EVALUATING TOPOLOGICAL ORDERING IN DIRECTED ACYCLIC GRAPHS

Suzana Antunović
University of Split, Croatia

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Evaluating topological ordering in directed acyclic graphs

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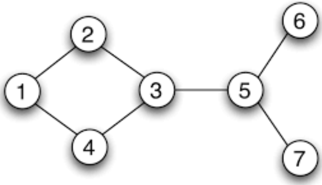
(**) Joint work with D. Vukičević

Graph vs. Network

"Network is a graph with meaning!"

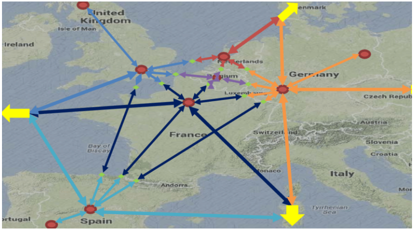


(a) Graph

(b) Network

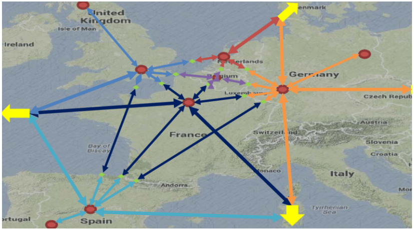
Graphs are everywhere!

Airports



Graphs are everywhere!

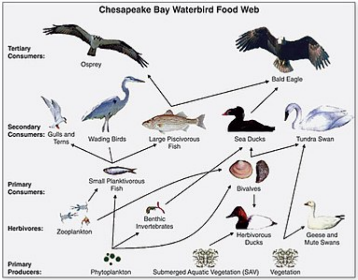
Airports



WWW

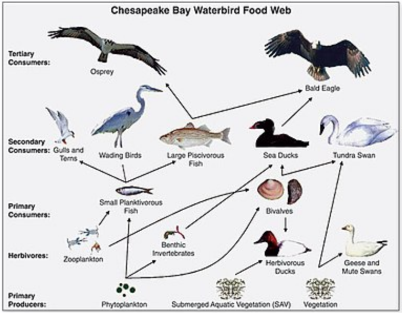
Graphs are everywhere!

Foodchain

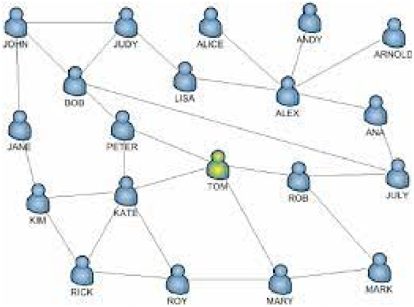


Graphs are everywhere!

Foodchain

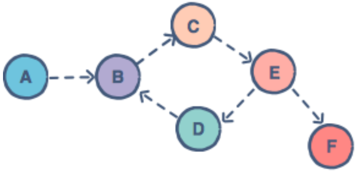


Socialnetwork

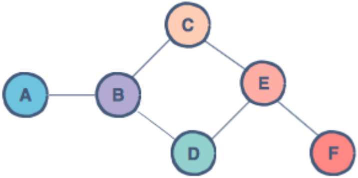


Directed vs. Undirected

Directed

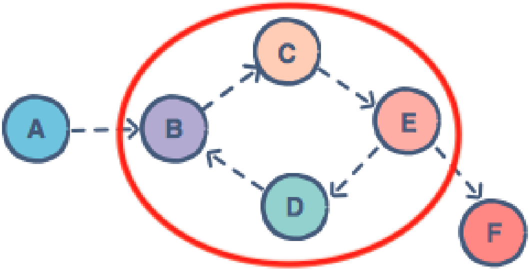


Undirected



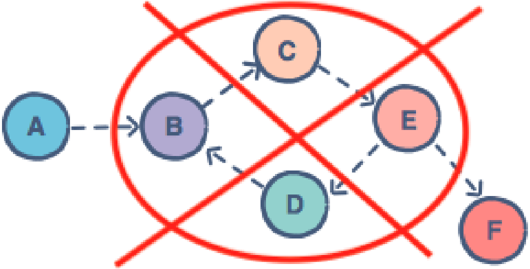
Directed acyclic graph

***D**irected **A**cyclic **G**raphs = DAG*



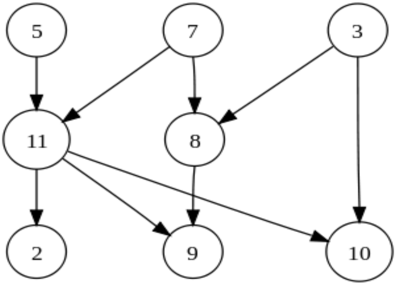
Directed acyclic graph

$$\boxed{\textit{Directed Acyclic Graphs} = \textit{DAG}}$$



Topological ordering

"Every DAG has a topological ordering."



- 5, 7, 3, 11, 8, 2, 9, 10

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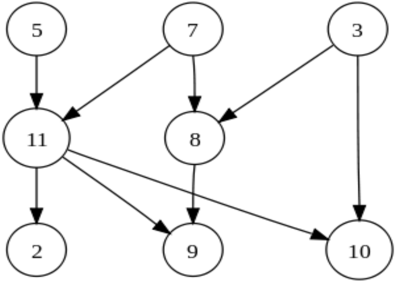

- 5, 7, 3, 11, 8, 2, 9, 10
- 3, 5, 7, 8, 11, 2, 9, 10

Topological ordering

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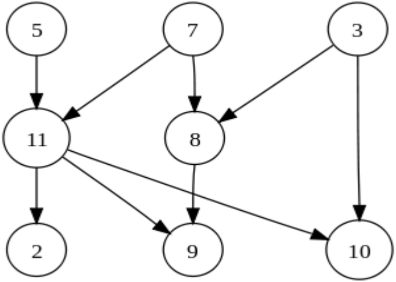

- 5, 7, 3, 11, 8, 2, 9, 10
- 3, 5, 7, 8, 11, 2, 9, 10
- 5, 7, 3, 8, 11, 10, 9, 2

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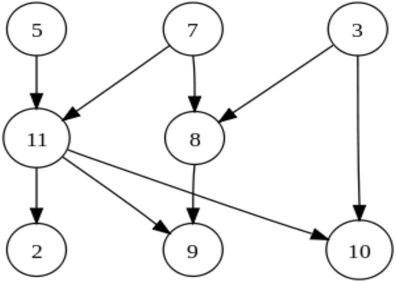

- 5, 7, 3, 11, 8, 2, 9, 10
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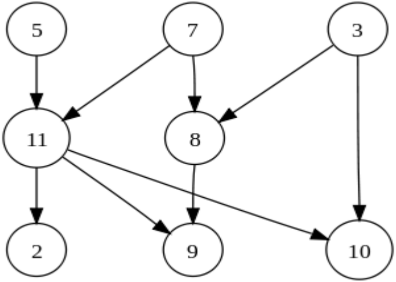

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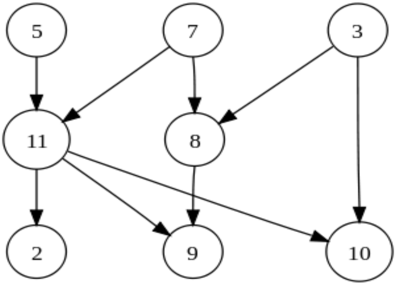

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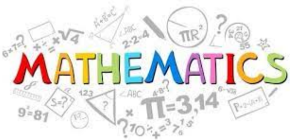
- 5, 7, 3, 8, 11, 10, 9, 2
- 7, 5, 11, 3, 10, 8, 9, 2
- 5, 7, 11, 2, 3, 8, 9, 10
- 3, 7, 8, 5, 11, 10, 2, 9

Topological ordering

Task scheduling problem

Linear ordering problem



Mathematical framework

- $G = (V, E)$ directed acyclic graph
- $P(G)$ set of all bijections $p : V(G) \to \{1, 2, \ldots, n\}$ such that for every directed edge $uv \in E(G)$ it holds

$$p(u) < p(v)$$

- $d_p(uv) = p(v) - p(u)$ distance between vetices u and v in topological order p

Basic definitions

For each vertex $v \in V(G)$ we define:

$$s_{p,G}(v) = \sum_{uv \in E(G)} d_p(uv)$$

$$a_{p,G}(v) = \begin{cases} \frac{s_{p,G}(v)}{d_G^-(v)} & \text{if } d_G^-(v) > 0, \\ 0 & \text{if } d_G^-(v) = 0 \end{cases}$$

$$m_{p,G}(v) = \max_{uv \in E(G)} \{d_p(uv)\},$$

Basic definitions

We expand these measures to the entire DAG G with n vertices. There are a number of ways to approach this problem.

1)

$$s_p^1(G) = \frac{1}{n} \sum_{v \in V(G)} s_{p,G}(v)$$

$$a_p^1(G) = \frac{1}{n} \sum_{v \in V(G)} a_{p,G}(v)$$

$$m_p^1(G) = \frac{1}{n} \sum_{v \in V(G)} m_{p,G}(v)$$

Basic definitions

We expand these measures to the entire DAG G with n vertices. There are a number of ways to approach this problem.

2)

$$s_p^\infty(G) = \max_{v \in V(G)} \{s_{p,G}(v)\}$$

$$a_p^\infty(G) = \max_{v \in V(G)} \{a_{p,G}(v)\}$$

$$a_p^\infty(G) = \max_{v \in V(G)} \{a_{p,G}(v)\}$$

Basic definitions

We expand these measures to the entire DAG G with n vertices. There are a number of ways to approach this problem.

3)

$$s_p^\alpha(G) = \left(\frac{1}{n} \sum_{v \in V(G)} s_{p,G}(v) \right)^{\frac{1}{\alpha}}$$

$$a_p^\alpha(G) = \left(\frac{1}{n} \sum_{v \in V(G)} a_{p,G}(v) \right)^{\frac{1}{\alpha}}$$

$$m_p^\alpha(G) = \left(\frac{1}{n} \sum_{v \in V(G)} m_{p,G}(v) \right)^{\frac{1}{\alpha}}$$

Basic definitions

Since the topological ordering od directed acyclic graph is generally not unique, we define

$$s^{\alpha}(G) = \min_{p \in P(G)} \left\{ s_p^{\alpha}(G) \right\}$$

$$a^{\alpha}(G) = \min_{p \in P(G)} \left\{ a_p^{\alpha}(G) \right\}$$

$$m^{\alpha}(G) = \min_{p \in P(G)} \left\{ m_p^{\alpha}(G) \right\}$$

Sample claim

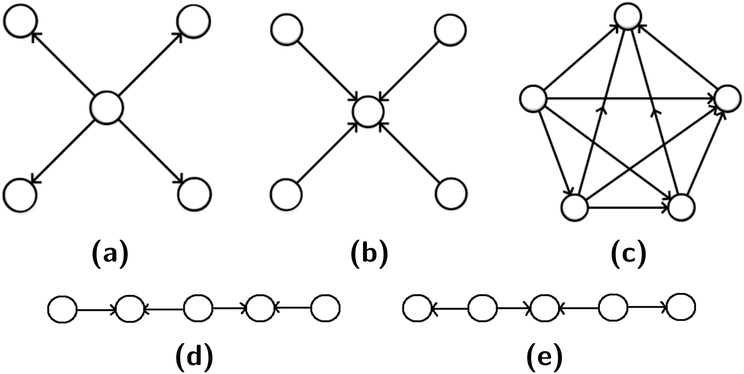
Theorem

Let G be a directed graph of type A with $n \geq 3$ vertices and $\alpha \geq 1$. It holds

$$\left(\frac{n-1}{n}\right)^{\frac{1}{\alpha}} \leq m^{\alpha}(G) \leq \left(\frac{1}{n}\sum_{i=1}^{n-1} i^{\alpha}\right)^{\frac{1}{\alpha}}.$$

The lower bound is obtained for $G = P_n$ and the upper bound for $G = O_n$.

Extremal results



Summary - type A

Table: Extremal values for the graphs of type A

Measure	Minimal	Maximal
$s^{\infty}(G)$	1	$\sum_{i=1}^{n-1} i$
$a^{\infty}(G)$	1	$n-1$
$m^{\infty}(G)$	1	$n-1$
$s^{\alpha}(G)$	$\left(\frac{n-1}{n}\right)^{\frac{1}{\alpha}}$	$\left[\frac{1}{n}\sum_{i=2}^{n}\left(\sum_{j=1}^{i-1} j\right)^{\alpha}\right]^{\frac{1}{\alpha}}$
$a^{\alpha}(G)$	$\min\left\{\frac{n}{2n^{\frac{1}{\alpha}}}, \frac{\alpha}{2(\alpha-1)}\left[\frac{(n-1)(\alpha-1)}{n}\right]^{\frac{1}{\alpha}}, \left(\frac{n-1}{n}\right)^{\frac{1}{\alpha}}\right\}$	$\left(\frac{1}{n}\sum_{i=1}^{n-1} i^{\alpha}\right)^{\frac{1}{\alpha}}$
$m^{\alpha}(G)$	$\left(\frac{n-1}{n}\right)^{\frac{1}{\alpha}}$	$\left(\frac{1}{n}\sum_{i=1}^{n-1} i^{\alpha}\right)^{\frac{1}{\alpha}}$

Summary - type B

Table: Extremal values for the graphs of type B

Measure	Minimal	Maximal
$s^{\infty}(G)$	4	$\sum_{i=1}^{n-1} i$
$a^{\infty}(G)$	2	$n-1$
$m^{\infty}(G)$	3	$n-1$
$s^{\alpha}(G)$	$\frac{2n-3}{n}$ **(*)**	$\left(\frac{1}{n}\sum_{i=2}^{n}\left(\sum_{j=1}^{i-1} j\right)^{\alpha}\right)^{\frac{1}{\alpha}}$
$a^{\alpha}(G)$	$\frac{1}{2}$ **(*)**	$\left(\frac{1}{n}\sum_{i=1}^{n-1} i^{\alpha}\right)^{\frac{1}{\alpha}}$
$m^{\alpha}(G)$	$\frac{n-1}{n}$ **(*)**	$\left(\frac{1}{n}\sum_{i=1}^{n-1} i^{\alpha}\right)^{\frac{1}{\alpha}}$

Thank you for your attention!