On a coloring problem in the plane

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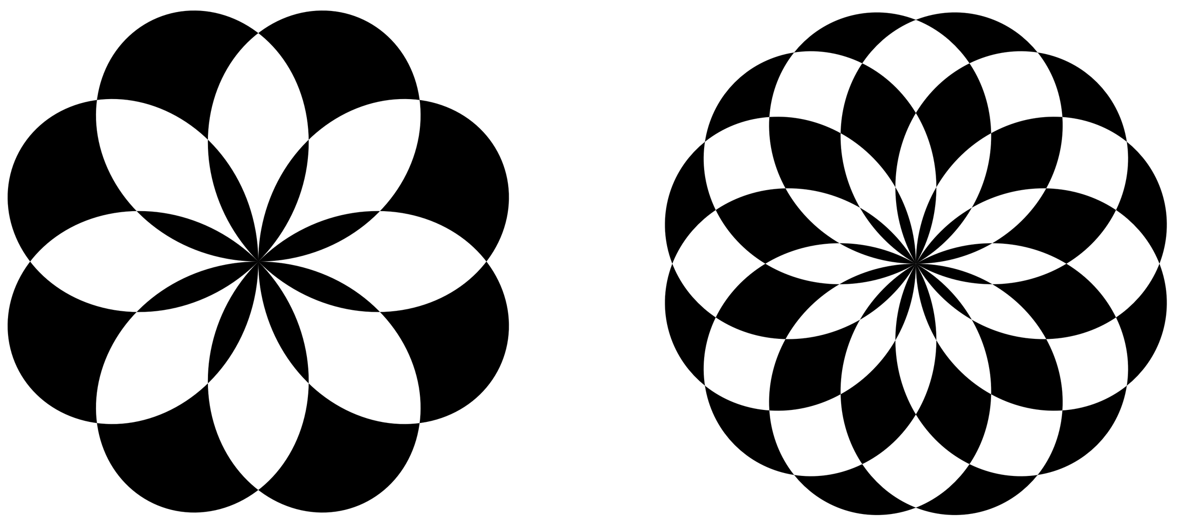
Proposed problem

Shown below (from left to right) are graphs of $r = \sin(4\theta/3)$ and $r = \sin(6\theta/5)$, where every other adjacent region (starting from the outside) is shaded black. Find the total shaded area for any such graph $r = \sin(k+1)\theta/k$, where $k > 0$ is an odd integer and θ ranges from 0 to $2k\pi$. [1] [2]



About rhodonea curve

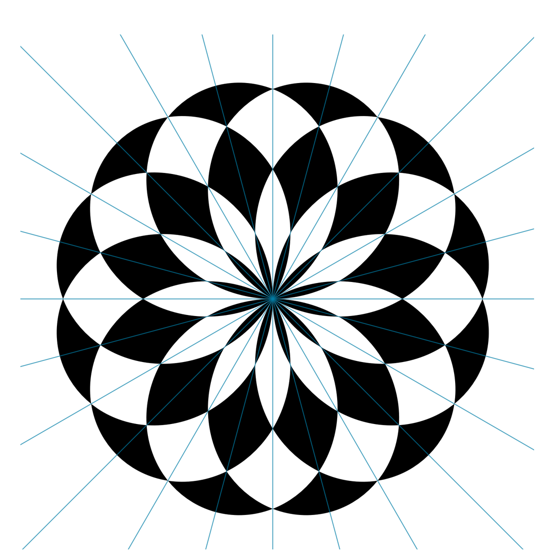
Rhodonea curve is a sinusoid specified by either the cosine or sine functions with no phase angle that is plotted in polar coordinates.

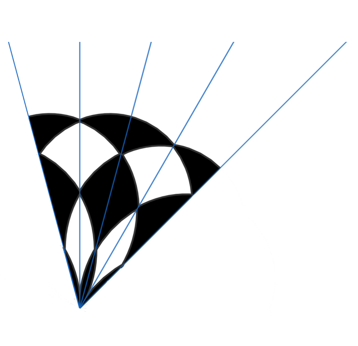
Let $r = \sin(m\theta/n)$, where n and $m > n$ are relatively prime, non-zero integeres.

- Graph of rhodonea curve is composed of petals.
- Petal is the shape formed by the graph of a half-cycle of the sinusoid.
- A cycle is a portion of a sinusoid that is one period $T = 2n\pi/m$ long and consists of a positive half-cycle, the continuous set of points, $T/2 = n\pi/m$.
- For an even integer m, the curve will be rose-shaped with $2m$ petals. For an odd integer m, the curve will be rose-shaped with m petals.

Consider the petal which is symmetric about the line $y = \mathrm{tg}\left(\frac{n\pi}{2m}\right)x$. All other petals are given by rotation of this petal about the pole by $\frac{k\pi}{m}$ radians.

Solution





- Graph is symmetric about every line which passes through the pole and self-intersections of the curve.
- Petal is symmetric about every line which passes through the pole and petal's peak, $y = \mathrm{tg}\left(\frac{(k+2l)\pi}{2(k+1)}\right)x$, where $l = 0, 1, 2, \ldots$.
- Consider sectors from $\frac{(l-1)\pi}{2(k+1)}$ to $\frac{l\pi}{2(k+1)}$, where $l = 1, \ldots, n$.

The total shaded area is equal to $\frac{\pi}{2}$ and obtained by formula:

$$P(k+1,k) = 2(k+1)\left[\sum_{l=1}^{k}(-1)^{l+1}\int_{\frac{(l-1)\pi}{2(k+1)}}^{\frac{l\pi}{2(k+1)}}\sin^2\left(\frac{(k+1)\theta}{k}\right)d\theta\right].$$

After integration and summation we have:

$$P(k+1,k) = \frac{\pi(-1)^{k+1} - 2k\sin(k\pi) - 2k(\cos(k\pi)+1)\mathrm{tg}\left(\frac{\pi}{2k}\right) + \pi}{4}.$$

Since k is an integer, it follows that $\sin(k\pi) = 0$, while $\cos(k\pi) = -1$ since k is an odd integer.
Thus we have:

$$P(k+1,k) = \frac{\pi}{2}.$$

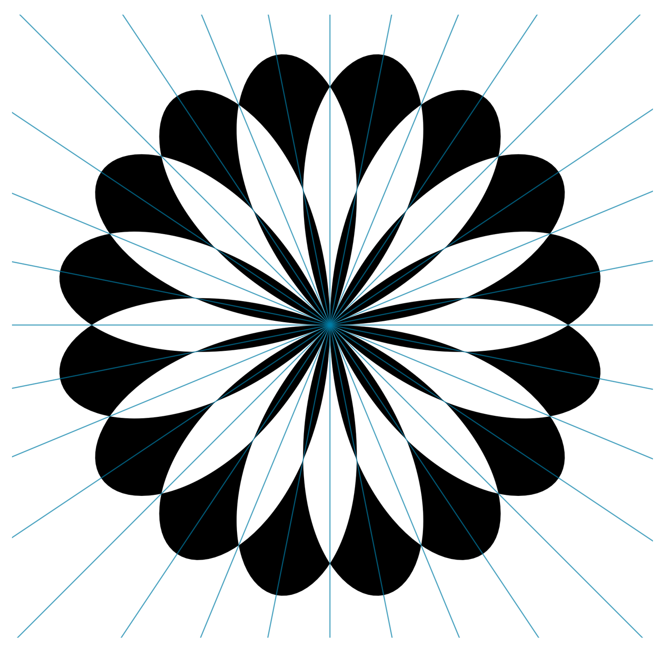
References

1. https://dresden.academic.wlu.edu/studentresearch/
2. G. Dresden, Problem 1221, College Math. J. 53 (2022) 152.

We thank Luka Podrug for his help in preparing the poster.

Generalization of proposed problem

Let $r = \sin(m\theta/n)$, where $n > 0$ is an odd integer, $m > n$ is an even integer, m and n are relatively prime.



- Graph is symmetric about every line which passes through the pole and self-intersections of the curve.
- Petal is symmetric about every line which passes through the pole and petal's peak, $y = \mathrm{tg}\left(\frac{(n+2k)\pi}{2m}\right)x$, where $k = 0, 1, 2, \ldots$.
- Consider sectors from $\frac{(k-1)\pi}{2m}$ to $\frac{k\pi}{2m}$, where $k = 1, \ldots, n$.

The total shaded area is equal to $\frac{\pi}{2}$ and obtained by formula:

$$P(m,n) = 2m\left[\sum_{k=1}^{n}(-1)^{k+1}\int_{\frac{(k-1)\pi}{2m}}^{\frac{k\pi}{2m}}\sin^2\left(\frac{m\theta}{n}\right)d\theta\right].$$

After integration and summation we have:

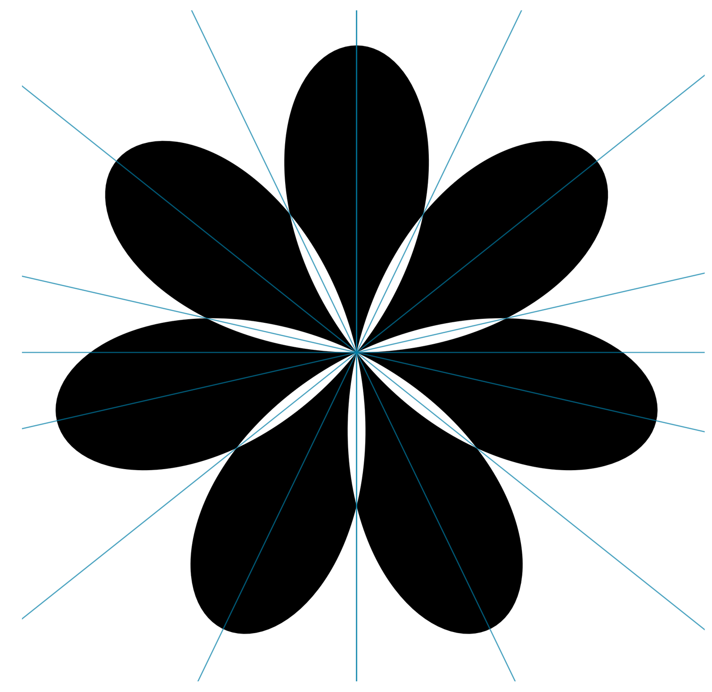
$$P(m,n) = 2m\frac{\pi(-1)^{n+1} - 2n\sin(n\pi) - 2n(\cos(n\pi)+1)\mathrm{tg}\left(\frac{\pi}{2n}\right) + \pi}{8m}.$$

Since n is an integer, it follows that $\sin(n\pi) = 0$, while $\cos(n\pi) = -1$ since n is an odd integer.
Thus we have:

$$P(m,n) = \frac{\pi}{2}.$$

What if m is an odd integer?

Let $r = \sin(m\theta/n)$, where $n > 0$ and $m > n$ are relatively prime, odd integers.



- Graph is symmetric about every line which passes through the pole and self-intersections of the curve.
- Petal is symmetric about every line which passes through the pole and petal's peak, $y = \mathrm{tg}\left(\frac{(n+4k)\pi}{2m}\right)x$, where je $k = 0, 1, 2, \ldots$.
- Consider sectors from $\frac{(n-2k-2)\pi}{2m}$ to $\frac{(n-2k)\pi}{2m}$, where $k = 0, \ldots, \frac{n-3}{2}$.

The total shaded area is obtained by formula:

$$P(m,n) = m\left[\sum_{k=0}^{\frac{n-3}{2}}(-1)^k\int_{\frac{(n-2k-2)\pi}{2m}}^{\frac{(n-2k)\pi}{2m}}\sin^2\left(\frac{m\theta}{n}\right)d\theta + (-1)^{\frac{n-1}{2}}\int_0^{\frac{\pi}{2m}}\sin^2\left(\frac{m\theta}{n}\right)d\theta\right].$$

After integration and summation we have:

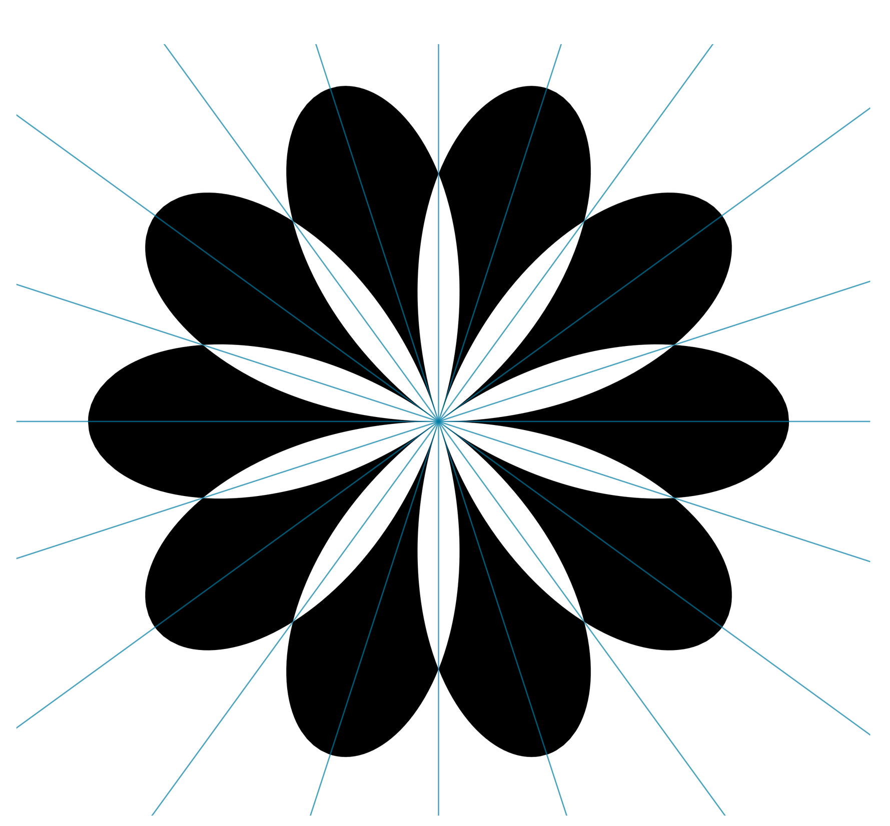
$$P(m,n) = \frac{m\left[2\left(i^{n+1}n\sin\left(\frac{\pi}{n}\right) + n\mathrm{tg}\left(\frac{\pi}{n}\right) + \pi\right) - n\cos\left(\frac{n\pi}{2}\right)\sec\left(\frac{\pi}{n}\right) + n\cos\left(\frac{2\pi}{n} - \frac{n\pi}{2}\right)\sec\left(\frac{\pi}{n}\right)\right]}{8m}.$$

After applying trigonometric identities we have:

$$P(m,n) = \frac{n\mathrm{tg}\left(\frac{\pi}{n}\right) + \pi}{4}.$$

What if n is an even integer?

Let $r = (\sin m\theta/n)$, where $n > 0$ is an even integer, $m > n$ and n are relatively prime. Hence m is an odd integer.



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- Petal is symmetric about every line which passes through the pole and petal's peak, $y = \mathrm{tg}\left(\frac{(n+2k)\pi}{2m}\right)x$, where $k = 0, 1, 2, \ldots$.
- Consider sectors from $\frac{(k-1)\pi}{2m}$ to $\frac{k\pi}{2m}$, where $k = 1, \ldots, n$.

The total shaded area is obtained by formula:

$$P(m,n) = 2m\left[\sum_{k=1}^{n}(-1)^k\int_{\frac{(k-1)\pi}{2m}}^{\frac{k\pi}{2m}}\sin^2\left(\frac{m\theta}{n}\right)d\theta\right].$$

After integration and summation we have:

$$P(m,n) = 2m\frac{\pi((-1)^n - 1) + 2n\sin(n\pi) + 2n(\cos(n\pi)+1)\mathrm{tg}\left(\frac{\pi}{2n}\right)}{8m}.$$

Since n is an integer, it follows that $\sin(n\pi) = 0$, while $\cos(n\pi) = 1$ since n is an even integer.
Thus we have:

$$P(m,n) = n\mathrm{tg}\left(\frac{\pi}{2n}\right).$$

Remarks

$P(m,n)$ tends to $\frac{\pi}{2}$ as n tends to infinity, whenever $P(m,n)$ is not a constant.

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