Constructing partial geometries with prescribed automorphism groups*

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Introduction

A partial geometry $pg(s, t, \alpha)$ is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$ such that:

- every line is incident with $s + 1$ points,
- every point is incident with $t + 1$ lines,
- every pair of points is incident with at most one line,
- for every non-incident point-line pair (P, ℓ), there are exactly α points on ℓ collinear with P.

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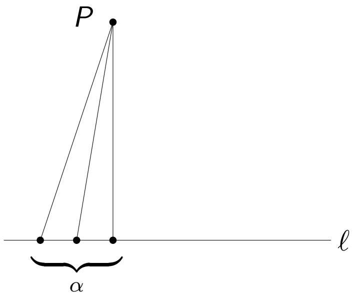

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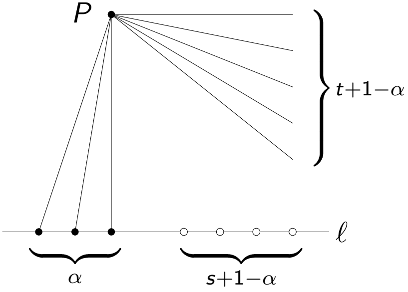

$$\alpha \le \min\{s+1, t+1\}$$

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The dual of a $pg(s,t,\alpha)$ is a $pg(t,s,\alpha)$. In the sequel we will always assume $\alpha < s \leq t$.

Necessary existence conditions

Counting arguments in a $pg(s,t,\alpha)$ yield:

$$v := |\mathcal{P}| = (s+1)\frac{(st+\alpha)}{\alpha}, \qquad b := |\mathcal{L}| = (t+1)\frac{(st+\alpha)}{\alpha}.$$

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The point graph of a $pg(s, t, \alpha)$ is strongly regular with parameters

$$SRG(v,\ s(t+1),\ s-1+t(\alpha-1),\ \alpha(t+1)).$$

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From this we get necessary conditions on the parameters:

$\alpha \mid (s+1)st$ and $(t+1)st$ (integrality of v and b)

$\alpha(s+t+1-\alpha) \mid st(s+1)(t+1)$ (multiplicities of eigenvalues of SRG)

$(s+1-2\alpha)t \le (s-1)(s+1-\alpha)^2$ (Krein inequalities for SRG)

Admissible parameters of small $pg(s,t,\alpha)$ ($v \leq 100$)

Part.geom.	Npg	Point gr.	Nsrg	Line gr.	Nsrg
$pg(2,2,1)$	1	$(15,6,1,3)$	1	$(15,6,1,3)$	1
$pg(2,4,1)$	1	$(27,10,1,5)$	1	$(45,12,3,3)$	78
$pg(3,4,2)$	0	$(28,15,6,10)$	4	$(35,16,6,8)$	3854
$pg(3,3,1)$	2	$(40,12,2,4)$	28	$(40,12,2,4)$	28
$pg(4,6,3)$	2	$(45,28,15,21)$	1	$(63,30,13,15)$	$+$
$pg(3,5,1)$	1	$(64,18,2,6)$	167	$(96,20,4,4)$	$+$
$pg(5,8,4)$	0	$(66,45,28,36)$	1	$(99,48,22,24)$	$+$
$pg(6,6,4)$	?	$(70,42,23,28)$	$+$	$(70,42,23,28)$	$+$
$pg(4,7,2)$	0	$(75,32,10,16)$	0	$(120,35,10,10)$	?
$pg(3,6,1)$	0	$(76,21,2,7)$	0	$(133,24,5,4)$	?
$pg(5,5,2)$	≥ 1	$(81,30,9,12)$	$+$	$(81,30,9,12)$	$+$
$pg(4,4,1)$	1	$(85,20,3,5)$	$+$	$(85,20,3,5)$	$+$
$pg(6,10,5)$	?	$(91,66,45,55)$	1	$(143,70,33,35)$	$+$
$pg(4,9,2)$	0	$(95,40,12,20)$	0	$(190,45,12,10)$	?
$pg(5,6,2)$	?	$(96,35,10,14)$	?	$(112,36,10,12)$	?
$pg(5,9,3)$	0	$(96,50,22,30)$	0	$(160,54,18,18)$	?

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J. H. van Lint, A. Schrijver, *Construction of strongly regular graphs, two-weight codes and partial geometries by finite fields*, Combinatorica **1** (1981), no. 1, 63–73. $\exists$ $pg(5, 5, 2)$

Partial geometries with prescribed automorphism groups

An automorphism of a partial geometry $\mathcal{G}$ is a permutation of the points mapping lines onto lines. The full automorphism group is denoted $\mathrm{Aut}(\mathcal{G})$. Any subgroup $G \leq \mathrm{Aut}(\mathcal{G})$ is called an automorphism group of $\mathcal{G}$.

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Construction method:

1. Choose a permutation group G on the set of points $\mathcal{P}$.
2. Compute the orbits of G on k-element subsets of $\mathcal{P}$.
3. Select orbits comprising lines of the partial geometry.

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B. McKay, A. Piperno, *nauty and Traces*, `https://pallini.di.uniroma1.it`

GAP – Groups, Algorithms, and Programming, `http://www.gap-system.org`

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Orbits by size:

$\lvert\mathcal{O}\rvert$	#
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54	5 670
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27	75
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9	1

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(orbits of sizes 54, 36, 27, 18, 9:) $5\,891 \approx 5.9 \cdot 10^3$

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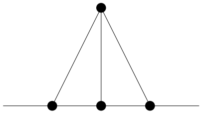
1. Lines of a partial linear space intersect in at most one point.
2. A partial geometry with $\alpha = 2$ does not contain the configuration:



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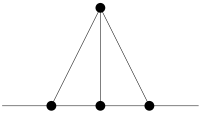

An orbit $\mathcal{O}$ is called good if $|X \cap Y| \leq 1$ for all $X, Y \in \mathcal{O}$, $X \neq Y$ and it does not contain the configuration above.

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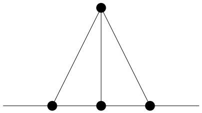

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How to generate the orbits?

V. Krčadinac, R. Vlahović Kruc, *Quasi-symmetric designs on* 56 *points*, Adv. Math. Commun. doi:10.3934/amc.2020086

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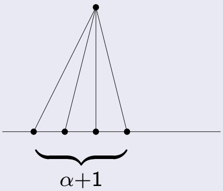
Theorem.

A partial linear space of order (s,t) is a $pg(s,t,\alpha)$ if and only if it does not contain the configuration:



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We want to select a subset of the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_n$ such that $\bigcup_{i=1}^{n} \mathcal{O}_i$ is the set of lines of a $pg(5,5,2)$. Necessary: $\sum_{i=1}^{n} |\mathcal{O}_i| = 81$.

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Define the compatibility graph with the 181 orbits as vertices and their sizes as weights. Orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ are joined by an edge if $|X \cap Y| \leq 1$ for all $X \in O_1$, $Y \in \mathcal{O}_2$ and $\mathcal{O}_1 \cup \mathcal{O}_2$ does not contain the forbidden configuration.

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Search for cliques of weight $b = 81$ in the compatibility graph.

S. Niskanen, P. R. J. Östergård, *Cliquer user's guide, version 1.0*, Communications Laboratory, Helsinki University of Technology, Espoo, Finland, Tech. Rep. T48, 2003.

Example: $pg(5,5,2)$, $v = b = 81$, $k = 6$, $|G| = 108$, orbits: 181

The compatibility graph has 181 vertices and 528 edges (density 3.2 %).

There are 384 cliques of weight 81, all of them correspond to $pg(5,5,2)$.

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Using `nauty`, one finds that there are **two** non-isomorphic $pg(5,5,2)$.

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$|\mathrm{Aut}(\mathcal{G}_2)| = 972$, a new partial geometry $pg(5,5,2)$!

Partial geometries with prescribed automorphism groups

Example: $pg(5,5,2)$, $v = b = 81$, $k = 6$, $|G| = 108$, orbits: 181

The compatibility graph has 181 vertices and 528 edges (density 3.2 %).

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V. Krčadinac, *A new partial geometry* $pg(5,5,2)$, 16 September 2020.
https://arxiv.org/abs/2009.07946

We give a computer-free description of the new $pg(5,5,2)$ using a four-dimensional vector space over $GF(3)$, by changing some lines of the geometry of van Lint and Schrijver.

A general construction of partial geometries

Let $\mathcal{R}$ be a projective plane of order q. A subset $\mathcal{A}$ of points is called a d-arc if d is the greatest number of collinear points in $\mathcal{A}$.

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The number of points in a d-arc is $|\mathcal{A}| \le dq - q + d$. Equality holds if and only if every line is either disjoint from $\mathcal{A}$, or intersects $\mathcal{A}$ in exactly d points. In this case $\mathcal{A}$ is called a maximal arc (for $d = 2$ a hyperoval).

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Theorem (J. A. Thas, W. D. Wallis, 1973).

Let $\mathcal{A}$ be a maximal d-arc in a projective plane of order q. The set of d-secants of $\mathcal{A}$ as POINTS and the set of points not in $\mathcal{A}$ as LINES constitute a partial geometry $pg(s, t, \alpha)$ for $s = q(d-1)/d$, $t = q - d$, and $\alpha = (q-d)(d-1)/d$.

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We want $s \leq t$ and hence only consider $d \leq \sqrt{q}$.

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$d = 2$ It contains hyperovals $\rightsquigarrow$ $pg(2,2,1)$ (smallest gen. quadrangle)

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R. Mathon, *The partial geometries* $pg(5,7,3)$, Congr. Numer. **31** (1981), 129–139.

Mathon proved that there are precisely **two** $pg(4,6,3)$ up to isomorphism. The one above has $\mathrm{Aut}(\mathcal{G}) = PGL(2,8) \rtimes \mathbb{Z}_3$ of order 1512. The other one does not come from a hyperoval in $PG(2,8)$ and has $\mathrm{Aut}(\mathcal{G}) = (\mathbb{Z}_3^2 \rtimes Q_8) \rtimes \mathbb{Z}_3$ of order 216. Both geometries can be obtained from a group of order 18 isomorphic to $S_3 \times \mathbb{Z}_3$. This is the "largest common subgroup" of their full automorphism groups.

A general construction of partial geometries

$q = 9$ There are **4** projective planes: $PG(2,9)$, Hall, dual Hall and Hughes.

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$q = 16$ There are 22 known projective planes.

$d = 2$ Hyperovals exist in 18 known planes $\rightsquigarrow pg(8,14,7)$ (93 non-isom.)

T. Penttila, G. F. Royle, M. K. Simpson, *Hyperovals in the known projective planes of order* 16, J. Combin. Des. **4** (1996), no. 1, 59–65.

M. Gezek, V. D. Tonchev, *On partial geometries arising from maximal arcs*, 30 August 2020. https://arxiv.org/abs/2008.13246

A general construction of partial geometries

$\lvert \mathrm{Aut} \rvert$	$\# pg(8,14,7)$	$\lvert \mathrm{Aut} \rvert$	$\# pg(8,14,7)$	$\lvert \mathrm{Aut} \rvert$	$\# pg(8,14,7)$
16320	1	64	8	6	2
320	1	32	3	4	4
144	1	16	59	3	1
112	2	14	2	2	3
80	1	8	5		

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Open problem: is there a $pg(8,14,7)$ not arising from a hyperoval in a plane of order 16 (analogue of Mathon's $pg(4,6,3)$)?

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$q = 16$ There are 22 known projective planes.

$d = 4$ Maximal 4-arcs exist in 18 known planes (not completely classified)

$\rightsquigarrow$ $pg(12,12,9)$ (59 non-isomorphic)

A general construction of partial geometries

$\lvert \mathrm{Aut} \rvert$	$\#pg(12,12,9)$	$\lvert \mathrm{Aut} \rvert$	$\#pg(12,12,9)$
408	1	32	21
144	3	24	12
96	3	16	9
68	1	4	5
48	4		

A general construction of partial geometries

$\|\mathrm{Aut}\|$	$\#pg(12,12,9)$	$\|\mathrm{Aut}\|$	$\#pg(12,12,9)$
408	1	32	21
144	3	24	12
96	3	16	9
68	1	4	5
48	4		

Work in progress. . .

- find $pg(6,6,4)$ or prove that they do not exist
- find $pg(8,14,7)$ not arising from a hyperoval
- find more $pg(12,12,9)$
- . . .

The End

Thanks for your attention!