Equidistant walks to infinity

Biserka Kolarec

University of Zagreb, Croatia

4th Croatian Combinatorial Days
Zagreb, September 22-23, 2022.

Equidistant walks

Arithmetical progressions can be considered as **equidistant walks to infinity** with a common difference d being "the length of the step":

$$a, a+d, (a+d)+d, \dots$$

Equidistant walks

Arithmetical progressions can be considered as **equidistant walks to infinity** with a common difference d being "the length of the step":

$$a, a+d, (a+d)+d, \ldots$$

Example

Equidistant walks with the length of the step equal to 2:

$$1, 3, 5, 7, 9, \ldots$$

$$2, 4, 6, 8, 10, \ldots$$

Grids of equidistant walks

A grid of odd numbers and a grid of alternate columns of odd and even numbers, respectively:

$$
\begin{array}{cccc}
1 & 3 & 5 & \cdots \\
3 & 5 & 7 & \cdots \\
5 & 7 & 9 & \cdots \\
7 & 9 & 11 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\qquad
\begin{array}{ccccc}
1 & 2 & 3 & 4 & \cdots \\
3 & 4 & 5 & 6 & \cdots \\
5 & 6 & 7 & 8 & \cdots \\
7 & 8 & 9 & 10 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Walks over grids

Given natural numbers R, D an (R, D)-**walk** is a walk that starts anywhere in a grid: R gives a move to the right and D a move down to get from one number of the walk to another.

Walks over grids

Given natural numbers R, D an (R, D)-**walk** is a walk that starts anywhere in a grid: R gives a move to the right and D a move down to get from one number of the walk to another.
There are:

- $(R, 0)$-walks or horizontal walks
- $(0, D)$-walks or vertical walks
- (R, D)-walks or diagonal walks (including a walk in a direction of the main diagonal $R = D$)

Walks over grids

Given natural numbers R, D an (R, D)-**walk** is a walk that starts anywhere in a grid: R gives a move to the right and D a move down to get from one number of the walk to another.
There are:

- $(R, 0)$-walks or horizontal walks
- $(0, D)$-walks or vertical walks
- (R, D)-walks or diagonal walks (including a walk in a direction of the main diagonal $R = D$)

A grid is called **equidistant** if all walks in it are equidistant.

Equidistant grids made of equidistant walks of odd and even numbers

1	6
3	8
5	10
7	12
9	14
11	16
$\vdots$	$\vdots$

Equidistant grids made of equidistant walks of odd and even numbers

$$\begin{matrix} 1 & 6 \\ 3 & 8 \\ 5 & 10 \\ 7 & 12 \\ 9 & 14 \\ 11 & 16 \\ \vdots & \vdots \end{matrix} \longrightarrow \begin{matrix} 1 & 6 & 11 & 16 & \cdots \\ 3 & 8 & 13 & 18 & \cdots \\ 5 & 10 & 15 & 20 & \cdots \\ 7 & 12 & 17 & 22 & \cdots \\ 9 & 14 & 19 & 24 & \cdots \\ 11 & 16 & 21 & 26 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Equidistant grids made of alternate equidistant walks

Let us have two equidistant walks that start with a_1 and a_1' and have lengths of the step equal to d, d', respectively. Let $s = a_1' - a_1$, k be the shift up. One can form a grid:

$$\begin{array}{ccccc}
a_1 & a_1' & a_1 + kd & a_1' + kd' & \cdots \\
a_1 + d & a_1' + d' & a_1 + d(k+1) & a_1' + d'(k+1) & \cdots \\
a_1 + 2d & a_1' + 2d' & a_1 + d(k+2) & a_1' + d'(k+2) & \cdots \\
a_1 + 3d & a_1' + 3d' & a_1 + d(k+3) & a_1' + d'(k+3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Equidistant grids made of alternate equidistant walks

Let us have two equidistant walks that start with a_1 and a_1' and have lengths of the step equal to d, d', respectively. Let $s = a_1' - a_1$, k be the shift up. One can form a grid:

$$\begin{array}{ccccc}
a_1 & a_1' & a_1 + kd & a_1' + kd' & \cdots \\
a_1 + d & a_1' + d' & a_1 + d(k+1) & a_1' + d'(k+1) & \cdots \\
a_1 + 2d & a_1' + 2d' & a_1 + d(k+2) & a_1' + d'(k+2) & \cdots \\
a_1 + 3d & a_1' + 3d' & a_1 + d(k+3) & a_1' + d'(k+3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

When is this grid an equidistant grid?

Equidistant grids made of alternate equidistant walks

Let us have two equidistant walks that start with a_1 and a_1' and have lengths of the step equal to d, d', respectively. Let $s = a_1' - a_1$, k be the shift up. One can form a grid:

$$\begin{array}{ccccc}
a_1 & a_1' & a_1 + kd & a_1' + kd' & \cdots \\
a_1 + d & a_1' + d' & a_1 + d(k+1) & a_1' + d'(k+1) & \cdots \\
a_1 + 2d & a_1' + 2d' & a_1 + d(k+2) & a_1' + d'(k+2) & \cdots \\
a_1 + 3d & a_1' + 3d' & a_1 + d(k+3) & a_1' + d'(k+3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

When is this grid an equidistant grid?

- $a_1 + kd - a_1' = s$ iff $kd = 2s$

Equidistant grids made of alternate equidistant walks

Let us have two equidistant walks that start with a_1 and a_1' and have lengths of the step equal to d, d', respectively. Let $s = a_1' - a_1$, k be the shift up. One can form a grid:

$$\begin{array}{ccccc}
a_1 & a_1' & a_1 + kd & a_1' + kd' & \cdots \\
a_1 + d & a_1' + d' & a_1 + d(k+1) & a_1' + d'(k+1) & \cdots \\
a_1 + 2d & a_1' + 2d' & a_1 + d(k+2) & a_1' + d'(k+2) & \cdots \\
a_1 + 3d & a_1' + 3d' & a_1 + d(k+3) & a_1' + d'(k+3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

When is this grid an equidistant grid?

- $a_1 + kd - a_1' = s$ iff $kd = 2s$
- $a_1' + kd' - (a_1 + kd) = s$ iff $d = d'$

The number of equidistant grids for prime s

The condition $dk = 2s$ relates the length of the step d of equidistant walks in columns, the column shift k and the difference s of initial terms.

For prime s, $s \neq 2$, there are four possible cases for d and k:

The number of equidistant grids for prime s

The condition $dk = 2s$ relates the length of the step d of equidistant walks in columns, the column shift k and the difference s of initial terms.

For prime s, $s \neq 2$, there are four possible cases for d and k:

- Cases $d = 1, k = 2s$ and $d = 2s, k = 1$

The number of equidistant grids for prime s

The condition $dk = 2s$ relates the length of the step d of equidistant walks in columns, the column shift k and the difference s of initial terms.

For prime s, $s \neq 2$, there are four possible cases for d and k:

- Cases $d = 1, k = 2s$ and $d = 2s, k = 1$
- The case $d = 2, k = s$ gives already treated equidistant grids:

$$\begin{array}{cccc} a_1 & a_1 + s & a_1 + 2s & \cdots \\ a_1 + 2 & a_1 + s + 2 & a_1 + 2s + 2 & \cdots \\ a_1 + 4 & a_1 + s + 4 & a_1 + 2s + 4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

The number of equidistant grids for prime s

The condition $dk = 2s$ relates the length of the step d of equidistant walks in columns, the column shift k and the difference s of initial terms.

For prime s, $s \neq 2$, there are four possible cases for d and k:

- Cases $d = 1, k = 2s$ and $d = 2s, k = 1$
- The case $d = 2, k = s$ gives already treated equidistant grids:

$$\begin{array}{cccc} a_1 & a_1 + s & a_1 + 2s & \cdots \\ a_1 + 2 & a_1 + s + 2 & a_1 + 2s + 2 & \cdots \\ a_1 + 4 & a_1 + s + 4 & a_1 + 2s + 4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

- The case $d = s, k = 2$ gives the equidistant grid of a symmetrical layout:

$$\begin{array}{cccc} a_1 & a_1 + s & a_1 + 2s & \cdots \\ a_1 + s & a_1 + 2s & a_1 + 3s & \cdots \\ a_1 + 2s & a_1 + 3s & a_1 + 4s & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

The number of equidistant grids for composite s

The number of all ordered pairs (d, k) such that $dk = 2s$ equals the number $D(2s)$ of all divisors of $2s$.

The number of equidistant grids for composite s

The number of all ordered pairs (d, k) such that $dk = 2s$ equals the number $D(2s)$ of all divisors of $2s$.
Indeed, we can arrange prime factors of $2s$ to an increasing sequence: $p_1, p_2, \ldots, p_{l-1}, p_l$ with $l = D(2s)$, $p_1 = 1$ and $p_l = 2s$.
Then pairs of (d, k) are given by:

- $(p_1, p_l), (p_2 p_{l-1}), \ldots$ and
- $(p_l, p_1), (p_{l-1}, p_2), \ldots$

The number of equidistant grids for composite s

The number of all ordered pairs (d, k) such that $dk = 2s$ equals the number $D(2s)$ of all divisors of $2s$.
Indeed, we can arrange prime factors of $2s$ to an increasing sequence: $p_1, p_2, \ldots, p_{l-1}, p_l$ with $l = D(2s)$, $p_1 = 1$ and $p_l = 2s$.
Then pairs of (d, k) are given by:

- $(p_1, p_l), (p_2 p_{l-1}), \ldots$ and
- $(p_l, p_1), (p_{l-1}, p_2), \ldots$

Remark: The sequence $D(n)$ can be found in the On-Line Encyclopedia of Integer Sequences with the label A000005.

List of all equidistant grids for $s = 5$, $a_1 = 1$

There are four divisors of $2s = 10$: $1, 2, 5, 10$. There are also four possible choices for (d, k): $(1, 10), (10, 1), (2, 5)$ and $(5, 2)$.

$$\begin{matrix} 1 & 6 & 11 & 16 & \cdots \\ 2 & 7 & 12 & 17 & \cdots \\ 3 & 8 & 13 & 18 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}, \quad \begin{matrix} 1 & 6 & 11 & 16 & \cdots \\ 11 & 16 & 21 & 26 & \cdots \\ 21 & 26 & 31 & 36 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix},$$

$$\begin{matrix} 1 & 6 & 11 & 16 & \cdots \\ 3 & 8 & 13 & 18 & \cdots \\ 5 & 10 & 15 & 20 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}, \quad \begin{matrix} 1 & 6 & 11 & 16 & \cdots \\ 6 & 11 & 16 & 21 & \cdots \\ 11 & 16 & 21 & 26 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}.$$

List of all equidistant grids for $s = 6$, $a_1 = 1$

There are six divisors of $2s = 12$: $1, 2, 3, 4, 6, 12$ and six choices for (d, k): $(1, 12), (12, 1), (2, 6), (6, 2), (3, 4)$ and $(4, 3)$.

$$\begin{matrix} 1 & 7 & 13 & 19 & \cdots \\ 2 & 8 & 14 & 20 & \cdots \\ 3 & 9 & 15 & 21 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}, \quad \begin{matrix} 1 & 7 & 13 & 19 & \cdots \\ 13 & 19 & 25 & 28 & \cdots \\ 25 & 31 & 37 & 40 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix},$$

$$\begin{matrix} 1 & 7 & 13 & 19 & \cdots \\ 3 & 9 & 15 & 21 & \cdots \\ 5 & 11 & 17 & 23 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}, \quad \begin{matrix} 1 & 7 & 13 & 19 & \cdots \\ 7 & 13 & 19 & 25 & \cdots \\ 13 & 19 & 25 & 31 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix},$$

$$\begin{matrix} 1 & 7 & 13 & 19 & \cdots \\ 4 & 10 & 16 & 22 & \cdots \\ 7 & 13 & 19 & 25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}, \quad \begin{matrix} 1 & 7 & 13 & 19 & \cdots \\ 5 & 11 & 17 & 23 & \cdots \\ 9 & 15 & 21 & 27 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}.$$

Zigzag walks over two equidistant walks

The following walks are of interest:

$$
\begin{array}{ccc}
a_1 & & a'_1 \\
a_1 + d & & a'_1 + d \\
\vdots & \searrow & \vdots \\
a_1 + rd & & a'_1 + rd \\
\vdots & \swarrow & \vdots \\
a_1 + (r+l)d & & a'_1 + (r+l)d \\
\vdots & \searrow & \vdots \\
a_1 + (2r+l)d & & a'_1 + (2r+l)d \\
\vdots & & \vdots
\end{array}
$$

Zigzag walks over two equidistant walks

The following walks are of interest:

$$
\begin{array}{ccc}
\color{red}{a_1} & & a_1' \\
a_1 + d & & a_1' + d \\
\vdots & \color{red}{\searrow} & \vdots \\
a_1 + rd & & \color{red}{a_1' + rd} \\
\vdots & \color{red}{\swarrow} & \vdots \\
\color{red}{a_1 + (r + l)d} & & a_1' + (r + l)d \\
\vdots & \color{red}{\searrow} & \vdots \\
a_1 + (2r + l)d & & \color{red}{a_1' + (2r + l)d} \\
\vdots & & \vdots
\end{array}
$$

When is a zigzag walk equidistant? Let $s = a_1' - a_1$ and assume $s > 0$. The zigzag walk is equidistant if and only if $2s = (l - r)d$. Consequently $l - r = \frac{2s}{d}$, so d must be a divisor of $2s$.

Zigzag walks placed into an equidistant grid

One can obtain an eqidistant grid in which given zigzag walk appears in the first row:

$$\begin{array}{cccccc} a_1 & a_1' + rd & a_1 + (r+l)d & a_1' + (2r+l)d & a_1 + (2r+2l)d & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Walks over products with overlapping factors

For given natural number n let us look at the equidistant walk over odd numbers:

$$2n-1, 2n+1, 2n+3, \ldots$$

Observe **products with overlapping factors** and form a walk:

$$(2n-1)(2n+1), (2n+1)(2n+3), (2n+3)(2n+5), \ldots$$

This walk is not equidistant.

Equidistant walks over products with overlapping odd and even factors

Look at equidistant walks of differences of two by two successive products with overlapping odd and even factors:

$$\begin{array}{lclc}
(2n-1)(2n+1)\nwarrow & & 2n(2n+2)\quad\nwarrow & \\
 & 8n+4 & & 8n+8 \\
(2n+1)(2n+3)\nwarrow\!\!\!\!\!\nearrow & & (2n+2)(2n+4)\nwarrow\!\!\!\!\!\nearrow & \\
 & 8n+12 & & 8n+16 \\
(2n+3)(2n+5)\nearrow & & (2n+4)(2n+6)\nearrow & \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Distance between equidistant walks over products with overlapping odd and even factors (1)

One can also look at distances between differences of products with overlapping factors with odd and even numbers after shifting the second by k ($k > 0$ denotes shifts up, $k < 0$ shifts down by $|k|$; here $k \geq 1 - n$).

$$\begin{array}{cc}
(2n-1)(2n+1) & (2n+2k)(2n+2(k+1)) \\
(2n+1)(2n+3) & (2n+2(k+1))(2n+2(k+2)) \\
\vdots & \vdots
\end{array}$$

Distance between equidistant walks over products with overlapping factors (2)

For the distance of differences of products with overlapping factors of odd numbers and products with overlapping factors of even numbers shifted by k the following holds:

Theorem

For a natural number n and an integer $k \geq 1 - n$,

$$[(2n+2(k+1))(2n+2(k+2)) - (2n+2k)(2n+2(k+1))] - \\ -[(2n+1)(2n+3) - (2n-1)(2n+1)] = 8k+4.$$

Distance between equidistant walks over products with overlapping odd and even factors (3)

- The notion of products of overlaping factors can be generalized: we can speak of products of n consecutive odd or even numbers that overlapp in m factors, $m < n$ i. e. of m-**overlapping** n-**products**.

Distance between equidistant walks over products with overlapping odd and even factors (3)

- The notion of products of overlaping factors can be generalized: we can speak of products of n consecutive odd or even numbers that overlapp in m factors, $m < n$ i. e. of m-**overlapping** n-**products**.
- In this notation the above overlapping products are 1-overlapping 2-products.

Distance between equidistant walks over products with overlapping odd and even factors (3)

- The notion of products of overlaping factors can be generalized: we can speak of products of n consecutive odd or even numbers that overlapp in m factors, $m < n$ i. e. of m-**overlapping** n-**products**.
- In this notation the above overlapping products are 1-overlapping 2-products.
- In general, if we have progression of differences of two succesive m-overlapping n-products of even numbers and progression of differences of two succesive m-overlapping n-products of odd numbers, their distance is constant.

THANK YOU FOR YOUR ATTENTION