Acyclic, star and injective colouring of H-free graphs

Jan Bok [1], Nikola Jedličková [1], Barnaby Martin [2], Daniël Paulusma [2], Siani Smith [2]

[1] Charles University, Prague, Czech Republic
[2] Durham University, Durham, UK

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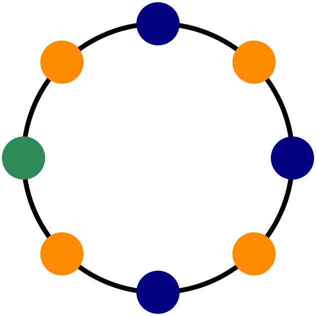
Acyclic colouring

Definition

An acyclic k-colouring of a graph G is a proper k-colouring such that the graph induced by the vertices of any two colour classes is a forest.

In other words, at least 3 different colours are assigned to the vertices of any cycle.



Definition

A star k-colouring of a graph G is a proper k-colouring such that the union of any two colour classes induces a star forest.

In other words any cycle and any path on four vertices is assigned at least 3 different colours.

Injective colouring

Definition

An injective k-colouring of a graph G is a proper k-colouring such that the union of any two colour classes induces a disjoint union of edges and vertices.

In other words any path on 3 vertices is assigned 3 different colours.

For a given graph G and its colourings:

injective k-colourings $\subseteq$ star k-colourings $\subseteq$ acyclic k-colourings.

We distinguish whether the number of colours is a part of the input or not.

Problem:	ACYCLIC (STAR/INJECTIVE) k-COLOURING
Input:	A graph G.
Question:	Does there exist an acyclic (star/injective) colouring of G with at most k colours?

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Definition

A graph G is said to be H-free for some graph H if G has no induced subgraph isomorphic to H.

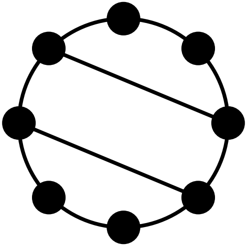

Figure: An example of a triangle-free (or K_3-free) graph.

Classical colouring problem for H-free graphs

- The problem was extensively studied for COLOURING and k-COLOURING problems.
- For k-COLOURING, there is not a complete complexity dichotomy yet.
- For COLOURING, there is a dichotomy.

Theorem (Král' et al. (2001))

The problem COLOURING is polynomial-time solvable for H-free graphs if H is an induced subgraph of P_4 or $P_3 + P_1$ and NP-complete otherwise.

Our results

Based on our joint paper accepted to ESA 2020 conference and its journal version (about to be submitted).

- In a similar way, we combine new and known results to show that each of the three problems are NP-complete for claw-free graphs and C_p-free graphs when k is fixed.
- We then study the remaining open cases to obtain complete complexity dichotomies for each problem when k is fixed and almost complete complexity classifications when k is part of the input.

Theorem

Let H be a graph. For the class of H-free graphs it holds that:

(i) Acyclic Colouring *is polynomial-time solvable if* $H \subseteq_i P_4$ *and NP-complete if H is not a forest or* $H \supseteq_i 19P_1, 3P_3, P_{11}$ *or* $2P_5$*;*

(ii) *For every* $k \geq 3$*,* Acyclic k-Colouring *is polynomial-time solvable if H is a linear forest and NP-complete otherwise.*

Our results - star colouring

Theorem

Let H be a graph. For the class of H-free graphs it holds that:

(i) STAR COLOURING *is polynomial-time solvable if $H \subseteq_i P_4$ and NP-complete for any $H \neq 2P_2$.*

(ii) *For every $k \geq 3$,* STAR k-COLOURING *is polynomial-time solvable if H is a linear forest and NP-complete otherwise.*

Our results - injective colouring

Theorem

Let H be a graph. For the class of H-free graphs it holds that:

(i) INJECTIVE COLOURING *is polynomial-time solvable if* $H \subseteq_i P_4$ *or* $H \subseteq_i P_1 + P_3$ *or* $H \subseteq_i 3P_1 + P_2$ *and NP-complete if H is not a forest or* $2P_2 \subseteq_i H$ *or* $6P_1 \subseteq_i H$.

(ii) *For every* $k \geq 4$, INJECTIVE k-COLOURING *is polynomial-time solvable if H is a linear forest and NP-complete otherwise.*

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(ii) *For every* $k \geq 4$, INJECTIVE k-COLOURING *is polynomial-time solvable if* H *is a linear forest and NP-complete otherwise.*

I will further focus in more detail on acyclic colouring.

Acyclic colouring of H-free graphs

Girth

Lemma

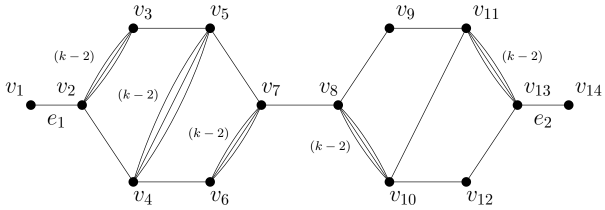
For every $g \geq 3$, ACYCLIC k-COLOURING *is NP-complete for graphs of girth at least g.*

- We reduce from ACYCLIC k-COLOURING, which is known to be NP-complete in general.
- We replace every edge in G by a suitable acyclically k-colourable graph F of high girth with $F + xy$ not being an acyclically k-colourable graph.
- Let G be an instance of ACYCLIC k-COLOURING. We pick an edge $uv \in E(G)$. In $G - uv$ we "glue" F by identifying u with x and y with v. We then prove that G has an acyclic k-colouring if and only if G' has an acyclic k-colouring.
- We repeat the construction for each edge of G to get a graph with high girth.
- Since we do not know how to construct F (there is just an existential result), we have to carefully distinguish between all the possible cases of acyclic colouring of F.

Lemma

For every $k \geq 3$, ACYCLIC k-COLOURING is NP-complete for line graphs.

- We shall make use of a result by Alon and Zaks.
- They have proved that deciding if a graph has an acyclic 3-edge colouring is NP-complete.
- We generalise this to acyclic k-edge colouring.



- Since we can translate edge colouring of a graph into vertex colouring of its line graph and $K_{1,3}$ is one of the forbidden induced subgraphs of the class of line graphs, we get the lemma.

Lemma

ACYCLIC COLOURING *is NP-complete for* $(19P_1, 3P_3, 2P_5, P_{11})$*-free graphs.*

- We reduce from 3-COLOURING with maximum degree 4 which is known to be NP-complete by Garey and Johnson.

A (general) polynomial result

Definition (BB-condition)

We say that a colouring c of a graph G satisfies the *balance biclique condition* (BB-condition) if c uses at least $k+1$ colours to colour G, where k is the order of a largest biclique that is contained in G as a (not necessarily induced) subgraph.

Theorem

Let H be a linear forest, and let π be a colouring property that can be expressed in MSO$_2$, such that every colouring with property π satisfies the BB-condition. Then, for every integer $k \geq 1$, k-$\textsc{Colouring}(\pi)$ *is linear-time solvable for H-free graphs.*

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Long story short...

Corollary

Let H be a linear forest. For every $k \geq 1$, Acyclic k-Colouring, Star k-Colouring *and* Injective k-Colouring *are polynomial-time solvable for H-free graphs.*

Putting it together: k fixed

Theorem

For every $k \geq 3$, Acyclic k-Colouring is polynomial-time solvable if H is a linear forest and NP-complete otherwise.

- First suppose that H contains an induced cycle C_p as its largest induced cycle.
  - If $p = 3$, then we use the result of Coleman and Cai, who proved that for every $k \geq 3$, Acyclic k-Colouring is NP-complete for bipartite graphs.
  - If $p \geq 4$, set $g = p + 1$ and use the lemma on girth.
- Now assume H has no cycle so H is a forest. If H has a vertex of degree at least 3, then H has an induced $K_{1,3}$. As every line graph is $K_{1,3}$-free, we can use the lemma on claws.
- Otherwise H is a linear forest and we use the general polynomial result.

Putting it together: k part of the input

Theorem

For the class of H-free graphs it holds that ACYCLIC COLOURING *is polynomial-time solvable if* $H \subseteq_i P_4$ *and NP-complete if H is not a forest or* $H \supseteq_i 19P_1, 3P_3, 2P_5$ *or* P_{11}

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- If $H \subseteq_i P_4$, then we use the result of Lyons that states that ACYCLIC COLOURING is polynomial-time solvable for P_4-free graphs (cographs).

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- If $H \supseteq_i 19P_1, 3P_3, 2P_5$ or P_{11}, then we use the lemma on linear forests.

Open problems

Open problems

There are still many gaps to fill.

- Determine the complexity of Acyclic Colouring for H-free graphs where H is a linear forest with $19P_1$, $3P_3$, P_{11}, or $2P_5$ not being an induced subgraph of H.
- For every $g \geq 4$, determine the complexity of Injective Colouring and Injective k-Colouring ($k \geq 4$) for graphs of girth at least g.
- For every $g \geq 4$, determine the complexity of Star k-Colouring ($k \geq 4$) for graphs of girth at least g.
- Determine the complexity of Star Colouring for $2P_2$-free graphs.
- Determine the complexity of Star Colouring for split graphs.

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We initiated a systematic complexity study and similar to the study of COLOURING we use the class of H-free graphs.

1. We give almost complete classifications for the computational complexity of ACYCLIC COLOURING, STAR COLOURING and INJECTIVE COLOURING for H-free graphs.
2. If the number of colours k is fixed, that is, not part of the input, we give full complexity classifications for each of the three problems for H-free graphs.

We conclude that for fixed k the three problems behave in the same way, but this is no longer true if k is part of the input.

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Thank you!