Pairwise balanced designs and periodic Golay pairs

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- D. Crnković, D. Dumičić Danilović, R. Egan, A. Švob, *Periodic Golay pairs and pairwise balanced designs*, J. Algebraic Combin. 55 (2022), 245-257.

Periodic Golay pair

Let $a = [a_0, \ldots, a_{v-1}]$ be a $\{\pm 1\}$-sequence of length v.

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For example: $s = 2, v = 8, a = [-1, 1, -1, 1, 1, -1, 1, -1] \rightarrow \mathrm{PAF}_2(a) = 0$

$$\begin{array}{cccccccrrrrrrrr} a_0 & a_1 & a_2 & \ldots & a_6 & a_7 & = & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ a_2 & a_3 & a_4 & \ldots & a_0 & a_1 & = & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{array}$$

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A pair (a, b) of $\{\pm 1\}$-sequences is a **periodic Golay pair** ($\mathrm{PGP}(v)$) if $\mathrm{PAF}_s(a) + \mathrm{PAF}_s(b) = 0$ for all $1 \leq s \leq v - 1$.

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a_0	a_1	a_2	a_3	b_0	b_1	b_2	b_3	(a, b) are $PGP(4)$
-1	1	1	1	-1	1	1	1	
1	1	1	-1	1	1	1	-1	$\mathrm{PAF}_1(a) + \mathrm{PAF}_1(b) = 0$
1	1	-1	1	1	1	-1	1	$\mathrm{PAF}_2(a) + \mathrm{PAF}_2(b) = 0$
1	-1	1	1	1	-1	1	1	$\mathrm{PAF}_3(a) + \mathrm{PAF}_3(b) = 0$

PGP(v)

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- M. J. E. Golay, *Multi-Slit Spectrometry*, J. Opt. Soc. Am. 39 (1949), 437–444.
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Recent progress on PGPs:

1. D. Ž. Đoković, I. S. Kotsireas, *Periodic Golay Pairs of Length 72*, in: C. J. Colbourn, (Ed.), Algebraic Design Theory and Hadamard Matrices, Springer Proceedings in Mathematics and Statistics, vol 133, Springer, Cham, 2015, pp. 83–92.
2. D. Ž. Đoković, I. S. Kotsireas, *Some new periodic Golay pairs*, Numer. Algor. 69 (2015), 523–530.
3. D. Ž. Đoković, I. S. Kotsireas, *Compression of periodic complementary sequences and applications*, Des. Codes Cryptogr. 74 (2015), 365–377.

Pairwise balanced design

Let K be a set of positive integers.
A **pairwise balanced design** $\mathrm{PBD}(v, K, \lambda)$ is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

- $|\mathcal{P}| = v$,
- if an element of $\mathcal{B}$ is incident with k elements of $\mathcal{P}$, then $k \in K$,
- every pair of distinct elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

Elements of $\mathcal{P}$ and $\mathcal{B}$ are called **points** and **blocks**, respectively.

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Incidence matrix

An **incidence matrix** of a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a $\{0,1\}$ matrix $M = [m_{i,j}]$ of type $|\mathcal{P}| \times |\mathcal{B}|$ defined by the rule

$$m_{i,j} = \begin{cases} 1, & \text{if } (P_i, B_j) \in \mathcal{I} \\ 0, & \text{if } (P_i, B_j) \notin \mathcal{I}. \end{cases}$$

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- Isomorphism between designs
- Automorphism of a design $\mathcal{D}$; $\mathrm{Aut}(\mathcal{D})$

The main result

- **By constructing PBDs** with appropriate parameters and a presumed cyclic automorphism group, **we can construct PGPs**.
- We completely classified PBDs with appropriate parameters and a presumed cyclic automorphism group, which correspond to PGP(v)s for all $v \leq 34$, and we noted that there are no periodic Golay pairs of length 36 and 38.
- We constructed a new PGP(74).

PBDs and periodic Golay pairs

Let $(a, b) \in \mathrm{PGP}(v)$, and let A and B be the circulant matrices with first rows a and b, respectively.
Then $\begin{bmatrix} A & B \end{bmatrix}$ is a $v \times 2v$ matrix where the dot product of any two distinct rows is zero, i.e. the top half of a Hadamard matrix.

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By replacing each 1 with 0 and each -1 with 1 in $\begin{bmatrix} A & B \end{bmatrix}$, an *incidence matrix* $\begin{bmatrix} A' & B' \end{bmatrix}$ of a $\mathrm{PBD}(v, \{k_a, k_b\}, \lambda)$ is obtained.

If the blocks label the columns and points label the rows of $\begin{bmatrix} A' & B' \end{bmatrix}$ we have v points, each incident with $r = k_a + k_b$ blocks and any pair of points is incident with $\lambda = r - \frac{v}{2}$ blocks.

PBDs and periodic Golay pairs

Example: $(a, b) \in \mathrm{PGP}(10)$

$a = [-1, -1, 1, 1, 1, 1, -1, 1, -1, 1], \quad b = [-1, -1, 1, 1, 1, 1, 1, -1, 1, 1]$

1	1	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0
0	1	1	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0
1	0	1	1	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	1
0	1	0	1	1	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0
1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0	0	0
0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0	0
0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0
0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0
0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1
1	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1

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1	1	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0
0	1	1	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0
1	0	1	1	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	1
0	1	0	1	1	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0
1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0	0	0
0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0	0
0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0
0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0
0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1
1	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1

$\mathcal{D} = \mathrm{PBD}(10,\{4,3\},2)$ with $k_a = 4$, $k_b = 3$, $r = 7$ and $C_{10} \leq \mathrm{Aut}(\mathcal{D})$.

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1	1	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0
0	1	1	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0
1	0	1	1	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	1
0	1	0	1	1	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0
1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0	0	0
0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0	0
0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0
0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	0
0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1
1	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1

$\mathcal{D} = \mathrm{PBD}(10,\{4,3\},2)$ with $k_a = 4$, $k_b = 3$, $r = 7$ and $C_{10} \leq \mathrm{Aut}(\mathcal{D})$.

So, by constructing $\mathrm{PBD}(v, K, \lambda)$ with presumed automorphism group C_v acting transitively on points and with two orbits on the set of blocks, we can construct the corresponding PGP.

Construction of PBDs - tactical decomposition

- Z. Janko, Coset enumeration in groups and constructions of symmetric designs, Combinatorics '90 (Gaeta, 1990), Ann. Discrete Math. 52 (1992), 275–277.
- D. Crnković, Symmetric (70,24,8) designs having $\mathrm{Frob}_{21} \times Z_2$ as an automorphism group, Glas. Mat. Ser. III 34 (54) (1999), 109–121.

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Let $\mathcal{D}$ be a $\mathrm{PBD}(v, K, \lambda)$ with a replication number r, and $G \leq \mathrm{Aut}(\mathcal{D})$. G-orbits of points $\mathcal{P}_1, \ldots, \mathcal{P}_m$, G-orbits of blocks $\mathcal{B}_1, \ldots, \mathcal{B}_n$, and $|\mathcal{P}_i| = \omega_i$, $|\mathcal{B}_j| = \Omega_j$, $1 \leq i \leq m$, $1 \leq j \leq n$, and γ_{ij} is the number of blocks of $\mathcal{B}_j$ incident with a representative of the point orbit $\mathcal{P}_i$.

The following equalities hold:

$$0 \leq \gamma_{ij} \leq \Omega_j, \quad 1 \leq i \leq m, 1 \leq j \leq n, \tag{1}$$

$$\sum_{j=1}^{n} \gamma_{ij} = r, \quad 1 \leq i \leq m, \tag{2}$$

$$\sum_{i=1}^{m} \frac{\omega_i}{\Omega_j} \gamma_{ij} \in K, \quad 1 \leq j \leq n, \tag{3}$$

$$\sum_{j=1}^{n} \frac{\omega_t}{\Omega_j} \gamma_{sj} \gamma_{tj} = \lambda \omega_t + \delta_{st} \cdot (r - \lambda), \quad 1 \leq s, t \leq m. \tag{4}$$

A $(m \times n)$-matrix $M = (\gamma_{ij})$ with entries satisfying conditions $(1) - (4)$ is called a **point orbit matrix** of a pairwise balanced design $\mathrm{PBD}(v, K, \lambda)$ with orbit length distributions $(\omega_1, \ldots, \omega_m)$ and $(\Omega_1, \ldots, \Omega_n)$.

Main construction

The construction of PBDs corresponding to PGPs of length v using orbit matrices:

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1. Find all possible combinations of numbers k_a and k_b of a $\mathrm{PBD}(v, \{k_a, k_b\}, \lambda)$ corresponding to PGPs.
   For a fixed combination of such numbers k_a and k_b, we are proceeding with the construction of PBD. The cyclic group $G \cong C_v$ acts transitively on points and has two orbits on the set of blocks. For the cyclic group G there is exactly one orbit matrix $M = [k_a \ k_b]$.

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2. Construction of $\mathrm{PBD}s(v, \{k_a, k_b\}, \lambda)$ for the orbit matrix M. A principal series $\{1\} = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_n = G$, $G_i \cong C_{v_1} \times \ldots \times C_{v_i}$, of the group G can be used to construct refinements of the matrix M.
   In ith iteration of the refinements we construct all the orbit matrices for the group G_{n-i}, having in mind the action of the group G.In the last iteration we obtain the orbit matrices for the trivial group *i.e.* incidence matrices of $\mathrm{PBDs}(v, \{k_a, k_b\}, \lambda)$.

- D. Ž. Doković, I. S. Kotsireas, Compression of periodic complementary sequences and applications, Des. Codes Cryptogr. 74 (2015), 365–377.

- $\mathrm{PGP}(v)$ for $v > 1$ can exist only if v is even.
- Length v of a $\mathrm{PGP}(v)$ must be a sum of two squares.
- If there exists a $\mathrm{PGP}(v)$ where $v = p^t u > 1$, $p \equiv 3 \mod 4$ is prime, and $\gcd(p, u) = 1$, then $u \geq 2p^{t/2}$. (Arasu, Xiang, 1992.)
- If $(a, b) \in \mathrm{PGP}(v)$, then where r_a and r_b denote the sum of the entries in a and b respectively, it holds that $r_a^2 + r_b^2 = 2v$. It follows that we can limit the possible choices of k_a and k_b so that $2(k_a - \frac{v}{2})^2 + 2(k_b - \frac{v}{2})^2 = v$.

Isomorph rejection

For *isomorph rejection* we use the elements of the normalizer $N_S(G)$ of the group $G \cong C_v$ in $\mathrm{Sym}(v)$.

Given a periodic Golay pair (a, b) of length v, we can construct a new periodic Golay pair of length v by applying certain equivalence operations.

- D. Ž. Doković, Equivalence classes and representatives of Golay sequences, Discrete Math. 189 (1998), 79–93.
- R. Egan, On equivalence of negaperiodic Golay pairs, Des. Codes Cryptogr., 85 (2017), 523–532.

Equivalence operations on PGPs

Let C be the circulant $v \times v$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We define that two periodic Golay pairs (a, b) and (c, d) of length v are **equivalent** if (c, d) is obtained from (a, b) by any combination of the following operations:

1. Swap a and b.
2. Replace a with aC.
3. Reverse a.
4. For any $k < v$ coprime to v replace both a and b with $[a_{ki}]_{0 \leq i \leq v-1}$ and $[b_{ki}]_{0 \leq i \leq v-1}$ respectively (decimation of a and b).
5. Negate every odd indexed entry of both a and b.

Results

With the construction of PBDs using orbit matrices all equivalence classes of PGPs of length v are constructed.

PBD classifications corresponding to PGPs

v	4	8	10	16	20	26	32	34	40
Isom.classes	3	4	8	62	448	816	10208	5856	≥ 565

The number of equivalence classes of PGPs of length v:

v	2	4	8	10	16	20	26	32	34	40
Classes	1	1	2	1	11	34	53	838	373	≥ 323

All our results:

`http://www.math.uniri.hr/~ddumicic/results/PGpairs_PBDs.html`

Results on PGP of length 74

The first two noneqivalent $\mathrm{PGP}(74)$ were constructed by:

- D. Ž. Đoković, I. S. Kotsireas, Some new periodic Golay pairs, Numer. Algor. 69 (2015), 523–530.

A search for pairs using the orbit matrix $M = [k_a \ k_b] = [38 \ 43]$ of a PBD$(74, \{38, 43\}, 42)$ with presumed automorphism group C_{74} returned a third inequivalent class represented by the pair:

$$[4, 1^2, 5, 1^5, 2, 3, 1^4, 2, 3, 1, 2, 3, 2, 1^4, 5, 2^3, 1, 2, 1, 2^2, 4, 2, 1^3, 2^2],$$
$$[2^2, 5, 2, 1^2, 3^2, 1, 7, 5, 1, 2, 1^2, 2^2, 1^4, 2^2, 1, 2, 3, 4, 1, 5, 1^3, 4, 1^2].$$

(Notation: For example the sequence $[1, 1, -, 1, -, -, -, 1, -, 1, -]$ would be written as $[2, 1^2, 3, 1^4]$).

Results on PGP of length 90

- Existence of a PGP(90)?

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We used the orbit matrices of the corresponding $\mathrm{PBD}(90, \{k_a, k_b\}, \lambda)$ under the action of the cyclic automorphism group $G \cong C_{90} \cong C_2 \times C_5 \times C_9$ which acts with the point and block orbit lengths distributions (90) and (90, 90), respectively.

$\mathrm{PBD}(90, \{39, 42\}, 36)$, $\mathrm{PBD}(90, \{39, 48\}, 42)$, $\mathrm{PBD}(90, \{42, 51\}, 48)$ and $\mathrm{PBD}(90, \{48, 51\}, 54)$

$$\{1\} \lhd C_5 \lhd C_2 \times C_5 \lhd C_2 \times C_5 \times C_9$$

r	81	87	93	99
# orbit matrices for $C_2 \times C_5$	362	361	356	363
# orbit matrices for C_5	16232	15331	16536	15330

Thank you for your attention!