Metric vs. edge metric dimension of a graph

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Metric vs. edge metric dimension of a graph

-joint work with M. Knor, R. Škrekovski, I.G. Yero and A.T.M. Toshi

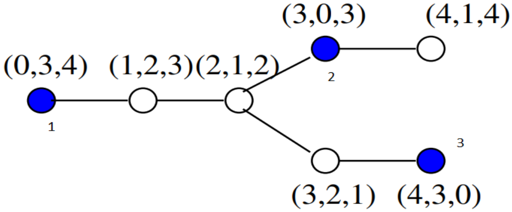
Metric dimension of a graph

G - simple connected graph
$d_G(u,v)$ - distance between vertices u and v in G
• Vertex z *identifies* a pair of vertices $u, v \in V(G)$ if

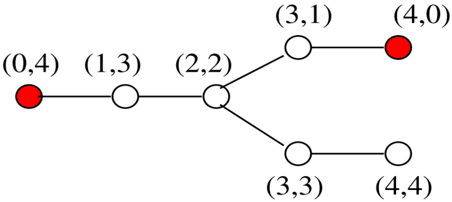
$$d_G(u,z) \neq d_G(v,z).$$

• Set $S \subseteq V(G)$is a *metric generator* for G if every two vertices $u, v \in V(G)$ are identified by a vertex of S
• The metric dimension $\dim(G)$ of G is the cardinality of the smallest metric generator (in this case it is called metric basis) for G.

Metric dimension of a graph



Metric dimension of a graph



Metric dimension of a graph

-origins:

P.J. Slater, Leaves of trees, Congr. Numer. 14 (1975) 549–559.

-application in network theory (location of an intruder), navigation of robots, chemistry, pattern recognition, image processing

-some other variants of metric generators

Question: are there some sets of vertices which uniquely identify all the edges of a graph?

YES - it is the edge metric generator

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Question: are there some sets of vertices which uniquely identify all the edges of a graph?
YES - it is the edge metric generator

Edge metric dimension of a graph

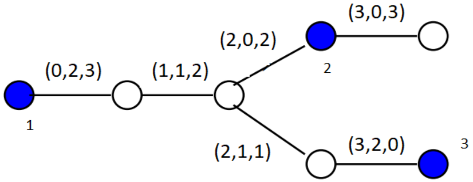
• Vertex z *distinguishes* a pair of edges $e, f \in E(G)$ if

$$d_G(e,z) \neq d_G(f,z) \qquad (d_G(e,z) = d_G(uv,z) = \min\{d_G(u,z), d_G(v,z)\}).$$

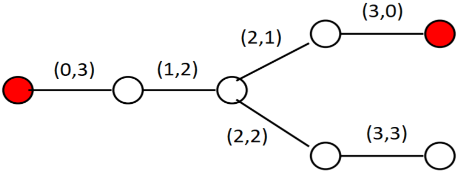
• Set $S \subseteq V(G)$is an *edge metric generator* for G if every two edges of G are distinguished by a vertex of S,

• The edge metric dimension $\mathrm{edim}(G)$ of G is the cardinality of the smallest edge metric generator (in this case it is called edge metric basis) for G.

Edge metric dimension of a graph



Edge metric dimension of a graph



Metric vs. edge metric dimension of a graph

Question: Is there any connection between metric generator and edge metric generator of a graph?

[2] A. Kelenc, N. Tratnik, and I. G. Yero, Uniquely identifying the edges of a graph: the edge metric dimension, Discrete Applied Mathematics, 251 (2018), 204–220

-several families of graphs for which $\dim(G) < \operatorname{edim}(G)$ or $\dim(G) > \operatorname{edim}(G)$ or $\dim(G) = \operatorname{edim}(G)$ were found

-only one for $\dim(G) > \operatorname{edim}(G)$: $C_{4r} \square C_{4t}$

Metric vs. edge metric dimension of a graph

We settle three open problems:

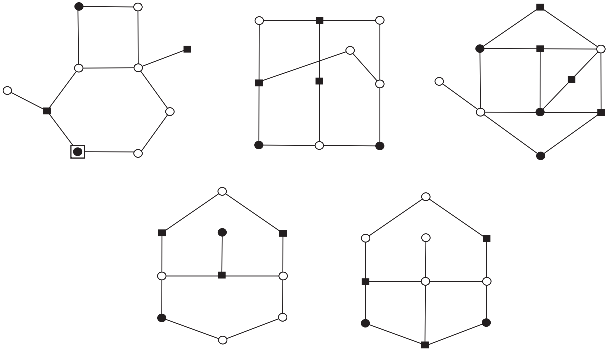
(i) For which integers $r, t, n \geq 1$, with $r, t \leq n - 1$, there exists a graph G of order n with $\dim(G) = r$ and $\mathrm{edim}(G) = t$?

(ii) Is it possible that $\dim(G)$ or $\mathrm{edim}(G)$ would be bounded from above by some constant factor of $\mathrm{edim}(G)$ or $\dim(G)$, respectively?

(iii) Are there any other families of graphs for which $\dim(G) > \mathrm{edim}(G)$?

Metric vs. edge metric dimension of a graph

The smallest possible graphs G for which $\mathrm{edim}(G) < \dim(G)$:



Metric vs. edge metric dimension of a graph

Our results:

Theorem 1 [R. Škrekovski, M. Knor, I.G. Yero, A.T.M. Toshi and S.M.]

Let $k_1, k_2 \geq 2$ and $k_1 \neq k_2$. Then there is an integer n_0 such that for every $n \geq n_0$ there exists a graph on n vertices with $\dim(G) = k_1$ and $\operatorname{edim}(G) = k_2$.

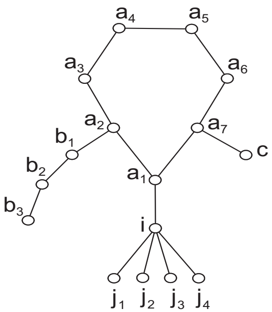
Theorem 2 [R. Škrekovski, M. Knor, I.G. Yero, A.T.M. Toshi and S.M.]

The ratio $\frac{\dim(G)}{\operatorname{edim}(G)}$ is not bounded from above.

Metric vs. edge metric dimension of a graph

Proof of Theorem 1.

Construction of an infinite family of graphs starts with this one:



The graph $G_{7,3,4}$

Metric vs. edge metric dimension of a graph

Proof of Theorem 1.

• Let $n_1 \geq 5$, $n_2 \geq 1$ and $n_3 \geq 2$. Then $\dim(G_{n_1,n_2,n_3}) = n_3$ if n_1 is odd, and $\dim(G_{n_1,n_2,n_3}) = n_3 + 1$ if n_1 is even.

• Let $n_1 \geq 5$, $n_2 \geq 1$ and $n_3 \geq 2$. Then $\mathrm{edim}(G_{n_1,n_2,n_3}) = n_3 + 1$ if n_1 is odd, and $\mathrm{edim}(G_{n_1,n_2,n_3}) = n_3$ if n_1 is even.

Metric vs. edge metric dimension of a graph

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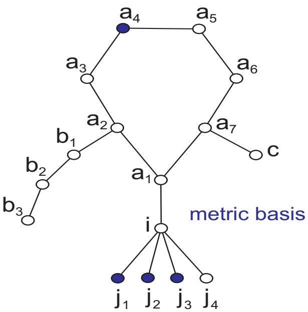

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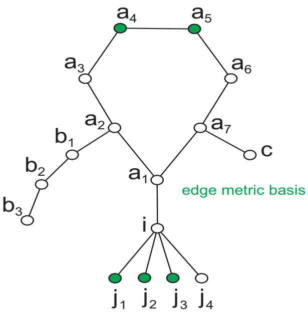
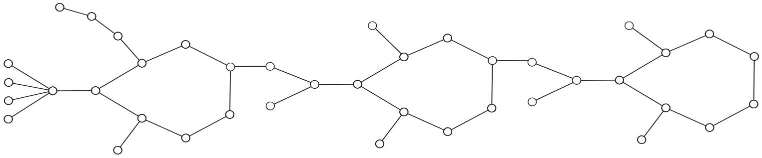


The graph $G_{7,3,4}$

Metric vs. edge metric dimension of a graph

Proof of Theorem 1.
We connect several copies of G_{n_1,n_2,n_3} by adding a few edges:



The graph $L^3_{7,3,4}$

Metric vs. edge metric dimension of a graph

Proof of Theorem 1.

• Let G_1 and G_2 be two graphs which are not paths, such that for any $i \in \{1, 2\}$, the graph G_i contains a metric basis S_i and an edge metric basis T_i satisfying the following conditions.

(1) There is $v_1 \in S_1 \cap T_1$.

(2) There are $v_2, u_2 \in S_2 \cap T_2$ such that $d_{G_2}(u_2, v_2) \geq d_{G_2}(u_2, z)$ for every $z \in V(G_2)$.

Let G be a graph obtained by adding the edge v_1v_2 to the disjoint union of the graphs G_1 and G_2. Then, $\dim(G) = \dim(G_1) + \dim(G_2) - 2$ and $\text{edim}(G) = \text{edim}(G_1) + \text{edim}(G_2) - 2$. Moreover, $S = S_1 \cup S_2 - \{v_1, v_2\}$ is a metric basis of G and $T = T_1 \cup T_2 - \{v_1, v_2\}$ is an edge metric basis of G.

Metric vs. edge metric dimension of a graph

Proof of Theorem 1.

- Let $n_1 \geq 5$, $n_2 \geq 1$, $n_3 \geq 2$ and $\ell \geq 1$. Then the following holds.

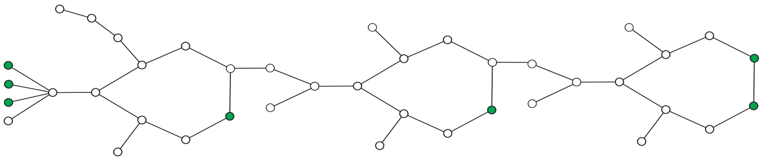
  (1) If n_1 is odd, then $\dim(L^{\ell}_{n_1,n_2,n_3}) = n_3$ and $\operatorname{edim}(L^{\ell}_{n_1,n_2,n_3}) = n_3 + \ell$.

  (2) If n_1 is even, then $\dim(L^{\ell}_{n_1,n_2,n_3}) = n_3 + \ell$ and $\operatorname{edim}(L^{\ell}_{n_1,n_2,n_3}) = n_3$.

Metric vs. edge metric dimension of a graph

Proof of Theorem 1.

Edge metric basis:



The graph $L^3_{7,3,4}$

Metric vs. edge metric dimension of a graph

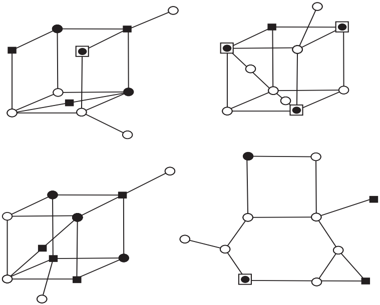
Proof of Theorem 1.

• If $k_1 < k_2$, then $\dim(L^{k_2-k_1}_{5,n_2,k_1}) = k_1$ and $\mathrm{edim}(L^{k_2-k_1}_{5,n_2,k_1}) = k_2$.
Hence, if $n_0 = |V(L^{k_2-k_1}_{5,1,k_1})|$, then for every $n \geq n_0$ the graph $L^{k_2-k_1}_{5,1+n-n_0,k_1}$ has the required properties.

• If $k_1 > k_2$, then $\dim(L^{k_1-k_2}_{6,n_2,k_1}) = k_1$ and $\mathrm{edim}(L^{k_1-k_2}_{6,n_2,k_2}) = k_2$.
Hence, if $n_0 = |V(L^{k_1-k_2}_{6,1,k_2})|$, then for every $n \geq n_0$ the graph $L^{k_1-k_2}_{6,1+n-n_0,k_2}$ has the required properties. ■

Further work

Characterizing the whole class of graphs G for which $\mathrm{edim}(G) < \dim(G)$.



Some graphs G with 11 vertices for which $\dim(G) > \mathrm{edim}(G)$.
The squared vertices form a metric basis and the circled bolded vertices form an edge metric basis.

Further work

Open problems:

- Characterize the class of unicyclic graphs G for which $\mathrm{edim}(G) < \dim(G)$.
- Characterize all the graphs (or maybe only the unicyclic ones) G for which $\mathrm{edim}(G) = \dim(G) - 1$.
- Characterize all the graphs G for which ($\mathrm{edim}(G) = 2$ and $\dim(G) = 3$) or ($\mathrm{edim}(G) = 3$ and $\dim(G) = 4$).
- Find some necessary and/or sufficient conditions for a connected graph G to satisfy that $\mathrm{edim}(G) < \dim(G)$.

End of slides

Thank you!

End of slides

Thank you!

References

[1] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, and D. R. Wood, On the metric dimension of Cartesian products of graphs, *SIAM J. Discrete Math.* **21** (2) (2007) 423–441.

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[3] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combin.* **2** (1976) 191–195.

[4] A. Kelenc, N. Tratnik, and I. G. Yero, Uniquely identifying the edges of a graph: the edge metric dimension, *Discrete Appl. Math.* **251** (2018) 204–220.

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[7] N. Zubrilina, On the edge dimension of a graph, *Discrete Math.* **341** (7) (2018) 2083–2088.