A New Algorithm for Finding Largest Small Polygons Using Symbolic Computations

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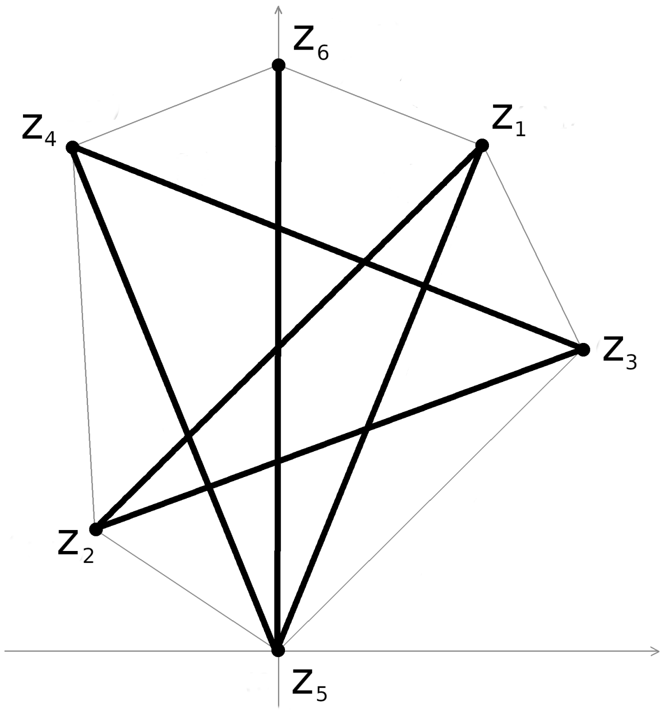
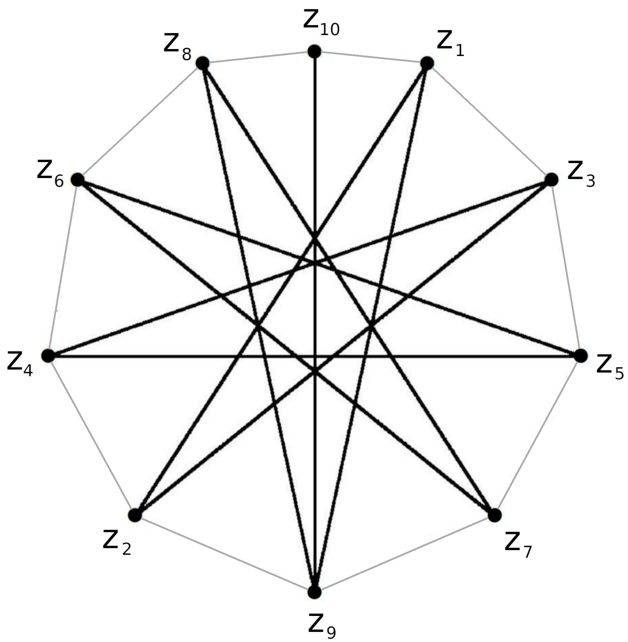
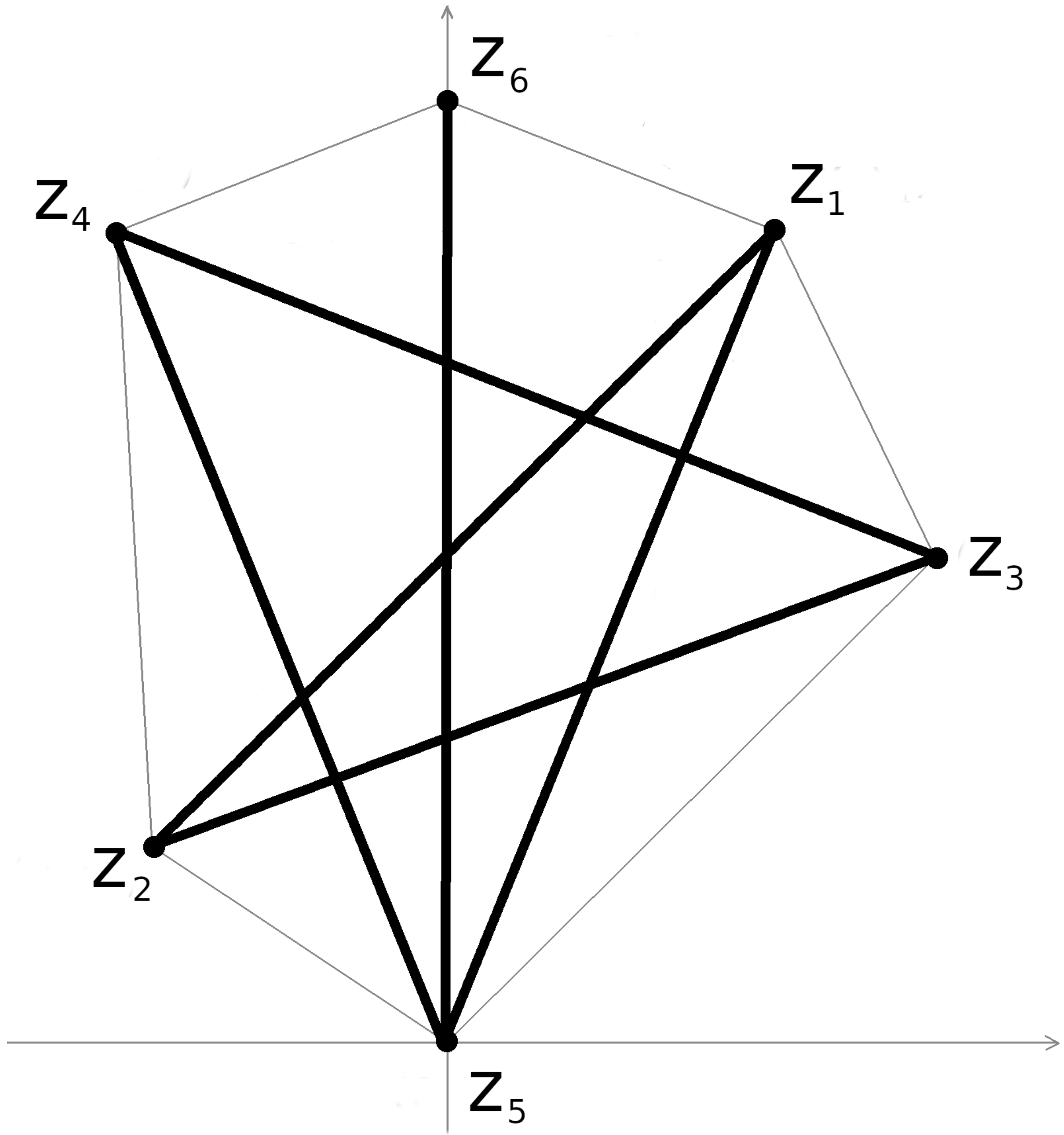
Abstract

A **small polygon** is a convex polygon of unit diameter. We are interested in **largest small polygons** which have the largest area for a given number of vertices n. For n odd the solution is a regular small polygon (Reinhardt, 1922.). It is conjectured (and proved for $n = 4$ (Reinhardt 1922. [3]) and $n = 6$ (Yuan 2004. [4], and our short proof recently) so far) that such polygons are **axially symmetric**. The **largest small hexagon** was studied by Graham (1975.) and the **largest small octagon** by Hansen et al. (2002.) using **global optimization**. Using our new algorithm we present here the exact area equation of degree 10, 42 and 152 for largest **small hexagon**, (axially symmetric) **small octagon** and (axially symmetric) **small decagon**.





1 Largest Small General Hexagon



For a general small hexagon P_6 of maximal area its **diameter graph** is a cycle $z_1 z_2 z_3 z_4 z_5$ with a pending edge $z_5 z_6$, as shown by Graham in [2].

We parametrize P_6 by using unit complex numbers z, u, v: $z_1 = z$, $z_2 = z + u$, $z_3 = -\overline{z} + v$, $z_4 = -\overline{z}$, $z_5 = 0$, $z_6 = I$ subject to the constraint $|z_2 - z_3| = 1$.

Then the equation for area of P_6 is given by (S denotes area)

$$\mathbf{A_6} := \frac{1}{2}\mathrm{Im}(\overline{z_3} z_1) + \mathrm{Re}(z_1) + \frac{1}{2}\mathrm{Im}(\overline{z_4} z_2) - S = \left(\mathbf{z} + \frac{\mathbf{1}}{\mathbf{z}}\right)\left(\frac{\mathbf{-1}}{\mathbf{2I}}\left(\mathbf{z} - \frac{\mathbf{1}}{\mathbf{z}}\right) + \mathbf{1}\right) + \frac{\mathbf{1}}{\mathbf{4I}}\left(\frac{\mathbf{1}}{\mathbf{zu}} - \mathbf{zu} + \frac{\mathbf{z}}{\mathbf{v}} - \frac{\mathbf{v}}{\mathbf{z}}\right) - \mathbf{S}\,.$$

The above constraint now reads as:

$$\mathbf{Eq_6} := \left(\mathbf{z} + \frac{\mathbf{1}}{\mathbf{z}} + \mathbf{u} - \mathbf{v}\right)\left(\mathbf{z} + \frac{\mathbf{1}}{\mathbf{z}} + \frac{\mathbf{1}}{\mathbf{u}} - \frac{\mathbf{1}}{\mathbf{v}}\right) - \mathbf{1}\,.$$

Then we do constrained optimization by our algorithm in two steps (using Maple):

1) We eliminate one variable, say v, by computing

$$r_{12} = \text{resultant}(\text{numer}(A_6), \text{numer}(Eq_6), v) / \left(uz^2\right) = (z^9 + 2z^7 + z^5)u^6 + \cdots$$

This gives an implicit equation $r_{12}(z, u, S) = 0$ for S.

2) We optimize this implicitly given area function by taking partial derivatives of r_{12} w.r.t. u and z by computing two discriminants:

$$\Delta_1 = \text{discrim}(r_{12}, u) = \text{resultant}\left(r_{12}, \frac{\partial r_{12}}{\partial u}, u\right) = P_6 \cdot \overline{P_6} \cdot P_{12}^2 \cdot P_{24}^2 (z^2 - z + 1)^2 (z^2 + z + 1)^2 z^{14}$$

where P_6, P_{12}, P_{24} have z-degrees $6, 12, 24$. The discriminant of $P_6 = P_6(z, S)$ (or $\overline{P_6}$) leads directly to the famous degree 10 Graham's area equation of largest small hexagon ([2]):

$$\mathbf{Q(S) = discrim(P_6, z) / \left(4(2S - 1)^2 (2S + 1)^2\right) = 4096S^{10} + 8192S^9 - \cdots + 11993}\,.$$

Remark1: Elimination in different order z, u, v leads to polynomials with 652 digit coefficients!

Remark2: For axially symmetric hexagon we replace v by $-1/u$ in A_6 and Eq_6 by eq_6:

$$eq_6 := z^{-1} + u^{-1} + z + u + 1.$$

Viewing this system as linear in z and z^{-1} we obtain the following simplest compact implicit equation $P(S)$, whose discriminant divided by $4(2S-1)^4$ equals to Graham's $Q(S)$:

$$\mathbf{P(S) := u(2uSI - (u+1-I)(1+u+u^2))^2 + (1+u+u^2)(u+1)^2 \cdot (2uSI - (u+1-I)(1+u+u^2)) + u(u+1)^4 = 0}\,.$$

2 Largest Small (Axially Symmetric) Octagon

For an axially symmetric largest small octagon P_8 of maximal area we parametrize vertices of its diameter graph by three unit complex numbers z, u, v:

$$z_1 = z, z_2 = z + u, z_3 = z_2 + v, z_6 = -1/z,$$
$$z_5 = z_6 - 1/u, z_4 = z_5 - 1/v, z_7 = 0, z_8 = I$$

with constraint $z_3 - z_4 = 1$. This leads to the following two equations:

$$\mathbf{A_8} = \frac{\mathbf{1}}{\mathbf{2I}}\left(\left(\mathbf{z} + \frac{\mathbf{1}}{\mathbf{z}} + \frac{\mathbf{1}}{\mathbf{u}}\right)\left(\frac{\mathbf{1}}{\mathbf{z}} + \frac{\mathbf{1}}{\mathbf{u}} + \frac{\mathbf{1}}{\mathbf{v}}\right) - \left(\mathbf{z} + \frac{\mathbf{1}}{\mathbf{z}} + \mathbf{u}\right)(\mathbf{z} + \mathbf{u} + \mathbf{v})\right) + \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{z} + \frac{\mathbf{1}}{\mathbf{z}}\right) - \mathbf{S}$$

and $\mathbf{eq_8} := \mathbf{z} + \dfrac{\mathbf{1}}{\mathbf{z}} + \mathbf{u} + \dfrac{\mathbf{1}}{\mathbf{u}} + \mathbf{v} + \dfrac{\mathbf{1}}{\mathbf{v}} - \mathbf{1}$.

Now we apply our algorithm:

$$r_{12} := \text{resultant}(\text{numer}(A_8), \text{numer}(eq_8), v)/u^3 z^3 = z^2(1-z)u^6 + \cdots$$

$$\Delta_1 := discrim(r_{12}, u)/\left((2z^4 - 4Sz^3 + 4S^2z^2 + 3z^2 - 4Sz + 2)^2 z^{10}\right) = 32Iz^{32} + \cdots$$

$$\begin{aligned}\Delta_2 := \text{discrim}(\Delta_1, z)/2^{147} = \; & (\mathbf{147573952589676412928 \cdot S^{42}} - \cdots) \\ & \cdot(1934281311383406679529881600000000000000S^{76} + \cdots)^2 \\ & \cdot(63597477762712675306753228 8\,S^{50} - \cdots)^3\end{aligned}$$

The output of the last command is of the form:

$$\boxed{\mathbf{\Delta_2 = A_8 Q_8^2 R_8^3}}\;.$$

By Th. 6.8 ([6]) on irreducible factors of iterated discriminants it follows that the factor $\mathbf{A_8}$ **is the sought area equation of degree 42** for the Hansen's largest small octagon.

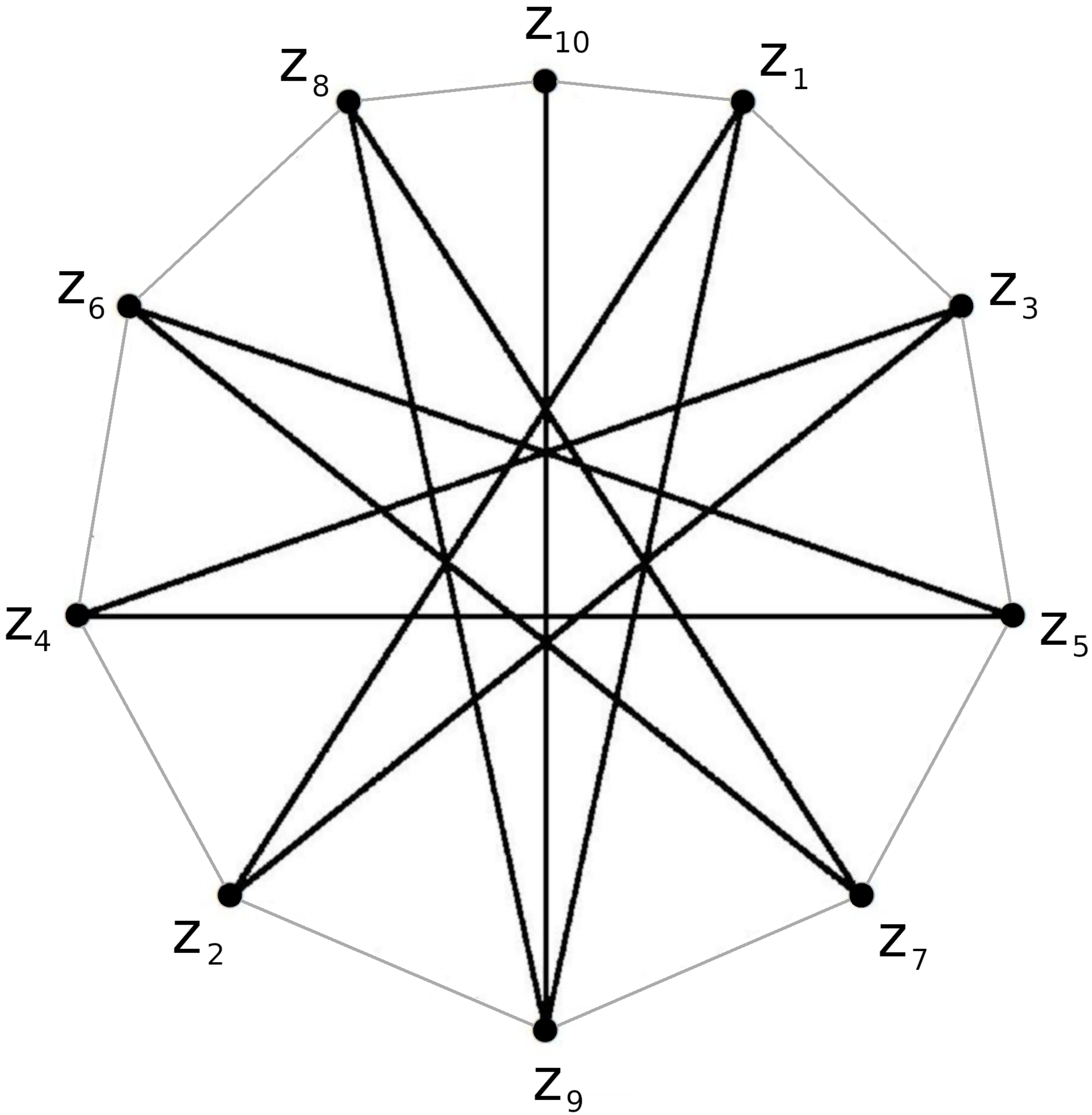
$$
\begin{aligned}
\mathbf{A8} :=\ & \\
& \mathbf{147573952589676412928 \cdot S^{42}} \\
& -44272185776902923878 4 \cdot S^{41} \\
& +260560260041147416576 0 \cdot S^{40} \\
& +767038677014935293132 8 \cdot S^{39} \\
& -198031201950824881192 96 \cdot S^{38} \\
& -902346445515520328335 36 \cdot S^{37} \\
& -531709183791524869465 7024 \cdot S^{36} \\
& -175940414306350846558 86336 \cdot S^{35} \\
& +297583954627030815782 99392 \cdot S^{34} \\
& +282207246119748476170 403840 \cdot S^{33} \\
& +335103297887714904283 021312 \cdot S^{32} \\
& -191792830770658778437 1240960 \cdot S^{31} \\
& -524030275888233572285 0746368 \cdot S^{30} \\
& +463161550709912144655 5746304 \cdot S^{29} \\
& +301141598745266485306 22218240 \cdot S^{28} \\
& -717500816118217966802 8030976 \cdot S^{27} \\
& -148064818635686576530 703515648 \cdot S^{26} \\
& -425518788297921320532 54275072 \cdot S^{25} \\
& +601318123428810231261 639475200 \cdot S^{24} \\
& +332708870397989105275 274002432 \cdot S^{23} \\
& -235889738935887683912 4819509248 \cdot S^{22} \\
& -680235061366055307103 034146816 \cdot S^{21} \\
& +745239256934692285875 3860567040 \cdot S^{20} \\
& -149186514413453909191 3264332800 \cdot S^{19} \\
& -154553479465468230258 54527832064 \cdot S^{18} \\
& +957486504044300438189 1485761536 \cdot S^{17} \\
& +201041980576999410488 10876698624 \cdot S^{16} \\
& -200270809479145717669 86403610624 \cdot S^{15} \\
& -161922708660050628360 01824866304 \cdot S^{14} \\
& +235881300612033363564 60301369344 \cdot S^{13} \\
& +800920668963918662182 2611818496 \cdot S^{12} \\
& -179358208579568143645 17526943744 \cdot S^{11} \\
& -237023873675284332563 5609948160 \cdot S^{10} \\
& +914703421371175991639 1887323136 \cdot S^{9} \\
& +367361764236902187872 898865664 \cdot S^{8} \\
& -307842863763637985028 0988117504 \cdot S^{7} \\
& +105551688808743610680 13425792 \cdot S^{6} \\
& +647330513128418259524 157203072 \cdot S^{5} \\
& -235235280294399556987 46202488 \cdot S^{4} \\
& -761430048779063209757 09476552 \cdot S^{3} \\
& +583370708172332860364 7313856 \cdot S^{2} \\
& +377304103834759651502 1000956 \cdot S \\
& -478425365462547737405 343343
\end{aligned}
$$

$$\mathbf{Q_8} = 19342813113834066795298816000000000000 \cdot S^{76} + \cdots + 11584100403774518202305003667230036656752805844 9$$

$$\mathbf{R_8} = 635974777627126753067532288 \cdot S^{50} + \cdots + 950890132718569056523540689874237176 2$$

Among all the glasses of octagonal shape with given diameter and height, this one, which is based on the Hansen's largest small octagon, has the maximal volume.

3 Largest Small (Axially Symmetric) Decagon



Computation of maximal area by old algorithm

For a small axially symmetric decagon, in terms of coordinates of vertices of its diameter graph the area equation is

$$\mathbf{A_{10} = x_7 y_7 + \left(x_7 + \frac{1}{2}\right)(y_5 - y_7) + \left(x_3 + \frac{1}{2}\right)(y_3 - y_5) + (x_1 + x_3)(y_1 - y_3) + x_1(1 - y_1) - S}\,.$$

Writing $y's$ in terms of $x's$ and using rational substitutions ($\mathbf{x_1 = \frac{2t}{1+t^2}}$, $\mathbf{x_3 = \frac{2s}{1+s^2} - \frac{1}{2}}$, $\mathbf{x_7 = \frac{2u}{1+u^2} - x_3}$) yields implicit area equation for the largest small decagon in terms of rational parameters s, t and u (599 terms):

$$A' = (4S^2 - 8S + 1)s^6 t^6 u^6 - \cdots + 4S^2 + 8S + 1\,.$$

Discriminant Δ_1 of A' w.r.t. variable u is of the form

$$\Delta_1 = -2^{34}(t^2 + 1)^{10}(s^2 + 1)^{10} P \cdot Q^2$$

for some irreducible polynomials $P = 2^{10}(S + 5)S^9 s^{32} t^{32} + \cdots$ and $Q = 4S(S + 1)s^4 t^4 + \cdots$.

Let Δ_2 be the discriminant of P w.r.t. t and let Δ_3 be the discriminant of a factor of degree 134 of Δ_2 w.r.t. s. The polynomial Δ_3 has the form

$$\boxed{\boldsymbol{\Delta_3 = -2^{16244}3^{72}5^2 A_{10} Q_{10}^2 R_{10}^3 (2S-1)^{32}},}$$

where $\mathbf{A_{10}}$ **is the sought polynomial in S of degree 152** (with 92– to 146–digit coefficients). The polynomial Q_{10} (resp. R_{10}) is huge with up to 2928–digit coefficients (resp. 426–digit). The root of A_{10} corresponding to the area of the largest small decagon is approximately

$$\mathbf{S_{10}^* \approx 0.749137345877830270622719 8}\ldots$$

(result in [5] agrees up to seven digits).

Computation of coordinates

Let Δ_{2s} be the discriminant of the polynomial P (a factor of Δ_1 above) w.r.t. s. Let P_1 be the analogous irreducible factor of Δ_{2s} and let its resultant w.r.t. variable S be E. Let E_1 be analogous irreducible factor of E. By eliminating t from the equation E_1 and the quadratic equation relating t and x_1 we obtain the equation for x_1 of degree 152. Its root corresponding to the abcissa x_1 of the first point of the largest small decagon is approximately

$$\mathbf{x_1^*} \approx \mathbf{0.21101203854137990861843 56}\ldots$$

(result in [5] agrees up to five digits).

4 Addendum

4.1 Graham's result

$$
\begin{aligned}
&R = 1125899906842624S^{40} + 9007199254740992S^{39} + 43523068273885184S^{39}+\\
&+96686654500110336S^{38} + 71892942171668480S^{36} - 203545990580404224S^{35}-\\
&-3231739551940608S^{34} + 2967153761027358720S^{33} + 3933037175129505792S^{32}-\\
&-10392801849559220224S^{31} - 26535490294760079360S^{30} + 6097086784076355993 6S^{29}+\\
&+12497127010966542745 6S^{28} - 29406155405292627558 4S^{27} - 37789210318528210534 4S^{26}+\\
&+10430077634353718886 40S^{25} + 12479341260277651865 60S^{24} - 45892046625445616025 60S^{23}-\\
&-53447928499775157043 2S^{22} + 13715332431765689073 664S^{21} - 13302644475428239835 136S^{20}-\\
&-11542363148164431609 856S^{19} + 30184905793343524306 944S^{18} - 11553929011986625003 520S^{17}-\\
&-20761961484788844621 824S^{16} + 23716893785532021784 576S^{15} - 14706887769816009359 36S^{14}-\\
&-11829726550983847370 752S^{13} + 70474104124464627527 68S^{12} + 59064832427255013376 0S^{11}-\\
&-22346950170997128744 96S^{10} + 83656057002703544422 4S^{9} + 38308661311225029504S^{8}-\\
&-11317144290755034547 2S^{7} + 29418059499453088640S^{6} + 666807807062928 64S^{5}-\\
&-14344056936110925 12S^{4} + 28386128184561030 4S^{3} - 20855579198349088S^{2}+\\
&+370175331659648S - 1886186974063
\end{aligned}
$$

$$
\begin{aligned}
&factor(R) = \left(4096S^{10} + 8192S^{9} + 1600S^{8} - 20608S^{7} + 20032S^{6} + 87360S^{5} - 105904S^{4}+\right.\\
&\left.+18544S^{3} + 11888S^{2} - 3416S + 41\right)\left(4096S^{10} + 8192S^{9} - 3008S^{8} - 30848S^{7} + 21056S^{6}+\right.\\
&\left.+146496S^{5} - 221360S^{4} + 1232S^{3} + 144464S^{2} - 78488S + 11993\right)\left(8192S^{10} + 12288S^{9}+\right.\\
&+66560S^{8} - 22528S^{7} - 138240S^{6} + 572928S^{5} - 90496S^{4} - 356032S^{3} + 113032S^{2} + 23420S-\\
&\left.-8179\right)\left(8192S^{10} + 20480S^{9} + 58368S^{8} - 161792S^{7} + 198656S^{6} + 199680S^{5} - 414848S^{4}-\right.\\
&\left.-4160S^{3} + 171816S^{2} - 48556S + 469\right)
\end{aligned}
$$

4.2 A8eq

$$
\begin{aligned}
&A8eq := \\
&\quad 147573952589676412928 \cdot S^{42} - 442721857769029238784 \cdot S^{41} + \\
&+ 2605602600411474165760 \cdot S^{40} + 7670386770149352931328 \cdot S^{39} - \\
&- 19803120195082488119296 \cdot S^{38} - 90234644551552032833536 \cdot S^{37} - \\
&- 531709183791524869465702 4 \cdot S^{36} - 1759404143063508465588633 6 \cdot S^{35} + \\
&+ 2975839546270308157829939 2 \cdot S^{34} + 28220724611974847617040384 0 \cdot S^{33} + \\
&+ 3351032978877149042830213 12 \cdot S^{32} - 1917928307706587784371240960 \cdot S^{31} - \\
&- 524030275888233572285074636 8 \cdot S^{30} + 4631615507099121446555746304 \cdot S^{29} + \\
&+ 30114159874526648530622218240 \cdot S^{28} - 7175008161182179668028030976 \cdot S^{27} - \\
&- 148064818635686576530703515648 \cdot S^{26} - 42551878829792132053254275072 \cdot S^{25} + \\
&+ 601318123428810231261639475200 \cdot S^{24} + 332708870397989105275274002432 \cdot S^{23} - \\
&- 2358897389358876839124819509248 \cdot S^{22} - 680235061366055307103034146816 \cdot S^{21} + \\
&+ 7452392569346922858753860567040 \cdot S^{20} - 1491865144134539091913264332800 \cdot S^{19} - \\
&- 15455347946546823025854527832064 \cdot S^{18} + 9574865040443004381891485761536 \cdot S^{17} + \\
&+ 20104198057699941048810876698624 \cdot S^{16} - 20027080947914571766986403610624 \cdot S^{15} - \\
&- 16192270866005062836001824866304 \cdot S^{14} + 23588130061203336356460301369344 \cdot S^{13} + \\
&+ 8009206689639186621822611818496 \cdot S^{12} - 17935820857956814364517526943744 \cdot S^{11} - \\
&- 2370238736752843325635609948160 \cdot S^{10} + 9147034213711759916391887323136 \cdot S^{9} + \\
&+ 367361764236902187872898865664 \cdot S^{8} - 3078428637636379850280988117504 \cdot S^{7} + \\
&+ 105551688808743610680134257 92 \cdot S^{6} + 647330513128418259524157203072 \cdot S^{5} - \\
&- 235235280294399556987462024 88 \cdot S^{4} - 761430048779063209757094765 52 \cdot S^{3} + \\
&+ 583370708172332860364731385 6 \cdot S^{2} + 37730410383475965150210009 56 \cdot S - \\
&- 4784253654625477374053433 43
\end{aligned}
$$

4.3 Sizes of coefficients

$map(length, coeffs(op(1, op(3, Delta3)), S));$

$\{1820, 1823, 1827, 1830, 1834, 1836, 1840, 1843, 1846, 1849, 1852, 1855, 1857, 1860, 1863, 1866,$
$1869, 1872, 1874, 1876, 1880, 1882, 1885, 1888, 1890, 1892, 1896, 1898, 1901, 1903, 1905, 1908,$
$1911, 1913, 1915, 1918, 1920, 1922, 1924, 1927, 1930, 1932, 1934, 1936, 1938, 1941, 1943, 1945,$
$1947, 1949, 1950, 1952, 1954, 1956, 1958, 1960, 1961, 1963, 1965, 1966, 1968, 1970, 1972, 1973,$
$1975, 1977, 1979, 1980, 1982, 1984, 1985, 1987, 1988, 1989, 1991, 1992, 1994, 1996, 1997, 1999,$
$2000, 2001, 2003, 2005, 2007, 2009, 2010, 2012, 2013, 2015, 2016, 2018, 2019, 2021, 2022, 2023,$
$2024, 2026, 2028, 2030, 2031, 2033, 2034, 2035, 2036, 2038, 2040, 2042, 2043, 2045, 2047, 2048,$
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$2316, 2317, 2318, 2319, 2320, 2321, 2322, 2323, 2324, 2325, 2326, 2327, 2328, 2329, 2330, 2331,$
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$2348, 2349, 2350, 2351, 2352, 2353, 2354, 2355, 2356, 2357, 2358, 2359, 2360, 2361, 2362, 2363,$
$2364, 2365, 2366, 2367, 2368, 2369, 2370, 2371, 2372, 2373, 2374, 2375, 2376, 2377, 2378, 2379,$
$2380, 2381, 2382, 2383, 2384, 2385, 2386, 2387, 2388, 2389, 2390, 2391, 2392, 2393, 2394, 2395,$
$2396, 2397, 2398, 2399, 2400, 2401, 2402, 2403, 2404, 2405, 2406, 2407, 2408, 2409, 2410, 2411,$

2412, 2413, 2414, 2415, 2416, 2417, 2418, 2419, 2420, 2421, 2422, 2423, 2424, 2425, 2426, 2427, 2428, 2429, 2430, 2431, 2432, 2433, 2434, 2435, 2436, 2437, 2438, 2439, 2440, 2441, 2442, 2443, 2444, 2445, 2446, 2447, 2448, 2449, 2450, 2451, 2452, 2453, 2454, 2455, 2456, 2457, 2458, 2459, 2460, 2461, 2462, 2463, 2464, 2465, 2466, 2467, 2468, 2469, 2470, 2471, 2472, 2473, 2474, 2475, 2476, 2477, 2478, 2479, 2480, 2481, 2482, 2483, 2484, 2485, 2486, 2487, 2488, 2489, 2490, 2491, 2492, 2493, 2494, 2495, 2496, 2497, 2498, 2499, 2500, 2501, 2502, 2503, 2504, 2505, 2506, 2507, 2508, 2509, 2510, 2511, 2512, 2513, 2514, 2515, 2516, 2517, 2518, 2519, 2520, 2521, 2522, 2523, 2524, 2525, 2526, 2527, 2528, 2529, 2530, 2531, 2532, 2533, 2534, 2535, 2536, 2537, 2538, 2539, 2540, 2541, 2542, 2543, 2544, 2545, 2546, 2547, 2548, 2549, 2550, 2551, 2552, 2553, 2554, 2555, 2556, 2557, 2558, 2559, 2560, 2561, 2562, 2563, 2564, 2565, 2566, 2567, 2568, 2569, 2570, 2571, 2572, 2573, 2574, 2575, 2576, 2577, 2578, 2579, 2580, 2581, 2582, 2583, 2584, 2585, 2586, 2587, 2588, 2589, 2590, 2591, 2592, 2593, 2594, 2595, 2596, 2597, 2598, 2599, 2600, 2601, 2602, 2603, 2604, 2605, 2606, 2607, 2608, 2609, 2610, 2611, 2612, 2613, 2614, 2615, 2616, 2617, 2618, 2619, 2620, 2621, 2622, 2623, 2624, 2625, 2626, 2627, 2628, 2629, 2630, 2631, 2632, 2633, 2634, 2635, 2636, 2637, 2638, 2639, 2640, 2641, 2642, 2643, 2644, 2645, 2646, 2647, 2648, 2649, 2650, 2651, 2652, 2653, 2654, 2655, 2656, 2657, 2658, 2659, 2660, 2661, 2662, 2663, 2664, 2665, 2666, 2667, 2668, 2669, 2670, 2671, 2672, 2673, 2674, 2675, 2676, 2677, 2678, 2679, 2680, 2681, 2682, 2683, 2684, 2685, 2686, 2687, 2688, 2689, 2690, 2691, 2692, 2693, 2694, 2695, 2696, 2697, 2698, 2699, 2700, 2701, 2702, 2703, 2704, 2705, 2706, 2707, 2708, 2709, 2710, 2711, 2712, 2713, 2714, 2715, 2716, 2717, 2718, 2719, 2720, 2721, 2722, 2723, 2724, 2725, 2726, 2727, 2728, 2729, 2730, 2731, 2732, 2733, 2734, 2735, 2736, 2737, 2738, 2739, 2740, 2741, 2742, 2743, 2744, 2745, 2746, 2747, 2748, 2749, 2750, 2751, 2752, 2753, 2754, 2755, 2756, 2757, 2758, 2759, 2760, 2761, 2762, 2763, 2764, 2765, 2766, 2767, 2768, 2769, 2770, 2771, 2772, 2773, 2774, 2775, 2776, 2777, 2778, 2779, 2780, 2781, 2782, 2783, 2784, 2785, 2786, 2787, 2788, 2789, 2790, 2791, 2792, 2793, 2794, 2795, 2796, 2797, 2798, 2799, 2800, 2801, 2802, 2803, 2804, 2805, 2806, 2807, 2808, 2809, 2810, 2811, 2812, 2813, 2814, 2815, 2816, 2817, 2818, 2819, 2820, 2821, 2822, 2823, 2824, 2825, 2826, 2827, 2828, 2829, 2830, 2831, 2832, 2833, 2834, 2835, 2836, 2837, 2838, 2839, 2840, 2841, 2842, 2843, 2844, 2845, 2846, 2847, 2848, 2849, 2850, 2851, 2852, 2853, 2854, 2855, 2856, 2857, 2858, 2859, 2860, 2861, 2862, 2863, 2864, 2865, 2866, 2867, 2868, 2869, 2870, 2871, 2872, 2873, 2874, 2875, 2876, 2877, 2878, 2879, 2880, 2881, 2882, 2883, 2884, 2885, 2886, 2887, 2888, 2889, 2890, 2891, 892, 2893}

4.4 Aeq10 := op(2, Delta3);

$Aeq10 := op(2, Delta3);$

$9407871652389753187964170144480501643901832571531552372745038198966433256229966771378769 1008S^{152}-$
$-6036717643616758295610342509374988554837009233399412772511399511003461339414228678301376 83968S^{151}+$
$+3151746798348979659802077212963803905326993552834709253765913235467612677336267632491684 7968256S^{150}-$
$-2604181627808055629331524573414770044629820836734590687282809052976584818950319788136465 60034816S^{149}+$
$+4640559190354763796951361381645043113896279990868044177384153215339472533649286455081767 735918592S^{148}-$
$-4304557892616252415974064300939704623898953229399469821685375804827780261086175770169624 5867741184S^{147}+$
$+2731445571038541597256962638542610456439059577904069573108656539941833807936366294696244 06256910336S^{146}-$
$-1058655202032968792575755674951588427296638077031233209798482111431964113002284029846387 036115697664S^{145}-$
$\ldots$

$+74578822372032351529299179587428053626366636762844327606991759386125665049593288636844316395337108663688$
$98716103333203292734905750800 0S^{4}-$
$-13375049592828902988764230367803264224214861704481682357510659152362505984397416660843930233056123730645$
$02669815121418262407898671412 8S^{3}+$
$+10659633508391497362882000083050705821400068190688468924091313006573792524886353268834224264347262278302$
$99632967984821057350931506198S^{2}-$
$-41559711898973036622753555013321849453348491961662045472133332755351309435492704885947540101318827391699$
$36059275320831881012577471 8S+$
$+55088978082313679795750772351197252489542724207499389193729835517655849747446641732852670627241971660905$
$32065793069744684948791 27$

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