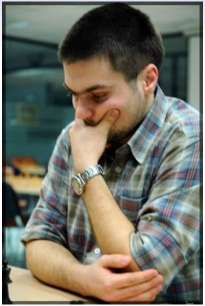
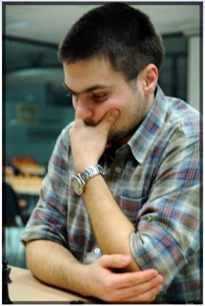
Notice that we constructed a counterexample of maximum degree five.




THANK YOU