Atiyah-Sutcliffe Conjectures for Almost Collinear Configurations and Some New Conjectures for Symmetric Functions

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Abstract

In 2001 Sir M. F. Atiyah formulated a conjecture $(C1)$ and later with P. Sutcliffe two stronger conjectures $(C2)$ and $(C3)$. These conjectures, inspired by physics (spin-statistics theorem of quantum mechanics), are geometrically defined for any configuration of points in the Euclidean three space. The conjecture $(C1)$ is proved for $n = 3, 4$ and for general n only for some special configurations (M. F. Atiyah, M. Eastwood and P. Norbury, D. Đoković).

Here we shall explain some new conjectures for symmetric functions which imply $(C2)$ and $(C3)$ for almost collinear configurations. Computations up to $n = 6$ are performed with a help of `Maple` and J. Stembridge's package `SF` for symmetric functions. For $n = 4$ we have also verified the conjectures $(C2)$ and $(C3)$ for some infinite families of tetrahedra.

Introduction

In the case of type (A) configurations of points (all but one point are collinear, [6]) Đoković arrived at a matrix $M_{n+1} =$

$$\begin{bmatrix} 1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & & 1 & \lambda_n \\ (-1)^n e_n & (-1)^{n-1} e_{n-1} & \cdots & \cdots & -e_1 & 1 \end{bmatrix}$$

where $0 < \lambda_1 < \lambda_2 \ldots < \lambda_n$ are real numbers and $e_k = e_k(\lambda_1, \ldots, \lambda_n)$, $1 \le k \le n$, is the k–th elementary symmetric function of $\lambda_1, \lambda_2, \ldots, \lambda_n$.
Inequality $[\det M_{n+1}(\lambda_1, \ldots, \lambda_n)]^{n-1} \ge$
$\ge \prod_{k=1}^{n} \det M_n(\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n)$
is equivalent to conjecture (C3) in this case.
For $n = 2$ this inequality takes the form

$$\begin{aligned} &1 + \lambda_2 e_1(\lambda_1, \lambda_2) + \lambda_1 \lambda_2 e_2(\lambda_1, \lambda_2) \ge \\ &\ge (1 + \lambda_2 e_1(\lambda_2))(1 + \lambda_1 e_1(\lambda_1)) \end{aligned}$$

that reduces to $(\lambda_2 - \lambda_1)\lambda_1 \ge 0$, so it is true.
Even for $n = 3$ the inequality is quite messy thanks to non-symmetric character of both sides. Let us start with the case $n = 2$. Take a look at the following inequality

$$1 + X_1(\xi_1 + \xi_2) + X_1 X_2 \xi_1 \xi_2 \ge (1 + X_1 \xi_1)(1 + X_2 \xi_2)$$

which is true if $X_1 \ge X_2 \ge 0$ and $\xi_1, \xi_2 \ge 0$. If we set $X_1 = \xi_1 = \lambda_2$, $X_2 = \xi_2 = \lambda_1$ then we get the original inequality.
Now, let $\xi_1, \ldots, \xi_n, X_1, \ldots, X_n, n \ge 1$ be two sets of commuting indeterminates. For any $l, 1 \le l \le n$ and any sequences $1 \le i_1 \le \cdots \le i_l \le n, 1 \le j_1 \le \cdots \le j_l \le n$ we define polynomials

$$\Psi_J^I = \Psi_{j_1, \ldots, j_l}^{i_1, \ldots, i_l} \in \mathbb{Q}[\xi_1, \ldots, \xi_n, X_1, \ldots, X_n]$$

as follows:

$$\Psi_J^I := \sum_{k=0}^{l} e_k(\xi_{j_1}, \xi_{j_2}, \ldots, \xi_{j_l}) X_{i_1} X_{i_2} \cdots X_{i_k},$$

where $l \ge 1$, $\Psi_\emptyset^\emptyset := 1$ $(j = 0)$ and e_k is the k-th elementary symmetric function.

Polynomials $\Psi_{12\cdots n}^{12\cdots n}$

In particular we have

$$\begin{aligned} \Psi_j^i &= 1 + \xi_j X_i, \\ \Psi_{j_1 j_2}^{i_1 i_2} &= 1 + (\xi_{j_1} + \xi_{j_2}) X_{i_1} + \xi_{j_1} \xi_{j_2} X_{i_1} X_{i_2}, \\ \Psi_{j_1 j_2 j_3}^{i_1 i_2 i_3} &= 1 + (\xi_{j_1} + \xi_{j_2} + \xi_{j_3}) X_{i_1} + \\ &\quad + (\xi_{j_1} \xi_{j_2} + \xi_{j_1} \xi_{j_3} + \xi_{j_2} \xi_{j_3}) X_{i_1} X_{i_2} + \\ &\quad + \xi_{j_1} \xi_{j_2} \xi_{j_3} X_{i_1} X_{i_2} X_{i_3}. \end{aligned}$$

Let us formulate a conjecture which implies the strongest Atiyah–Sutcliffe's conjecture for type (A) configurations of points ([6]).

Conjecture 1. For any $n \ge 1$, let $X_1 \ge X_2 \ge \ldots \ge X_n \ge 0$, $\xi_1, \xi_2, \ldots, \xi_n \ge 0$, be any nonnegative real numbers. Then

$$L_n := \left(\Psi_{12\cdots n}^{12\cdots n}\right)^{n-1} \ge \prod_{k=1}^{n} \Psi_{12\cdots\hat{k}\cdots n}^{12\cdots\hat{k}\cdots n} =: R_n$$

where $12 \cdots \hat{k} \cdots n$ denotes the sequence $12 \cdots (k-1)(k+1) \cdots n$. The equality obviously holds true iff $X_1 = X_2 = \cdots = X_n$.

Let $\widehat{\Psi}_{12\ldots\underline{k}\ldots n}^{12\ldots k\ldots n} := \xi_k(X_2 - X_1) + \Psi_{12\ldots n}^{12\ldots n}$ be obtained from $\Psi_{12\ldots n}^{12\ldots n}$ by replacing only one term $\xi_k X_1$ by $\xi_k X_2$, hence $\widehat{\Psi}_{12\ldots\underline{k}\ldots n}^{12\ldots k\ldots n}$ are still positive. Let $\widehat{L}_n := \prod_{k=2}^{n} \widehat{\Psi}_{12\ldots\underline{k}\ldots n}^{12\ldots k\ldots n}$. Since clearly $L_n \ge \widehat{L}_n$, our stronger conjecture reads as

Conjecture 2. $\widehat{L}_n \ge R_n$ $(n \ge 1)$
with equality iff $X_2 = X_3 = \cdots = X_n$.

More generally, we conjecture that the difference $\widehat{L}_n - R_n$ is a polynomial in the differences $X_2 - X_3$, $X_3 - X_4$, …, $X_{n-1} - X_n$ with coefficients in $\mathbb{Z}_{\ge 0}[X_1, \ldots, X_n, \xi_1, \ldots, \xi_n]$.
We have some monotonicity properties:

Theorem 3. *Let for* $1 \le r \le n$

$$\Delta_r := \partial_{X_r} \Psi_{1\ldots n}^{1\ldots n} \cdot \Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots\widehat{k}\ldots n} - \Psi_{1\ldots n}^{1\ldots n} \cdot \partial_{X_r} \Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots\widehat{k}\ldots n},$$

Then we have the following explicit formulas

(i) *for any* r, $1 \le r < k (\le n)$ *we have* $\Delta_r =$

$$\xi_k \sum_{0 \le i < r \le j \le n} s_{(2^i 1^{j-i-1})}^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots X_j +$$

$$+ \sum_{0 \le i < r, k \le j < n} e_i e_j^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_r \cdots \widehat{X}_k \cdots X_j (X_k - X_{j+1})$$

(ii) *for any* r, $(1 \le) k < r \le n$ *we have* $\Delta_r =$

$$-\left(\sum_{0 \le i < r \le j \le n} s_{(2^i 1^{j-i-1})}^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j + \right.$$

$$\left. + \sum_{0 \le i < k, r \le j < n} e_i^{(k)} e_j X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j (X_{j+1} - X_k) \right)$$

where $s_\lambda^{(k)}$ *denotes the* λ*–th Schur function of* $\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n$ *(*ξ_k *omitted).*

Corollary 4. *Let* $X_1 \ge \cdots \ge X_n \ge 0$, $\xi_1, \ldots, \xi_n \ge 0$ *be as before. Then*

(i) for any r, $1 \le r < k$ $(\le n)$ *we have*

$$\frac{\Psi_{1\ldots n}^{1\ldots n}}{\Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots\widehat{k}\ldots n}} \ge \frac{\Psi_{1\ldots\ r\ \ r+1\ \ldots n}^{1\ldots\ r+1\ r+1\ \ldots n}}{\Psi_{1\ldots\ r\ \ r+1\ \ldots\widehat{k}\ldots n}^{1\ldots\ r+1\ r+1\ \ldots\widehat{k}\ldots n}}$$

(ii) for any r, $(1 \le)$ $k < r$ $(\le n)$ *we have*

$$\frac{\Psi_{1\ldots n}^{1\ldots n}}{\Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots\widehat{k}\ldots n}} \ge \frac{\Psi_{1\ldots\ r-1\ \ r\ \ldots n}^{1\ldots\ r-1\ r-1\ \ldots n}}{\Psi_{1\ldots\widehat{k}\ldots\ r-1\ \ r\ \ldots n}^{1\ldots\widehat{k}\ldots\ r-1\ r-1\ \ldots n}}$$

Conjecture 5. Let $X_1 \ge \cdots \ge X_n \ge 0$, $\xi_1, \ldots \xi_n \ge 0$. Then the following inequality for symmetric functions in $\xi_1, \ldots, \xi_n$

$$\prod_{k=1}^{n-1} \Psi_{1\ 2\ldots k\ \ k+1\ldots n}^{1\ 2\ldots k\ \ k\ \ldots n} \ge \prod_{k=1}^{n} \Psi_{1\ 2\ldots\widehat{k}\ldots n}^{1\ 2\ \ldots\ n-1}$$

holds true. (Implies the strongest (C3) for type (A) configurations; checked up to $n = 5$ by using MAPLE and symmetric function package of J. Stembridge.)

Lemma 6. *For any* k, $(1 \le k \le n)$, *we have*

$$\Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots k\ldots n} = \sum_{j=0}^{n-1} a_j \xi_k^{n-1-j}$$

where

$$\begin{aligned} &a_{n-1} = 1 + X_1 e_1 + X_1 X_2 e_2 + \ldots + X_1 \cdots X_{n-1} e_{n-1}, \\ &a_{n-2} = -X_1 - X_1 X_2 e_1 - \ldots - X_1 \cdots X_{n-1} e_{n-2}, \\ &\cdots \\ &a_0 = (-1)^{n-1} X_1 \cdots X_{n-1}, \\ &\textit{i.e. } a_{n-1-j} = (-1)^j \sum_{i=j}^{n-1} X_1 \cdots X_i e_{i-j} \end{aligned}$$

It follows that the right hand side

$$R_n = \prod_{k=1}^{n} \Psi_{1\ 2\ \ldots\ \widehat{k}\ \ldots\ n}^{1\ 2\ \ldots\ k\ \ldots\ n-1} = \prod_{k=1}^{n} \left(\sum_{j=0}^{n-1} a_j \xi_k^{n-1-j} \right)$$

can be understood as a resultant $R_n = Res(f, g)$ of the following two polynomials

$$f(x) = \sum_{j=0}^{n-1} a_j x^{n-1-j}$$

$$g(x) = \prod_{i=1}^{n} (x - \xi_i) = \sum_{j=0}^{n} (-1)^j e_j x^{n-j}$$

Using Sylvester formula we get a matrix Δ_n with entries

$$\delta_{ij} = \begin{cases} \sum_{k=0}^{i-j} (-1)^{i+j} X_1 \cdots X_{i-k} e_{j-k}, & n-1 \ge i \ge j \ge 0 \\ \sum_{k=0}^{n} (-1)^{j-i} X_1 \cdots X_{j-i+k} e_{j-i+k+1}, & n-1 \ge j > i \ge 0 \end{cases}$$

By elementary operations we get

$$\Delta_n = \det(\delta'_{ij})_{1 \le i,j \le n-1}$$

where

$$\delta'_{ij} = \begin{cases} \sum_{k=j+1}^{n-1} (-1)^{i+j+1} X_1 \cdots X_{k-1} (X_k - X_i) e_{k+1}, & \text{for } i < j \\ \Psi_{1\ 2\ \ldots\ n}^{1\ \ldots\ i\ i\ \ldots\ n-1}, & \text{for } i = j \\ \sum_{k=0}^{j-1} (-1)^{i+j} X_1 \cdots X_{i-k-2} (X_{i-k-1} - X_i) e_k, & \text{for } i > j \end{cases}$$

Corollary 7. *The conjecture 5 is equivalent to a Hadamard type inequality for the (non Hermitian) matrix* $(\delta'_{ij})_{1 \le i,j \le n-1}$, *i.e.*

$$\det(\delta'_{ij}) \le \prod_{i=1}^{n-1} \delta'_{ii}.$$

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