List homomorphism problems for signed graphs

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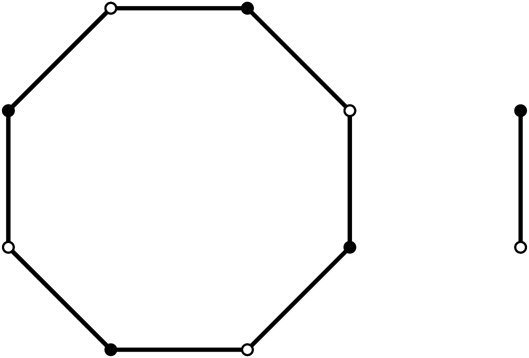
Graph homomorphism

Definition

A *graph homomorphism* between two graphs G and H is a mapping $f : V(G) \to V(H)$ such that for every edge $uv \in E(G)$, $f(u)f(v) \in E(H)$.

- That means that graph homomorphisms are adjacency-preserving mappings between the vertex sets of two graphs.



Signed graphs

Signed graphs were first studied by König and Harary and an intensive research has begun with several papers of Zaslavsky in the 1980s.

Definition (Signed graph)

A *signed graph* is a graph G, with possible loops and multiple edges (at most two loops per vertex and at most two edges between a pair of vertices), together with a mapping $\sigma : E(G) \to \{+, -\}$, assigning a sign ($+$ or $-$) to each edge of G, so that different loops at a vertex have different signs, and similarly for multiple edges between the same two vertices. We denote such a signed graph by (G, σ), and call G its *underlying graph* and σ its *signature*.

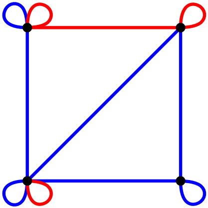


Figure: An example of a signed graph.

Switching operation

Definition (Switching and switching equivalence)

The *switching operation* can be applied to any vertex of a signed graph. It results in multiplying signs of all its incident edges by -1. We further say that two graphs are *switching-equivalent* if we can obtain one from the other by a sequence of switchings.

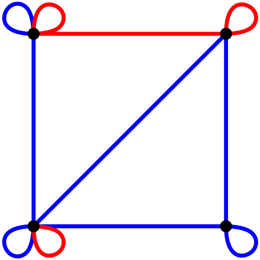


Figure: An example of switching-equivalent graphs.

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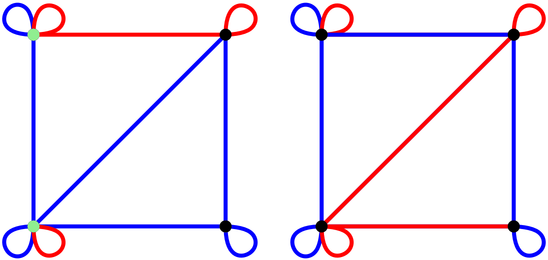

Figure: An example of switching-equivalent graphs.

Definition (Negative and positive cycles)

The sign of a cycle is the product of the signs of its edges. We distinguish between a *negative* cycle and a *positive* cycle.

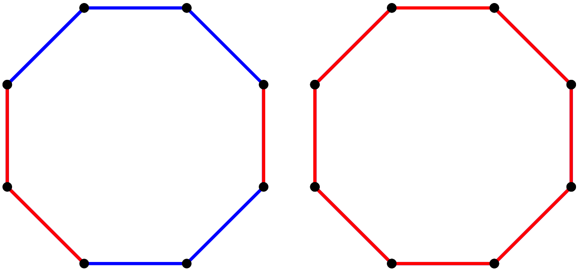

Figure: A negative cycle on the left and a positive cycle on the right.

Balancedness

Definition (Balanced and anti-balanced graph)

We say that a signed graph is *balanced* if every cycle in the graph is positive and *anti-balanced* if every cycle in the graph is negative.

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- It is known that a signed graph is balanced if and only if it is switching-equivalent to a signed graph with all edges positive. Analogously for anti-balanced graphs.

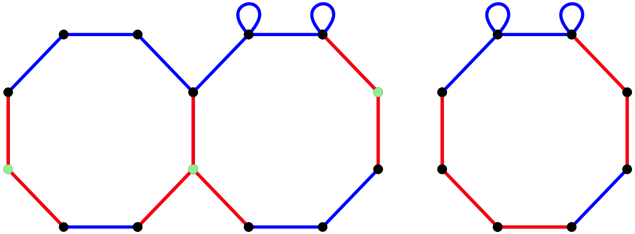

Figure: A balanced graph on the left. A non-balanced graph on the right.

Homomorphism of signed graphs

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Definition

We say that a mapping $f : V(G) \to V(H)$ is a *homomorphism* of the signed graph (G, σ) to the signed graph (H, π), written as $f : (G, \sigma) \to (H, \pi)$, if there exists a signed graph (G, σ'), switching equivalent to (G, σ), such that whenever uv is a positive edge in (G, σ'), then (H, π) contains a positive edge joining $f(u)$ and $f(v)$, and whenever uv is a negative edge in (G, σ'), then (H, π) contains a negative edge joining $f(u)$ and $f(v)$.

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- Note that without the possibility of switching we get homomorphisms of 2-edge-coloured graphs.

Problem

Let (H, π) be a fixed signed graph. The problem $\textsc{S-Hom}((H, \pi))$ is defined as follows:

INPUT:	A signed graph (G, σ).
QUESTION:	Is there a homomorphism of (G, σ) to (H, π)?

Signed core and dichotomy for $\textsc{S-Hom}((H, \pi))$

One of the most important results is settling the dichotomy of $\textsc{S-Hom}((H, \pi))$ by Brewster and Siggers, conjectured by Brewster, Foucaud, Hell and Naserasr.

To state the result of Brewster and Siggers, we give a definition of *s-core*.

Definition (Signed core)

A signed graph (G, σ) is a signed core (or *s-core*) if every homomorphism $f : (G, \sigma) \to (G, \sigma)$ is a bijection.

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Theorem (Brewster, Siggers, published in Discrete Mathematics, 2018)

The problem $\text{S-Hom}((H, \pi))$ is polynomial if (H, π) has an s-core with at most 2 edges and NP-complete otherwise.

List variant of the problem

Problem

Let (H, π) be a fixed signed graph. The List-S-Hom((H, π)) problem is defined as follows:

Input: A signed graph (G, σ) with lists $L(v) \subseteq V(H)$ for every $v \in V(G)$.

Question: Does there exist a homomorphism f from (G, σ) to (H, π) such that $f(v) \in L(v)$ for every $v \in V(G)$?

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- It is clear that every NP-complete case of $\textsc{S-Hom}((H, \pi))$ remains NP-complete for $\textsc{List-S-Hom}((H, \pi))$ so we can focus on signed graphs (H, π) whose s-cores have at most two edges.

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There are, however, many complex signed graphs with this property.

Our goals, results and an outline of the talk

The dichotomy conjecture for CSP was proved by Bulatov and Zhuk. Our problem can be formulated as CSP and therefore, we know that there is a dichotomy in terms of complexity. However, here we are interested in graph-theoretic classifications instead of algebraic ones.

Our goals, results and an outline of the talk

The dichotomy conjecture for CSP was proved by Bulatov and Zhuk. Our problem can be formulated as CSP and therefore, we know that there is a dichotomy in terms of complexity. However, here we are interested in graph-theoretic classifications instead of algebraic ones.

We have a full dichotomy for the following cases of target graphs (H, π):

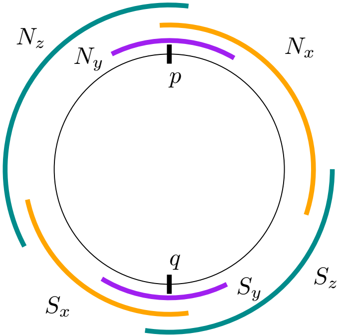
1. there are no bicoloured edges in (H, π),
2. the graph H is an irreflexive tree, and
3. the graph H is a reflexive tree.

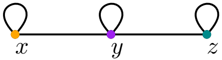
No bicoloured edges

Definition

Let C be a circle with two specified points p and q on C. A *bi-arc* is an ordered pair of arcs (N, S) on C such that N contains p but not q, and S contains q but not p. A graph H is a *bi-arc* graph if there exists a family of bi-arcs $\{(N_x, S_x) : x \in V(H)\}$ such that, for every $x, y \in V(H)$, not necessarily distinct, the following holds:

- if x and y are adjacent, then neither N_x intersects S_y nor N_y intersects S_x;
- if x and y are not adjacent, then N_x intersects S_y and N_y intersects S_x.





No bicoloured edges: a dichotomy

Theorem

Suppose (H, π) is a connected signed graph without bicoloured edges. If the underlying graph H is a bi-arc graph, and (H, π) is balanced or anti-balanced, then the problem $\textsc{List-S-Hom}((H, \pi))$ *is polynomial-time solvable. Otherwise, the problem is NP-complete.*

Irreflexive trees

Irreflexive trees: NP-complete cases I

Lemma

If (H, π) contains the forbidden induced subgraph F_1 (see below) in the underlying graph, then $\textsc{List-S-Hom}((H, \pi))$ *is NP-complete.*

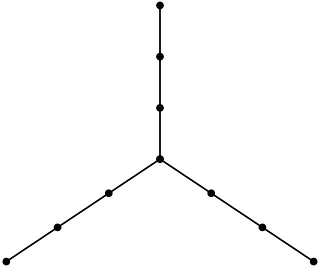


Figure: The forbidden induced subgraph F_1 in the underlying graph.

Irreflexive trees: NP-complete cases II

Lemma

If (H, π) contains one of the forbidden induced subgraphs from the figure below, then $\textsc{List-S-Hom}((H, \pi))$ *is NP-complete.*

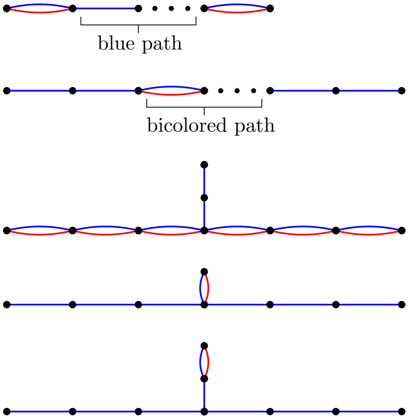


Figure: Forbidden induced subgraphs.

Irreflexive trees: structure of polynomial cases

If H is an irreflexive 2-caterpillar with respect to the spine $P = v_1 \ldots v_k$, the bicoloured edges of (H, π) form a connected subgraph, and there exists an integer d, with $1 \leq d \leq k$, such that:

- all edges on the path $v_1 v_2 \ldots v_d$ are bicoloured, and all edges on the path $v_d v_{d+1} \ldots v_k$ are blue,
- the edges of all subtrees rooted at $v_1, v_2, \ldots, v_{d-1}$ are bicoloured, except possibly edges incident to leaves, and
- the edges of all subtrees rooted at $v_{d+1}, \ldots, v_k$ are all blue,

then we call (H, π) a *good* 2-*catterpillar* and v_d is the *dividing vertex*.

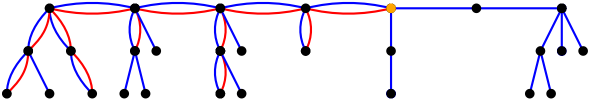
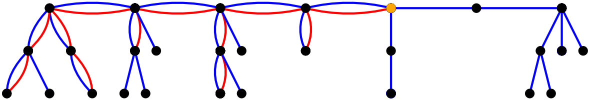


Figure: An example of an irreflexive good 2-catterpillar.

Lemma

Let (H, π) be an irreflexive tree. If (H, π) does not contain any of the forbidden induced subgraphs from the previous figures, then (H, π) is a good 2-catterpillar.



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Irreflexive trees: dichotomy

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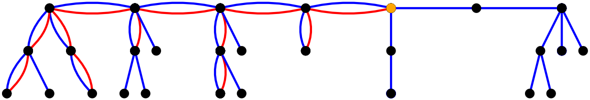

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Theorem

Let (H, π) be an irreflexive tree. If (H, π) is a good 2-catterpillar, then the problem $\textsc{List-S-Hom}((H, \pi))$ *is polynomial. Otherwise it is NP-complete.*

Reflexive trees

Reflexive trees: NP-complete cases I

Lemma

If (H, π) contains the forbidden induced subgraph F_2 (see below) in the underlying graph, then $\textsc{List-S-Hom}((H, \pi))$ *is NP-complete.*

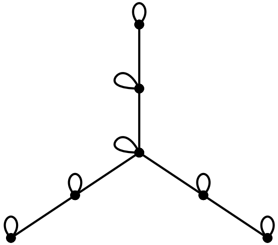


Figure: The forbidden induced subgraph F_2 in the underlying graph.

Reflexive trees: NP-complete cases II

Lemma

If (H, π) *contains one of the forbidden induced subgraphs from the figure below, then* $\textsc{List-S-Hom}((H,\pi))$ *is NP-complete.*

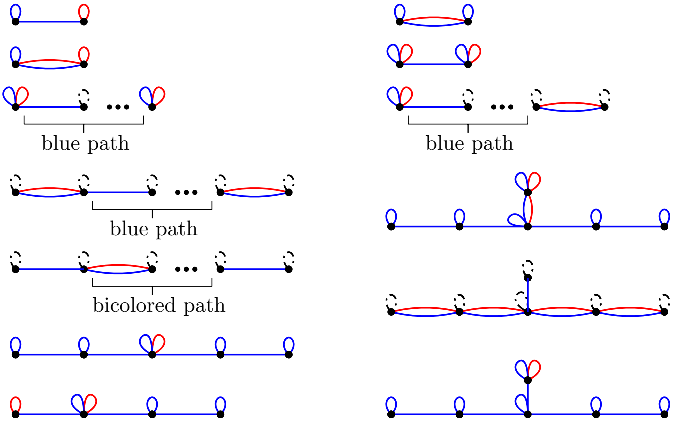


Figure: Forbidden induced subgraphs. Dashed loops can be positive, negative or bicoloured.

Reflexive trees: structure of polynomial cases

We say that (H, π) is a *good caterpillar with respect to the spine* $v_1 \dots v_k$ if H is a catterpillar, if the bicoloured edges of (H, π) form a connected subgraph, the unicoloured non-loop edges all have the same colour c, and there exists an integer d, with $1 \leq d \leq k$, such that

- all edges on the path $v_1 v_2 \dots v_d$ are bicoloured, and all edges on the path $v_d v_{d+1} \dots v_k$ are unicoloured with colour c,
- all loops at the vertices $v_1, \dots, v_{d-1}$ and all non-loop edges of the subtrees rooted at these vertices are bicoloured,
- all loops at the vertices $v_{d+1}, \dots, v_k$ and all edges and loops of the subtrees rooted at these vertices are unicoloured with colour c,
- if v_d has a bicoloured loop, then all children of v_d with bicoloured loops are adjacent to v_d by bicoloured edges,
- if v_d has a unicoloured loop of colour c, then all children of v_d have unicoloured loops of colour c, and are adjacent to v_d by unicoloured edges,
- if $d < k$, then the loops of all children of v_d adjacent to v_d by unicoloured edges also have colour c.

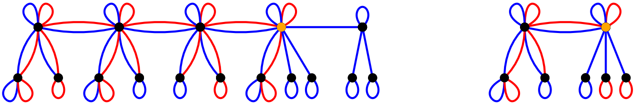
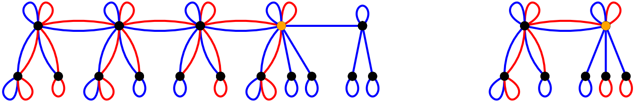


Figure: Examples of good catterpillars.

Reflexive trees: dichotomy

Lemma

Let (H, π) be a reflexive tree. If (H, π) does not contain any of the forbidden induced subgraphs from the previous figures, then (H, π) is a good catterpillar.



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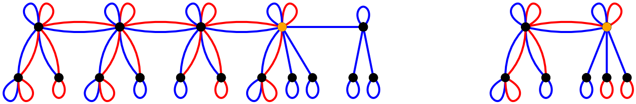

Figure: Examples of good catterpillars.

Theorem

Let (H, π) be a reflexive tree. If (H, π) is a good catterpillar, then the problem $\textsc{List-S-Hom}((H, \pi))$ *is polynomial and NP-complete otherwise.*

Conclusion

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In the paper, we proved a full complexity dichotomy and described polynomial instances for the following cases of targets (H, π):

- there are no bicoloured edges in (H, π),
- the graph H is an irreflexive tree, and
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https://arxiv.org/abs/2005.05547

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Thank you for attention!