Permutations avoiding a simsun pattern

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joint work with M. Barnabei, F. Bonetti, and N. Castronuovo (Bologna)

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contains the pattern 231, while it avoids the pattern 321.

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contains the pattern 231, while it avoids the pattern 321.

$S_n(\tau)$ is the subset of τ-avoiding permutations in S_n

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contains the simsun pattern 321, while it avoids the simsun pattern 132.

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A permutation π avoids the **simsun pattern** τ if π does not cointain the consecutive pattern τ neither do the the restriction of π to any interval $[k]$. For example, the permutation

$$\pi = 3\,2\,5\,1\,4$$

contains the simsun pattern 321, while it avoids the simsun pattern 132.

$S_n(\tau^S)$ is the subset of all permutations in S_n that avoid the simsun pattern τ

Simsun patterns of legth 3

S. Sundaram (1994)

$|S_n(321^S)| = E_{n+1}$, the $(n+1)$-th Euler number
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Our contribution

Enumeration of $S_n(132^S, \Sigma)$ and $S_n(213^S, \Sigma)$ for every $\Sigma \subseteq S_3$

Simsun patterns of legth 3

If $\rho' = \text{rev}(\rho)$ and $\Sigma' = \{\text{rev}(\sigma) | \sigma \in \Sigma\}$, then

$$|S_n(\rho^S, \Sigma)| = |S_n((\rho')^S, \Sigma')|$$

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$\Rightarrow$ the study of the sets $S_n(321^S, \Sigma)$ $S_n(132^S, \Sigma)$, and $S_n(213^S, \Sigma)$ for every $\Sigma \subseteq S_3$ completes the enumeration of all sets of permutations avoiding a simsun patter of length 3 together with a set of classical patterns $\Sigma \subseteq S_3$

$$|S_n(123^S, \Sigma)| = |S_n(321^S, \Sigma')|$$
$$|S_n(132^S, \Sigma)| = |S_n(231^S, \Sigma')|$$
$$|S_n(213^S, \Sigma)| = |S_n(312^S, \Sigma')|.$$

The simsun pattern 132

Lemma

A permutation π avoids the simsun pattern 132 if and only if each occurrence of 132 in π is part of an occurrence of the pattern 2413.

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$\Rightarrow$ if Σ contains at least one of those 4 patterns, then

$$S_n(132^S, \Sigma) = S_n(132, \Sigma)$$

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$$S_n(132^S, \Sigma) = S_n(132, \Sigma)$$

We will study $S_n(132^S, \Sigma)$ only for $\Sigma \subseteq \{123, 321\}$

The simsun pattern 132

We exploit the bijection between $S_n(132^S)$ and the set of partitions of $\{1, 2, \ldots, n\}$

$$\sigma = 4\,7\,8\,3\,5\,1\,2\,4\,6$$

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$$P(\sigma) = \{\{1, 2, 4, 6\}, \{3, 5\}, \{4, 7, 8\}\}$$

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- If $\Sigma = \{321\}$, each permutation $\sigma \in S_n(132^S, 321)$ is of the form $\sigma = w_1 w_2$ and corresponds to a partition of $\{1, 2, \dots, n\}$ into at most two blocks

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Enumeration of permutations in $S_n(132^S, 321)$

$|S_n(132^S, 321)| = 2^{n-1}$

The set $S_n(132^S, 123)$

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Enumeration of permutations in $S_n(132^S, 123)$

$|S_n(132^S, 123)|$ is the n-th Motzkin number M_n

The set $S_n(132^S, 123)$

$$\sigma = 12\,8\,11\,6\,10\,5\,9\,3\,2\,7\,1\,4$$

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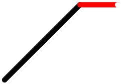

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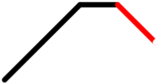

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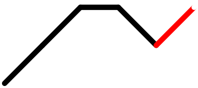

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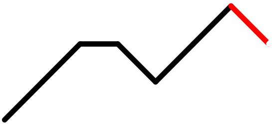

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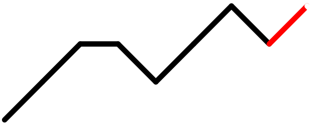

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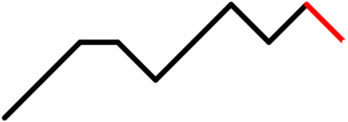

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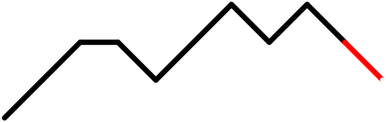

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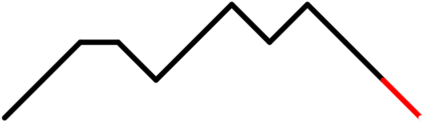

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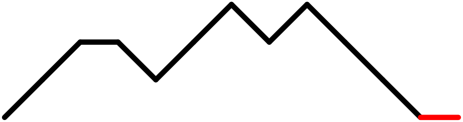

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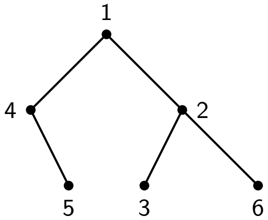
Binary increasing trees

A **binary increasing tree (b.i.t.)** is a plane, rooted, binary tree in which each of the n nodes bears a different positive integer label from 1 to n and labels increase along any descending path.



The bijection ϕ

$$\pi = 1$$

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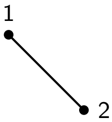
$\pi = 1 \quad 2$



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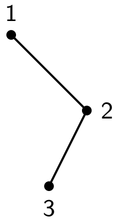
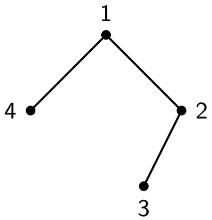


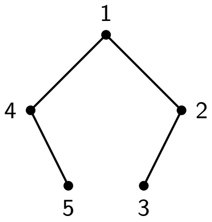
The bijection ϕ

$\pi = 4 \quad 1 \; 3 \; 2$



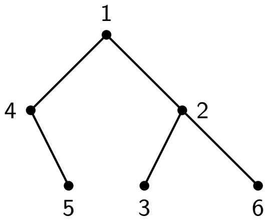
The bijection ϕ

$$\pi \quad = \quad 4 \quad 5 \quad 1 \quad 3 \quad 2$$



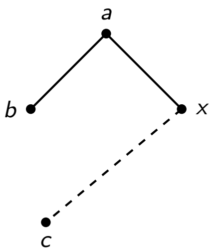
The bijection ϕ

$$\pi \quad = \quad 4 \quad 5 \quad 1 \quad 3 \quad 2 \quad 6$$



$\beth$-trees

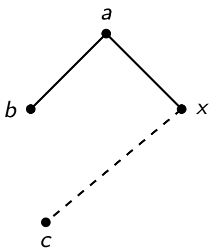
A $\beth$-tree is a binary increasing tree of the following form



where $a < b < c$, $x \leq c$ and where the nodes labelled with x and c are connected by an arbitrarily long sequence of left edges.

ℶ-trees

A ℶ-tree is a binary increasing tree of the following form



where $a < b < c$, $x \leq c$ and where the nodes labelled with x and c are connected by an arbitrarily long sequence of left edges. The vertices labelled x and c may coincide.

The simsun pattern 213

$T_n(\sqsupset)$ is the set of all binary increasing trees with n nodes and not containing any $\sqsupset$-subtree.

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Characterization of permutations avoiding the simsun pattern 213

The map ϕ defined above is a bijection between $S_n(213^S)$ and $T_n(\sqsupset)$, for every $n \in \mathbb{N}$

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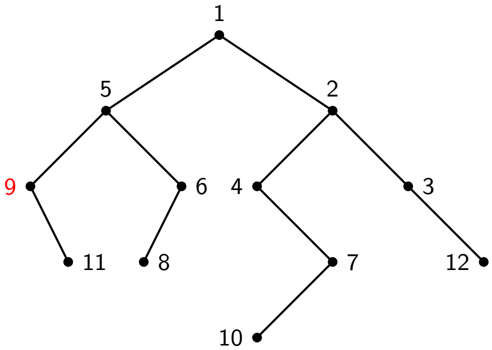
We determine $|T_n(\sqsupset)|$ rather than computing $|S_n(213^S)|$

Enumeration of ℶ-trees

The leftmost label of a b.i.t. is the label of the leftmost node in the left branch starting at the root

$T =$



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$t_{n,\ell}$ = number of b.i.t in $T_n(\beth)$ whose leftmost label is ℓ

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$t_{n,\ell} = |\{T \in T_n(\beth) | \phi^{-1}(T)(1) = \ell\}|$

Enumeration of $\daleth$-trees

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$t_{n,\ell} = |\{T \in T_n(\daleth) | \phi^{-1}(T)(1) = \ell\}|$

Enumeration of $\daleth$-trees according to size and leftmost label

The numbers $t_{n,\ell}$ satisfy the following recurrence

$$t_{n,\ell} = \begin{cases} \sum_{k=1}^{n-1} \sum_{i,j} \binom{\ell-j-2}{i-1} \binom{n-\ell}{k-i} t_{k,i} t_{n-1-k,j} & \text{if } \ell \geq 2 \\ \sum_j t_{n-1,j} & \text{if } \ell = 1 \end{cases} \qquad \forall n \geq 2$$

with initial conditions $t_{0,0} = t_{1,1} = 1$ and $t_{0,i} = t_{1,i} = 0$ if $i > 0$

The simsun pattern 213

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The pattern 3124 contains the classical patterns 123, 213 and 312.
$\Rightarrow$ if Σ contains at least one of those 3 patterns, then

$$S_n(213^S, \Sigma) = S_n(213, \Sigma)$$

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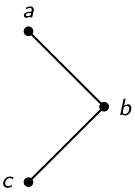
$$S_n(213^S, \Sigma) = S_n(213, \Sigma)$$

We will study the sets $S_n(213^S, \Sigma)$ only for $\Sigma \subseteq \{132, 231, 321\}$.

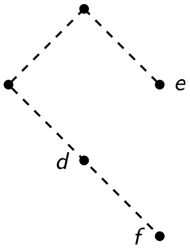
Right combs

A *right comb* is a binary increasing tree in $T_n(\sqsupset)$ that also avoids the following subtrees:

$\alpha =$



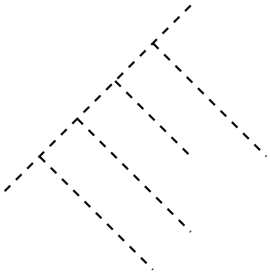
$\beta =$



where solid edges have length 1 and dashed edges have arbitrary length, with $d < e < f$.

Right combs

The shape of a right-comb is:



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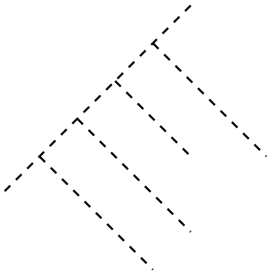


RCT_n is the set of right combs with n nodes and a_n its cardinality.

The set $S_n(213^S, 132)$

Characterization of $\phi(S_n(213^S, 132))$

$\phi(S_n(213^S, 132)) = RCT_n$. Hence

$$|S_n(213^S, 132)| = a_n$$

Recurrence for the sequence a_n

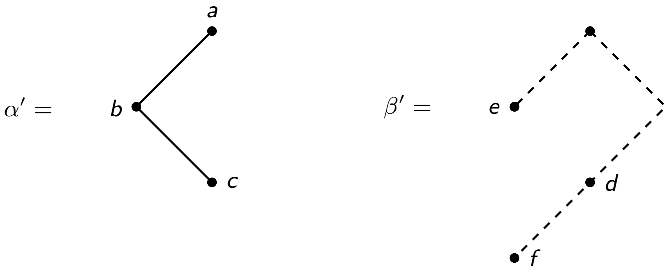
The sequence $\{a_n\}_{n\geq 0}$ satisfies

$$a_n = 2a_{n-1} + \sum_{i=1}^{n-2} a_i \cdot (a_{n-i-1} - a_{n-i-2}) \qquad \forall n \geq 2$$

with $a_0 = a_1 = 1$.

Left combs

A *left comb* is a binary increasing tree in $T_n(\sqsupset)$ that also avoids the following subtrees:



where solid edges have length 1 and dashed edges have arbitrary length, with $d < e < f$.

The set $S_n(213^S, 231)$

LCT_n is the set of left combs with n nodes and b_n its cardinality.

Characterization of $\phi(S_n(213^S, 231))$

$\phi(S_n(213^S, 231)) = LCT_n$. Hence

$$|S_n(213^S, 231)| = b_n$$

Equicardinality result (SURPISING)

The sequences $|LCT_n|$ and $|RCT_n|$ satisfy the same recurrence with the same initial condition. Hence:

$$|S_n(213^S, 231)| = |S_n(213^S, 132)|$$

The set $S_n(213^S, 321)$

$S_n(213^S, 321) \Rightarrow$ we focus on permutation themselves rather than studying the associated b.i.t.

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- $\rho = \sigma x_1 \dots x_i n \dots x_k$, $i > 1$
- $\rho = x_1 n x_2 \dots x_k$ if σ is empty (otherwise we would create an occurence of the consecutive pattern 213).

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Recurrence for $A_{n,k}$

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for all $n \geq 3$ and $k \geq 1$, where

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The sequence $\{|S_n(213^S, 321)|\}_{n\geq 0}$, is not present on the OEIS

The set $S_n(213^S, 132, 231)$

We have that $\phi(S_n(213^S, 132, 231)) = RCT_n \cap LTC_n$

The trees in this set are of the form:

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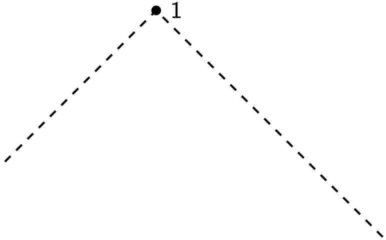

with the left and the right branches possibly empty.

The set $S_n(213^S, 132, 231)$

Enumeration of $S_n(213^S, 132, 231)$

$$|S_n(213^S, 132, 231)| = \begin{cases} 2^{n-2} + 1 & \text{if } n \geq 2 \\ 1 & \text{if } n = 0, 1. \end{cases}$$

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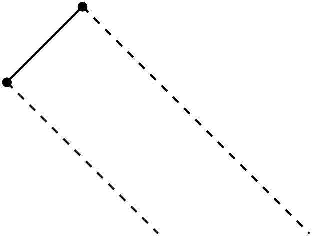
There are two possibilities:

- the right branch is empty
- if it is not empty, the right son of the root must have label 2 in order to avoid the $\sqsupset$-configuration. We choose labels for the other vertices in the left branch as an arbitrary subset of $\{3, \ldots, n\}$

The set $S_n(213^S, 132, 321)$

$\phi(S_n(213^S, 132, 321)) =$ set of right combs with the left branch starting at the root of length at most 1



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$$|S_n(213^S, 132, 321)| = \begin{cases} \frac{n^2-3n+6}{2} & \text{if } n \geq 2 \\ 1 & \text{if } n = 0, 1. \end{cases}$$

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This follows from the fact that labels in the bottommost right branch must be in consecutive order to avoid the β-configuration that is forbidden in a right comb

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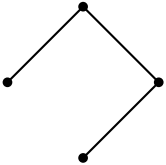
- the tree must be left combs (they belong to $\phi(S_n(213^S, 231))$
- the right branches have length at most 1 (no 321)
- the tree must avoid



(this configuration corresponds to either a 321 or a 213^S)

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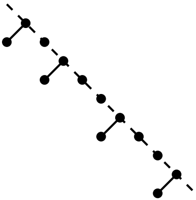

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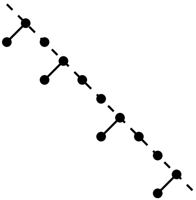


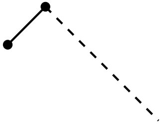
Enumeration of $S_n(213^S, 231, 321)$

$c_n = |S_n(213^S, 231, 321)|$ satisfies

$$c_n = c_{n-1} + c_{n-3} + c_{n-4} + \ldots + c_0.$$

The set $S_n(213^S, 132, 231, 321)$

The b.i.t. associated with a permutation in $S_n(213^S, 132, 231, 321)$ is of the form



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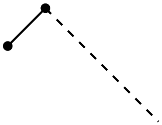

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Enumeration of $S_n(213^S, 132, 231, 321)$

$$|S_n(213^S, 132, 231, 321)| = \begin{cases} n-1 & \text{if } n \geq 3 \\ n & \text{if } n = 1, 2 \\ 1 & \text{if } n = 0. \end{cases}$$

Connection with barred generalized patterns

A **generalized pattern** is a classical pattern τ some of whose consecutive letters may be underlined.

A permutation π **contains the generalized pattern** τ if it contains τ in the classical sense and the elements corresponding to τ_i and τ_{i+1} are consecutive in π if $\tau_i\tau_{i+1}$ is underlined in τ. For example, the permutation

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contains the generalized pattern $\underline{32}1$, while it avoids the generalized pattern $3\underline{21}$.

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Connection with barred generalized patterns

Connection between simsun and barred generalized patterns

With the previous notation,

$$S_n(132^S) = S_n(24\bar{1}3)$$

$$S_n(213^S) = S_n(3\bar{1}\underline{24})$$

Thank you for your attention!