Polyomino tilings: from combinatorics to topology (joint work with Edin Liđan)

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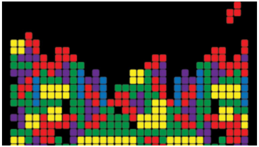
'There should be no such thing as boring mathematics.'
Edsger Dijkstra



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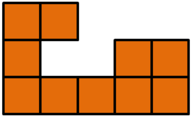

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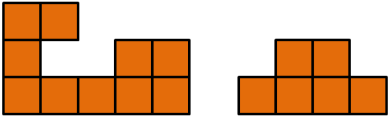

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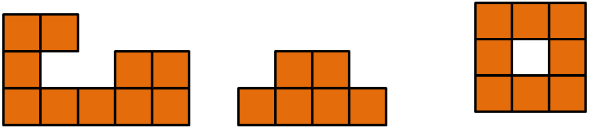

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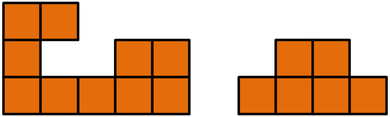

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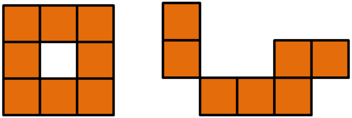

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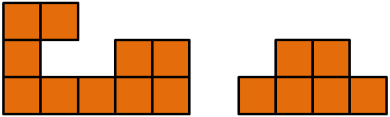

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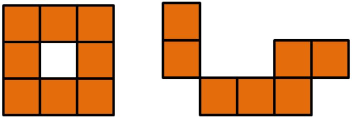

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A polyomino is called *free* if does not contain any 'holes' (or if its fundamental group is trivial).

n-ominoes

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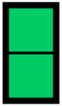

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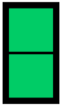

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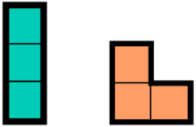

Figure: Trominoes

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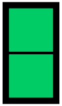

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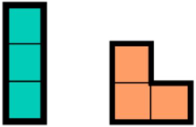

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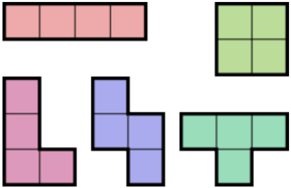

Figure: Tetrominoes

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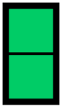

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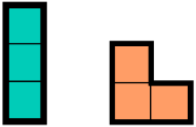

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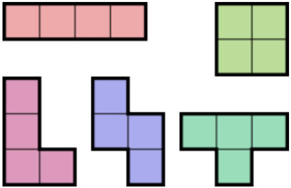

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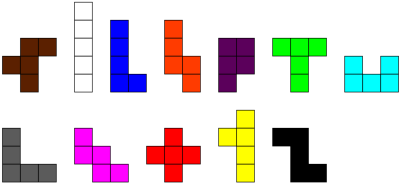

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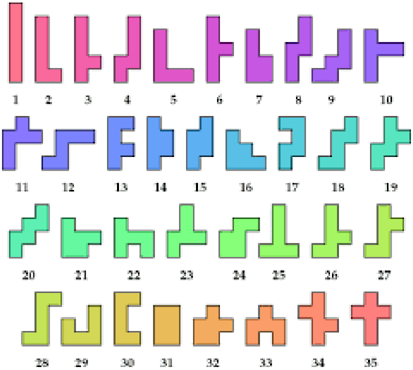

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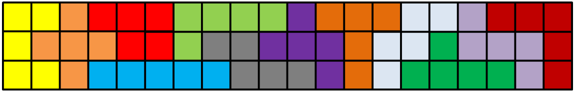

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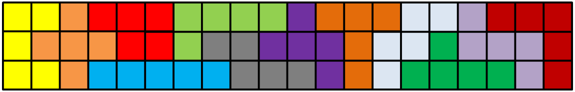

There are numerous generalizations of this question towards symmetrical and asymmetrical tilings, higher dimension analogs, polyomino types in other regular lattice grids (triangular, hexagonal), etc. The problem in the general case is NP-hard and we can give definite answer only in a limited number of cases.

Number of distinct polyomino tilings

Theorem (Fischer - Temperley & Kasteleyn, 1961)

Number of distinct polyomino tilings of $2m \times 2n$ board by dominoes is

$$\prod_{j=1}^{m}\prod_{i=1}^{n}\left(4\cos^2\frac{j\pi}{2m+1}+4\cos^2\frac{i\pi}{2n+1}\right).$$

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For $m=2$ this number is $n+1$-th Fibonacci number.

Simplicial complexes

An *abstract simplicial complex* K on a vertex set $[m] = \{1, 2, \ldots, m\}$ is a collection of subsets of $[m]$ such that,

(i) for each $i \in [m]$, $\{i\} \in [m]$,

(ii) for every $\sigma \in K$, if $\tau \subset \sigma$ then $\tau \in K$.

We assume that $\varnothing \in K$.

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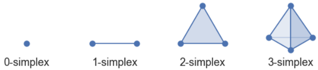

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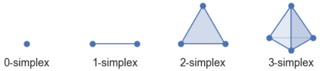

Figure: Simplices

The elements of K are called *faces*. The dimension of a face σ of a simplicial complex K is defined as $\dim \sigma = |\sigma| - 1$, where $|\sigma|$ denotes the cardinality of σ.

Simplicial complexes

Faces of dimension 0 are called *vertices* while faces of dimension 1 are called edges. *The dimension of* K, $\dim K$, is defined as the maximum dimension of the faces of K.

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A face of K is *maximal* if it is not contained as a subset in any other face of K. The maximal faces are also called *facets*. A complex K is *pure* if all of its facets are of the same dimension.

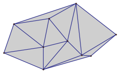
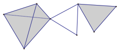


Figure: A pure simplicial complex



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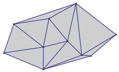

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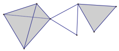

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A simplicial complex containing no face of dimension greater than 1 is called *graph*.

Link and star of a simplex

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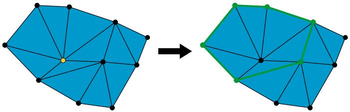

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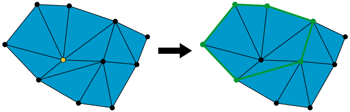

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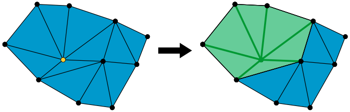

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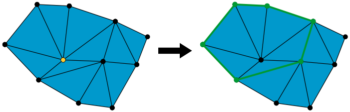

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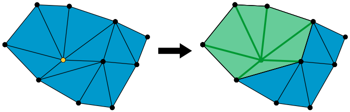

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The flag complex of K is the clique complex of $\mathrm{sk}^1(K)$.

The f-vector of a simplicial complex

The f-vector of an $(n-1)$-dimensional simplicial complex K^{n-1} is the integer vector

$$\mathbf{f}(K^{n-1}) = (f_{-1}, f_0, f_1, \ldots, f_{n-1}),$$

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$$\chi(K^{n-1}) = f_0 - f_1 + \cdots + (-1)^{n-1} f_{n-1} = (-1)^{n-1}\mathbf{f}(-1) + 1$$

is called the *Euler characteristic* of a simplicial complex K^{n-1}.

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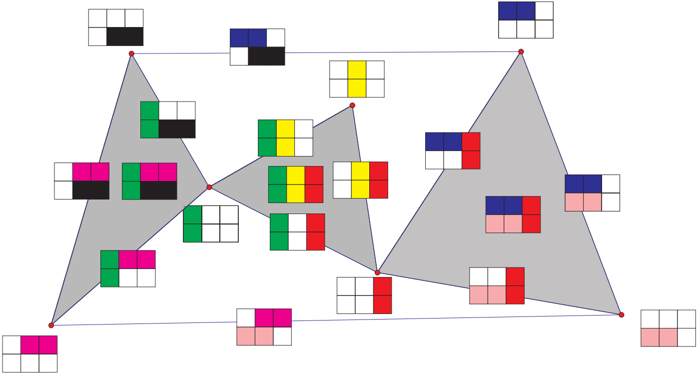
$\dim(K(M;\mathcal{T}))+1$ *is maximal number of polyomino shapes from* $\mathcal{T}$ *that may be placed on* M *without overlapping.*

An example of $K(M;\mathcal{T})$

Example

If we consider tiling of 2×3 board by dominoes then a geometrical realization of corresponding simplicial complex is



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There is an integer $l(\mathcal{T})$ *such that the number of vertices in* $K(M;\mathcal{T}) \setminus \mathrm{lk}_{K(M;\mathcal{T})} v$ *is not greater than* $l(\mathcal{T})$ *for all regions* M *in a plane and any vertex of* $K(M;\mathcal{T})$

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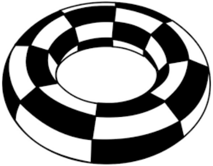
Obviously, up to translation there is only finitely many positions in the plane grid such that two polyomino shapes from $\mathcal{T}$ overlap. Since $\mathcal{T}$ is a finite set, the claim follows. $\square$

Simple connectivity

Let $\mathcal{T}$ be a given finite set of polyomino shapes and let $K(m,n;\mathcal{T})$ and $T(m,n;\mathcal{T})$ be the simplicial complexes of polyomino tilings of the board $m \times n$ in the plane and on the torus, respectively.



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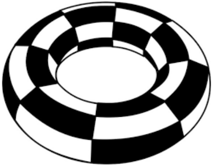

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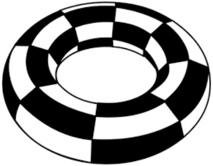

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A space X is simply connected if , $\pi_1(X) = 0$, or equivalently X is non-empty and each continuous map $S^1 \to X$ is nullhomotopic.

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□

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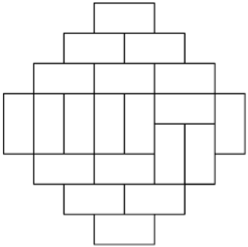

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By Proposition 3.3 for any r vertices $v_1,\ldots,v_r$ of $K(m,n;\mathcal{T})$ there are at most $r \cdot l(\mathcal{T})$ vertices lying outside $\mathrm{lk}_{K(m,n;\mathcal{T})} v_1 \cup \ldots \mathrm{lk}_{K(m,n;\mathcal{T})} v_1$ so for sufficiently large m and n there is a vertex v such that $v_1,\ldots,v_r \in \mathrm{lk}_{K(m,n;\mathcal{T})} v$.

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Each 6^n-conic simplicial complex is n-connected.

Corollary

For any increasing sequence of regions M_i and a finite fixed set of polyomino shapes $\mathcal{T}$, there is a sequence of positive integers $p_k(M_i,\mathcal{T})$ such that $\pi_k K(M_n;\mathcal{T})$ is trivial for all $n\geq p_k(M_i;\mathcal{T})$.

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1. for the plane boards $1 \times n$ it holds $p_0(K(1,n;2) = 5, p_1(K(1,n;2) = 8, p_2(K(1,n;2) = 8$

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Problem

Estimate $p_k(M_n,\mathcal{T})$ in terms of n for a given increasing sequence of regions M_n and the set of polyomino shapes $\mathcal{T}$

Homotopy type of $K(M;\mathcal{T})$

Lemma

[Cone Lemma] Let K be a simplicial complex and v a vertex of K. If $\mathrm{lk}_K v$ is contractible in $\mathrm{del}_K v$ then

$$K \simeq \mathrm{del}_K v \vee \Sigma(\mathrm{lk}_K v)$$

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Proposition

For $n \geq i_1 + i_k$ we have that
$K(1,n;i_1,i_2,\ldots,i_k) \simeq \bigvee_{j=2}^{k} \Sigma K(1,n-i_j;i_1,i_2,\ldots,i_k) \vee$
$\bigvee_{r=2}^{i_1} \bigvee_{l=1}^{k} \Sigma K(1,n-i_l-r;i_1,i_2,\ldots,i_k)$

Homotopy type of $K(M;\mathcal{T})$

It follows that

$$X \simeq Y \vee \bigvee_{r=1}^{i_1} \bigvee_{j=1}^{k} \Sigma K(1, n-i_j-r+1; i_1, i_2, \ldots, i_k),$$

where Y is obtained from X by deleting all of the simplices containing one of the vertex u_j^r where $1 \leq j \leq k$, and $2 \leq r \leq i_1$. But, as we already removed all vertices outside $\mathrm{lk}_K u_1$, Y is the simplicial cone with the vertex in u_1 and therefore, contractible. The claim now follows.

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If $n < i_1 + i_k$ observe that we cannot place $1 \times i_k$ I-mino and a tile from $\mathcal{T}$ on the board without overlapping. Thus,

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Proposition

If $n < i_1 + i_k$ then
$K(1, n; i_1, \ldots, i_k) \simeq K(1, n; i_1, \ldots, i_{k-1}) \vee \bigvee_{n-i_k+1} S^0.$

Homotopy type of $K(M;\mathcal{T})$

It follows that the homotopy type of $K(1,n;i_1,\dots,i_k)$ is determined by the homotopy types of $K(1,l;i_1)$ for all $i_1 \le l \le 2i_1 - 1$.

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$K(l; i_1)$ is contractible for $l = i_1$ and $K(l; i_1)$ is homotopic to the wedge of $l - i_1$ spheres S^0 for $i_1 < l \leq 2i_1 - 1$.

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Three proposition above together imply the following result, which generalizes the result of Kozlov.

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Theorem

$K(n;i_1,\ldots,i_k)$ has the homotopy type of a wedge of spheres.

Homotopy type of $K(M;\mathcal{T})$

Let $i_1 < i_2 < \cdots < i_k$ be positive integers and $\mathcal{T}$ consists of $1 \times i_j$ I-minoes for all $1 \leq j \leq k$ and let us denote by $T(m, n; i_1, i_2, \ldots, i_k)$ the simplicial complex of tilings of the board $1 \times n$ on the torus by $\mathcal{T}$.

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Theorem

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f-vectors

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Theorem

$$\mathbf{f}_k(T(1,n;d)) \quad = \quad (d-1)\binom{n+(1-d)k-k}{k}+\binom{n-(d-1)(k+1)}{k+1}.$$

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Table: Values of $\mathbf{f}(K(2,n;2))$ for small values of n

n	$\mathbf{f}_0$	$\mathbf{f}_1$	$\mathbf{f}_2$	$\mathbf{f}_3$	$\mathbf{f}_4$	$\mathbf{f}_5$	$\mathbf{f}_6$	$\mathbf{f}_7$
2	4	2						
3	7	11	3					
4	10	29	26	5				
5	13	56	94	56	8			
6	16	92	234	263	114	13		
7	19	137	473	815	667	223	21	
8	22	191	838	1982	2504	1577	424	34

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2. $$\mathbf{f}_1(K(2,n;2)) = \frac{9n^2-27n+22}{2}$$

3. $$\mathbf{f}_{n-1}(K(2,n;2)) = \prod_{i=0}^{n-1} a_i a_{n-1-i} + 2 \prod_{0\leq i<j\leq n-1} a_i a_{n-1-j}$$

where $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$.

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Let us denote by $L(m,n;d)$ the simplicial complex of tiling of the board $m \times n$ without the upper left cell by $1 \times d$ I-omino.

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Proposition

For $L(2,n;2)$ we have that

1. $$\mathbf{f}_0(L(2,n;2)) = 3n-4, n\geq 2$$
2. $$\mathbf{f}_1(L(2,n;2)) = \frac{9n^2-39n+48}{2}, n\geq 3$$
3. $$\mathbf{f}_{n-1}(L(2,n;2)) = \sum_{i=0}^{n-1} a_{n-1-i} = a_{n+1}-1$$

where $a_0=a_1=1$ and $a_n=a_{n-1}+a_{n-2}$ for all $n\geq 2$.

f-vectors

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Corollary

For a fixed k, $\mathbf{f}_k(K(2,n;2))$ and $\mathbf{f}_k(L(2,n;2))$ are polynomial functions of n of degree k.

The Euler Characteristic

Theorem (Kozlov)

$$K(1,n;2) \simeq \begin{cases} \bullet & n \equiv 2 \pmod 3 \\ S^{[\frac{n}{3}]-1} & n \not\equiv 2 \pmod 3 \end{cases}$$

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Corollary

$$\sum_{i=1}^{[\frac{n}{2}]} (-1)^{i-1} \binom{n-i}{i} = \begin{cases} 1 & n \equiv 2 \pmod 3 \\ 1 + (-1)^{[\frac{n}{3}]-1} & n \not\equiv 2 \pmod 3 \end{cases}$$

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Corollary

$$\sum_{i=1}^{[\frac{n}{3}]} (-1)^{i-1} \binom{n-2i}{i} = b_n + 1,$$

where b_n *is the nth term of the sequence given by* $b_3 = 1, b_4 = 2, b_5 = 3, b_6 = 3, b_7 = 2$ *and for all* $n \geq 8$

$$b_n + b_{n-4} + b_{n-5} = 1.$$


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