New Class of Binomial Sums and Their Applications

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Abstract

We introduce a notion that we call a "M-sum" and use it to examine the divisibility properties of some binomial sums.

There is a close relationship between M-sum and D-sum. By using D sums, we give recurrence relations for M-sums.

We present three applications of our new sums. The first application is for a known alternating binomial sum studied by Calkin.

The second application is for the following binomial sum S: $\mathbb{Z}^{l+2}_{>=0} \to \mathbb{Z}$,

$$S(2n, m, a_1, a_2, \ldots, a_l) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \prod_{i=1}^{l} \binom{a_i + k}{k} \binom{a_i + 2n - k}{2n - k};$$

which arises as a generalization of a known identity connected with a famous Dixon's formula. We prove that, if m is a positive integer, then $S(2n, m, a_1, a_2, \ldots, a_l)$ is divisible by $\binom{2n}{n}$ and $\binom{a_i+n}{n}$ for all $i = 1, l$.

As a third application, we present an interesting connection between our new sums and one Theorem by Guo, Jouhet and Zeng.

1 Introduction

Let m, n, and k be non-negative integers such that $m \geq 1$. We consider the sum

$$S(n, m) = \sum_{k=0}^{n} \binom{n}{k}^m F(n, k), \tag{1}$$

where $F(n, k)$ is an integer-valued function that depends only on n and k.

The aim is to examine some divisibility properties of sums of the form $S(n, m)$. To do this, we introduce the notion of "M sums".

Definition 1. Let n, j, and t be non-negative integers such that $j \leq \lfloor \frac{n}{2} \rfloor$. Then the M sums for $S(n,m)$ are as follows:

$$M_S(n,j,t) = \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} \binom{n}{j+v}^t F(n, j+v). \tag{2}$$

Obviously, for $m \geq 1$, the equation

$$S(n,m) = M_S(n, 0, m-1) \tag{3}$$

holds.

Hence, we can see the sum $M_S(n,j,m-1)$ as a generalization of the sum $S(n,m)$.

Let n, j, and t be as in Definition 1. We present our main theorem:

Theorem 2.

$$M_S(n,j,t+1) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n, j+u, t).$$

Theorem 2 gives a recursive definition of a M_S sum.

For proving Theorem 2, we use a method of "D sums".

2 Background

Recently, in our previous paper [6], the notion of "D sums" was introduced.

Let n, j, and t be as in Definition 1; and let S be as in Eq. (1).

Then the D sums for $S(n,m)$ are

$$D_S(n,j,t) = \sum_{l=0}^{n-2j} \binom{n-j}{l} \binom{n-j}{j+l} \binom{n}{j+l}^t F(n, j+l). \tag{4}$$

For $m \geq 2$, by Eq. (4), it follows that

$$S(n,m) = D_S(n, 0, m-2). \tag{5}$$

Furthermore, D sums satisfy the following two recurrence relations [6, Thm. 2, p. 2], [8, Eqns. (14) and (15), p. 4]:

$$D_S(n,j,t+1) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n}{j+u} \binom{n-j}{u} D_S(n, j+u, t), \tag{6}$$

$$D_S(n,j,0) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} \binom{n-j-u}{j+u} \sum_{v=0}^{n-2j-2u} \binom{n-2j-2u}{v} F(n, j+u+v). \tag{7}$$

In 1998, Calkin [1, Thm. 1] proved that the alternating binomial sum $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin's result [5, Thm. 1.2, Thm. 1.3, p. 2].

The first application of D sums [6, Section 8] was for proving Calkin's result [1, Thm. 1]. Also, by using D sums, it was proved [6] that $\sum_{k=0}^{2n}\binom{2n}{k}^m|n-k|$ is divisible by $n\binom{2n}{n}$ for all non-negative integers n and all positive integers m.

Recently, by the same method, it was proved [8, Thm. 1] that $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m\binom{2k}{k}\binom{4n-2k}{2n-k}$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m.

Furthermore, it was proved [8, Corollary 4] that $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m C_k C_{2n-k}$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m.

We will show that there is a close relationship between M-sums and D-sums.

3 Proof of Theorem 2

We need one auxiliary result.

We will use a symmetry of binomial coefficients and the well-known binomial identity

$$\binom{a}{b}\binom{b}{c}=\binom{a}{c}\binom{a-c}{b-c}; \tag{8}$$

where a, b, c are non-negative integers such that $a \geq b \geq c$.

Lemma 3. *Let n and j be non-negative integers; and let t be a positive integer. Then the following equation*

$$M_S(n,j,t)=\binom{n}{j}D_S(n,j,t-1). \tag{9}$$

holds.

Proof. By Eq. (2), we know that

$$M_S(n,j,t)=\sum_{v=0}^{n-2j}\binom{n-j}{j}\binom{n-2j}{v}\binom{n}{j+v}^t F(n,j+v). \tag{10}$$

Starting from Eq. (10), we have gradually:

$$
\begin{aligned}
M_S(n,j,t) &= \sum_{v=0}^{n-2j} \binom{n-j}{n-2j}\binom{n-2j}{v}\binom{n}{j+v}^t F(n,j+v) && \text{(by symmetry)}\\
&= \sum_{v=0}^{n-2j} \binom{n-j}{v}\binom{n-j-v}{n-2j-v}\binom{n}{j+v}^t F(n,j+v) && \text{(by Eq. (8))}\\
&= \sum_{v=0}^{n-2j} \binom{n-j}{v}\binom{n-j-v}{j}\binom{n}{j+v}^t F(n,j+v) && \text{(by symmetry)}\\
&= \sum_{v=0}^{n-2j} \binom{n-j}{v}\binom{n-j-v}{j}\binom{n}{j+v}\binom{n}{j+v}^{t-1} F(n,j+v) && \\
&= \sum_{v=0}^{n-2j} \binom{n-j}{v}(\binom{n-j-v}{j}\binom{n}{n-j-v})\binom{n}{j+v}^{t-1} F(n,j+v) && \text{(by symmetry)}\\
&= \sum_{v=0}^{n-2j} \binom{n-j}{v}\binom{n}{j}\binom{n-j}{n-2j-v}\binom{n}{j+v}^{t-1} F(n,j+v) && \text{(by Eq. (8))}\\
&= \binom{n}{j}\sum_{v=0}^{n-2j} \binom{n-j}{v}\binom{n-j}{j+v}\binom{n}{j+v}^{t-1} F(n,j+v) && \text{(by symmetry)}
\end{aligned}
$$

By the definition of D sums (see Eq. (4)) and the last equation above, we conclude that

$$M_S(n,j,t) = \binom{n}{j} D_S(n,j,t-1).$$

This completes the proof of Lemma 3.

Now, we are ready for the proof of main theorem 2. We will use Eqns. (6), (7), and Lemma 3.

By Eq. (6), it follows that

$$D_S(n,j,t+1) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u}\binom{n}{j+u} D_S(n,j+u,t). \tag{11}$$

Let us recall that t in Eq. (11) is a non-negative integer.

By Lemma 3, it follows that

$$\binom{n}{j+u} D_S(n,j+u,t) = M_S(n,j+u,t+1). \tag{12}$$

By Eq. (12), Eq. (11) becomes as follows:

$$D_S(n,j,t+1) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t+1). \tag{13}$$

By multiplication of both sides of Eq. (13) with $\binom{n}{j}$, it follows that

$$\binom{n}{j} D_S(n,j,t+1) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t+1) \tag{14}$$

By Lemma 3, Eq. (14) becomes as follows:

$$M_S(n,j,t+2) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t+1) \tag{15}$$

By setting $t := t+1$, Eq. (15) becomes

$$M_S(n,j,t+1) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t).$$

If t is a positive integer, then Eq. (15) proves Theorem 2.
For $t = 0$, we will use Eq. (7).
By Eq. (7), it follows that

$$\begin{aligned} D_S(n,j,0) &= \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} \binom{n-(j+u)}{j+u} \sum_{v=0}^{n-2(j+u)} \binom{n-2(j+u)}{v} F(n,(j+u)+v) \\ &= \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,0). \end{aligned}$$

By the last equation above, it follows that

$$D_S(n,j,0) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,0). \tag{16}$$

By multiplication of both sides of Eq. (16) with $\binom{n}{j}$, it follows that

$$\binom{n}{j} D_S(n,j,0) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,0). \tag{17}$$

By Lemma 3, we know that

$$M_S(n,j,1) = \binom{n}{j} D_S(n,j,0). \tag{18}$$

By using Eq. (18), Eq. (17) becomes, as follows:

$$M_S(n,j,1) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n, j+u, 0). \tag{19}$$

It is readily verified that Eq. (19) proves Theorem 2 for the case $t = 0$.

Hence, Eqns. (15) and (19) complete the proof of Theorem 2.

□

4 How do we use "M" sums ?

In one particular situation, Theorem 2 implies a simple consequence which is important for us.

Let n be a fixed non-negative integer. Let t_0 be a non-negative integer, and let j be an arbitrary integer in the range $0 \le j \le \lfloor \frac{n}{2} \rfloor$. Suppose that $q = q(n)$ is a positive integer which divides $M_S(n,j,t_0)$ sums for all j in the given range. We want q to be as large as possible. Then it can be shown, by Theorem 2, that q divides $M_S(n,j,t_0+1)$ for all j in the given range.

By induction, it follows that q divides $M_S(n,j,t)$ for all t such that $t \ge t_0$ and for all j in the given range. By Eq. (3), it follows that q divides $S(n,t+1)$ for all t such that $t \ge t_0$.

See also [8, Section 5, p. 7].

5 The first application for the alternating sum

We give a proof of Calkin's result [1, Thm. 1] by using M sums.

Let n be a non-negative integer and let m be a positive integer. Let

$$S_1(2n,m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m .$$

Obviously, the sum $S_1(2n,m)$ is an instance of the sum (1), where $F_1(2n,k) = (-1)^k$.

We use the well-known identity [2, Eq. (1.25), p. 4]

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} = \begin{cases} 0, & \text{if } n > 0; \\ 1, & \text{if } n = 0, \end{cases} \tag{20}$$

where n is a non-negative integer.

By Eq. (20), we conclude that $S_3(2n,1)$ is divisible by $\binom{2n}{n}$.

Furthermore, it is known

$$S_1(2n,2) = (-1)^n \binom{2n}{n}, \qquad \text{(Kummer's formula)}$$
$$S_1(2n,3) = (-1)^n \binom{2n}{n}\binom{3n}{2n}. \qquad \text{(Dixon's formula)}$$

Therefore, it follows that $S_1(2n,m)$ is divisible by $\binom{2n}{n}$ for $1 \le m \le 3$.
Let us calculate $M_{S_1}(2n,j,0)$ sum; where j is a non-negative integer such that $j \le n$.
Without loss of generality, let us assume that n is a positive integer.
By Def. (1), it follows that

$$\begin{aligned} M_{S_1}(2n,j,0) &= \binom{2n-j}{j} \sum_{v=0}^{2n-2j} \binom{2n-2j}{v} \binom{2n}{j+v}^0 F_1(2n,j+v) \\ &= \binom{2n-j}{j} \sum_{v=0}^{2n-2j} \binom{2n-2j}{v} (-1)^{j+v}. \end{aligned}$$

By the last equation above, it follows that

$$M_{S_1}(2n,j,0) = (-1)^j \binom{2n-j}{j} \sum_{v=0}^{2(n-j)} (-1)^v \binom{2(n-j)}{v}. \tag{21}$$

By Eqns. (20) and (21), it follows that

$$M_{S_1}(2n,j,0) = \begin{cases} 0, & \text{if } 0 \le j < n; \\ (-1)^n, & \text{if } j = n. \end{cases} \tag{22}$$

Let us calculate $M_{S_1}(2n,j,1)$ sum. We will use Theorem 2 and Eq. (22).
By setting $t = 0$ in Theorem 2, it follows that

$$M_{S_1}(2n,j,1) = \binom{2n}{j} \sum_{u=0}^{n-j} \binom{2n-j}{u} M_{S_1}(2n,j+u,0). \tag{23}$$

By Eq. (22), it follows that

$$M_{S_1}(2n,j+u,0) = \begin{cases} 0, & \text{if } 0 \le u < n-j; \\ (-1)^n, & \text{if } u = n-j. \end{cases} \tag{24}$$

By Eq. (24), Eq. (23) becomes gradually

$$
\begin{aligned}
M_{S_1}(2n,j,1) &= (-1)^n \binom{2n}{j}\binom{2n-j}{n-j} \\
&= (-1)^n \binom{2n}{2n-j}\binom{2n-j}{n} && \text{(by symmetry)} \\
&= (-1)^n \binom{2n}{n}\binom{n}{n-j} && \text{(by Eq. (8)).}
\end{aligned}
$$

By the lst equation above, it follows that

$$M_{S_1}(2n,j,1) = (-1)^n \binom{2n}{n}\binom{n}{j}. \tag{25}$$

By setting $j=0$ in Eq. (25) and by using Eq. (3), we obtain Kummer's formula.

Furthermore, Eq. (25) suggets setting $q_1(2n) = \binom{2n}{n}$.

By Theorem 2 and induction, it can be shown that $M_{S_1}(2n,j,t)$ is divisible by $\binom{2n}{n}$ for all positive integers t and all integers j such that $0 \le j \le n$.

Let $m \ge 2$. By Eq. (3), we know that

$$S_1(2n,m) = M_{S_1}(2n,0,m-1).$$

Since $m-1 \ge 1$, the sum $M_{S_1}(2n,0,m-1)$ is divisible by $\binom{2n}{n}$. By Eq. (3), $S_1(2n,m)$ is divisible by $\binom{2n}{n}$ for $m \ge 2$.

This proves Calkin's result.

Remark 4. By using M-sums, we also can prove Dixon's formula [4, 7]. Namely, by using Theorem 2, Eq. (25), and the Vandermonde identity, it can be shown that

$$M_{S_1}(2n,j,2) = (-1)^n \binom{2n}{n}\binom{2n}{j}\binom{3n-j}{2n}. \tag{26}$$

By setting $j=0$ in Eq. (26) and by using Eq. (3), we obtain Dixon's formula.

6 Generalization of new sums

We begin with the following definition:

Definition 5. Let m, n, k, a_1, a_2, $\ldots$, a_l be non-negative integers such that $m \ge 1$. We consider the sum:

$$S(n,m,a_1,a_2,\ldots,a_l) = \sum_{k=0}^{n} \binom{n}{k}^m F(n,k,a_1,a_2,\ldots,a_l), \tag{27}$$

where $F(n,k,a_1,a_2,\ldots,a_l)$ is an integer-valued function.

The aim is to examine some divisibility properties of sums of the form $S(n, m, a_1, a_2, \ldots, a_l)$. To do this, we introduce a natural generalization of a notion of M sums.

Definition 6. Let n, j, t, a_1 ,a_2, …, a_l be non-negative integers such that $j \leq \lfloor \frac{n}{2} \rfloor$. Then the M sums for $S(n, m, a_1, a_2, \ldots, a_l)$ are as follows:

$$M_S(n, j, t, a_1, a_2, \ldots, a_l) = \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} \binom{n}{j+v}^t F(n, j+v, a_1, a_2, \ldots, a_l). \quad (28)$$

Obviously, for $m \geq 1$, the equation

$$S(n, m, a_1, a_2, \ldots, a_l) = M_S(n, 0, m-1, a_1, a_2, \ldots, a_l) \quad (29)$$

holds.

The Eq. (29) is similar with Eq. (3).

Let n, j, t, a_1, a_2, …, a_l be as in Definition 6. Then the following theorem is true:

Theorem 7.

$$M_S(n, j, t+1, a_1, a_2, \ldots, a_l) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n, j+u, t, a_1, a_2, \ldots, a_l).$$

The proof of Theorem 7 is similar with the proof of Theorem 2.

7 The second application of new sums

Let us consider the following binomial sum S: $\mathbb{Z}^{l+2}_{>=0} \to \mathbb{Z}$

$$S(2n, m, a_1, a_2, \ldots, a_l) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \prod_{i=1}^{l} \binom{a_i+k}{k} \binom{a_i+2n-k}{2n-k};$$

where l is a positive integer and n, m, a_1, a_2, …, a_l are non-negative integers.

It is known [3, Eq. (6.56), p. 29] that

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{a+k}{k} \binom{a+2n-k}{2n-k} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{a}{k} \binom{a}{2n-k}, \quad (30)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{a}{k} \binom{a}{2n-k} = (-1)^n \binom{2n}{n} \binom{a+n}{2n}, \quad (31)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{a+k}{k} \binom{a+2n-k}{2n-k} = (-1)^n \binom{2n}{n} \binom{a+n}{2n}. \quad (32)$$

Note that Eq. (31) is Dixon's formula.

By using M sums, we can prove the following theorem:

Theorem 8. *If m is a positive integer, then $S(2n, m, a_1, a_2, \ldots, a_l)$ is divisible by $\binom{2n}{n}$ and $\binom{a_i+n}{n}$ for all $i = 1, l$.*

8 The third application of new sums

In [5, Section 4, Theorem 4.1 ,p. 8], Guo, Jouhet, and Zeng presented, among other, a theorem which gave an another proof of Calkin's result [1, Thm. 1]. Their theorem has a direct consequence:

Theorem 9. *Let l be a positive integer greater than 1, and let k, n_1, n_2, ...,n_{l+1} be non-negative integers. Then the sum*

$$\sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^{l} \binom{n_i + n_{i+1}}{n_i + k}$$

is divisible by $\binom{n_1+n_l}{n_1}$, where $n_{l+1} = n_1$.

We assert that the following implication is true:

Theorem 10. *Let l be a fixed positive integer greater than 2, and let k, n_1, n_2, ...,n_{l+1} be non-negative integers.. Let us suppose that the sum*

$$\sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^{l} \binom{n_i + n_{i+1}}{n_i + k}$$

is divisible by $\binom{n_1+n_l}{n_1}$, where $n_{l+1} = n_1$. Then, by using M sums, it can be proved that the following sum

$$\sum_{k=-n_1}^{n_1} (-1)^k \binom{2n_1}{k + n_1}^m \cdot \prod_{i=1}^{l-1} \binom{n_i + n_{i+1}}{n_i + k}$$

is divisible by $\binom{2n_1}{n_1}$ and $\binom{n_1+n_{l-1}}{n_1}$; where m is a positive integer and $n_l = n_1$.

References

[1] N. J. Calkin, Factors of sums of powers of binomial coefficients, *Acta Arith.* **86** (1998), 17–26.

[2] H. W. Gould, *Combinatorial Identities*, published by the author, revised edition, 1972.

[3] H. W. Gould and J. Quaintance, Combinatorial identities: Table I: intermediate techniques for summing finite series, preprint, 2010. Available at https://www.math.wvu.edu/~gould/Vol.4.PDF,

[4] V. J. W. Guo, A simple proof of Dixon's identity, *Discrete Math.* **268** (2003), 309–310.

[5] V. J. W. Guo, F. Jouhet, and J. Zeng, Factors of alternating sums of products of binomial and q-binomial coefficients, *Acta Arith.* **127** (2007), 17–31.

[6] J. Mikić, A method for examining divisibility properties of some binomial sums, *J. Integer Sequences* **21** (2018), Article 18.8.7.

[7] J. Mikić, A Proof of Dixon's Identity, *J. Integer Sequences* **19** (2016), Article 16.5.3.

[8] J. Mikić, On Certain Sums Divisible by the Central Binomial Coefficient, *J. Integer Sequences* **23** (2020), Article 20.1.6.

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