Edge-coloring (sub)cubic graphs with 5 colors

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Proper edge-coloring

- Adjacent edges receive distinct colors;
- Edges of every color form a matching;
- The smallest k for which a graph G admits an edge-coloring with k colors is the **chromatic index** of G, $\chi'(G)$;
- By Vizing's theorem [15], for every subcubic graph G it holds

$$3 \leq \chi'(G) \leq 4$$

(subcubic graphs being the graphs with max. degree 3)

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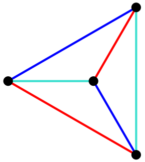

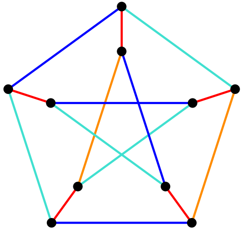

Suppose that we have **5** colors.

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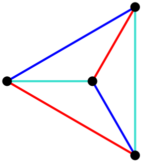
What can we do with an extra color?

Acyclic edge-coloring

- Proper edge-coloring with no bichromatic cycle;



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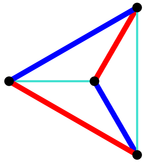

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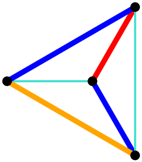

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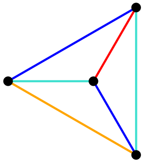

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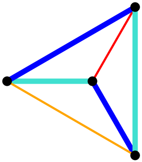

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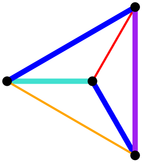

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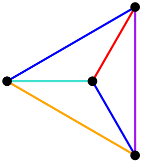

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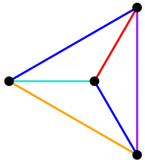

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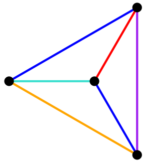

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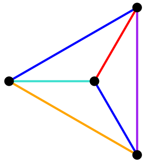

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- Due to Burnstein (1979) [6]:
  For every subcubic graph G we have $\chi'_a(G) \leq 5$.
- In fact, by Andersen, Máčajová & Mazák (2012) [2]:
  If G is not K_4 or $K_{3,3}$, then we have $\chi'_a(G) \leq 4$.

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- Andersen (1992) [1], and Horák, Qing, and Trotter (1993) [9]:
  For every subcubic graph G, we have

$$\chi'_s(G) \leq 10\,.$$

5 colors for strong edge-coloring?

Theorem 1 (BL, Máčajová, Škoviera & Soták [11])

A cubic graph G is a cover of the Petersen graph if and only if

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A graph G *covers* a graph H if there is a graph homomorphism from G to H that is locally bijective.

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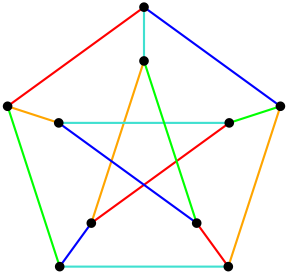

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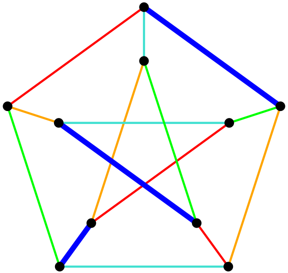

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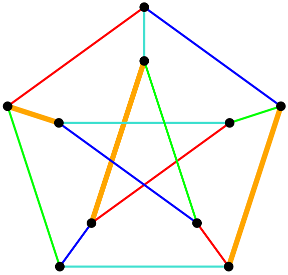

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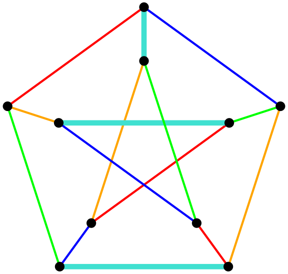

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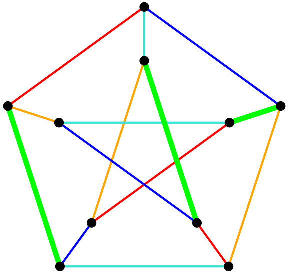

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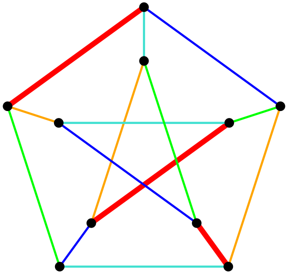

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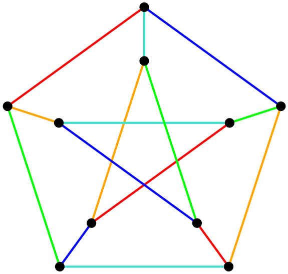
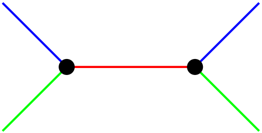


Normal edge-coloring

- Proper edge-coloring in which for every edge *uv* the number of distinct colors on the edges adjacent to *uv* together with the color of *uv* is either 3 (poor edge) or 5 (rich edge);
- Poor edge:



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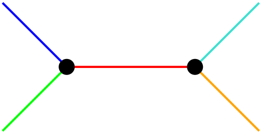

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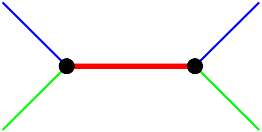

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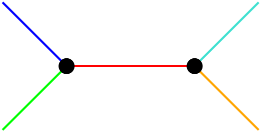

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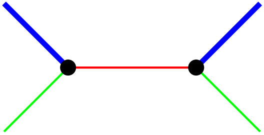

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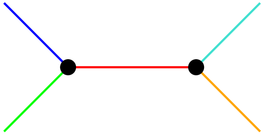

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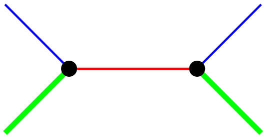

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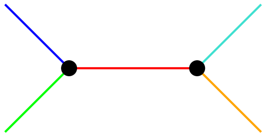

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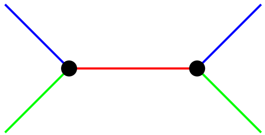

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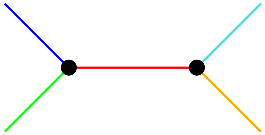

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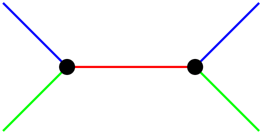

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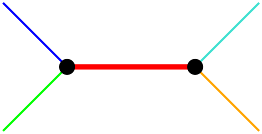

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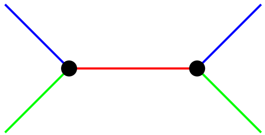

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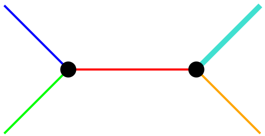

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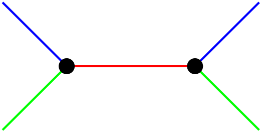

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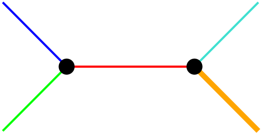

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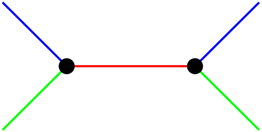

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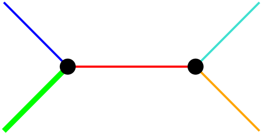

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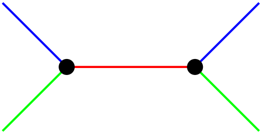

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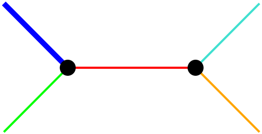

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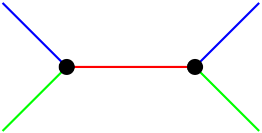

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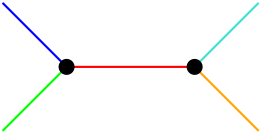

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Conjecture 2 (Jaeger [10])

Every bridgeless cubic graph admits a normal edge-coloring with at most 5 *colors.*

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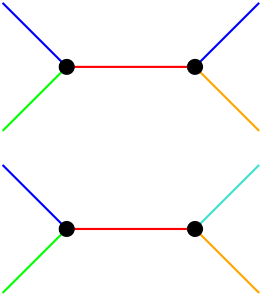
Every bridgeless cubic graph admits a normal edge-coloring with at most 5 colors.

- Equivalent to the Petersen Coloring Conjecture;
- Mazzuoccolo & Mkrtchyan (2018) [12]:
  Every subcubic graph admits a normal 7-edge-coloring.

Adjacent vertex-distinguishing edge-coloring

- Proper edge-coloring where **adjacent vertices meet different sets of colors**;
  → Each edge sees 4 or 5 colors;



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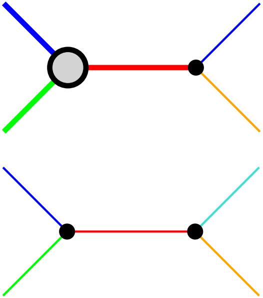

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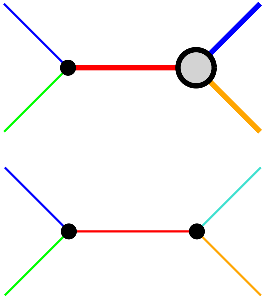

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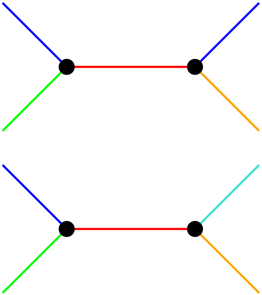

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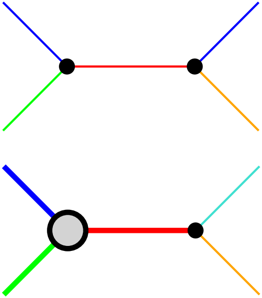

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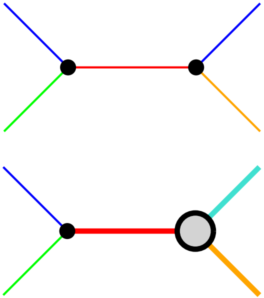
- Proper edge-coloring where **adjacent vertices meet different sets of colors**;
  → Each edge sees 4 or 5 colors;



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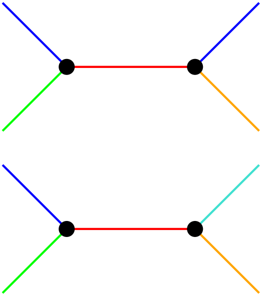

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Adjacent vertex-distinguishing edge-coloring

Theorem 3 (Balister, Györi, Lehel & Schelp [3])

For every subcubic graph G without isolated edges we have

$$\chi'_{\mathrm{avd}}(G) \leq 5 .$$

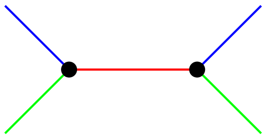
- Also known as neighbor-distinguishing edge-coloring;
- Equivalent to 2-intersection edge-coloring for cubic graphs [5];
- Two natural generalizations...

k-distance vertex-distinguishing edge-coloring

- Proper edge-coloring where **vertices at distance** **exactly** k **meet different sets of colors**;

k-distance vertex-distinguishing edge-coloring

- Proper edge-coloring where **vertices at distance exactly** k **meet different sets of colors**;
- We are interested in 2-distance vertex-distinguishing edge-coloring $\rightarrow$ and the corresponding invariant $\chi'_{2dd}(G)$;
- It means, it is possible to have poor edges:



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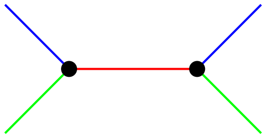
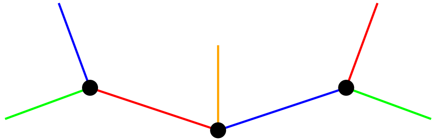


But it is forbidden to have:



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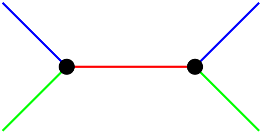

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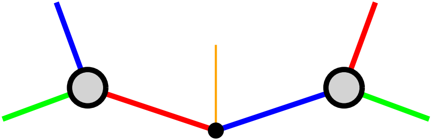

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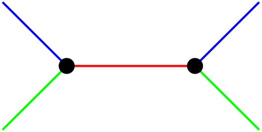

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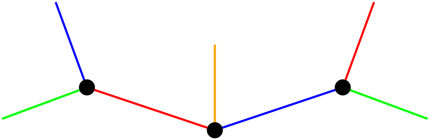

2-distance vertex-distinguishing edge-coloring

- Victor et al. [14]:
  For every subcubic graph G, it holds $\chi'_{2dd}(G) \leq 6$;

2-distance vertex-distinguishing edge-coloring

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For every subcubic graph G, it holds that

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- If true, also tight;
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  for every subcubic graph G with $\mathrm{mad}(G) < \frac{8}{3}$;

2-distance vertex-distinguishing edge-coloring

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- If true, also tight;
- Confirmed by Victor et al. [13]:
  for every subcubic graph G with $\mathrm{mad}(G) < \frac{8}{3}$;
- Not much is known for cubic graphs in general;

k-strong edge-coloring

- Proper edge-coloring where **vertices at distance at most k meet different sets of colors**;

k-strong edge-coloring

- Proper edge-coloring where **vertices at distance** **at most** k **meet different sets of colors**;
- We are interested in 2-strong edge-coloring
  $\rightarrow$ and the corresponding invariant $\chi'_{2s}(G)$;

k-strong edge-coloring

- Proper edge-coloring where **vertices at distance at most** k **meet different sets of colors**;
- We are interested in 2-strong edge-coloring
  $\rightarrow$ and the corresponding invariant $\chi'_{2s}(G)$;
- We are not aware of any results for (sub)cubic graphs
  $\rightarrow$ Just $\chi'_{2s}(G) \leq \chi'_{s}(G)$;

k-strong edge-coloring

- Proper edge-coloring where **vertices at distance at most k meet different sets of colors**;
- We are interested in 2-strong edge-coloring
  $\rightarrow$ and the corresponding invariant $\chi'_{2\mathrm{s}}(G)$;
- We are not aware of any results for (sub)cubic graphs
  $\rightarrow$ Just $\chi'_{2\mathrm{s}}(G) \leq \chi'_s(G)$;

Conjecture 5 (Holub, BL, Mihaliková, Mockovčiaková & Soták)

For every subcubic graph G on at least 9 vertices, it holds.

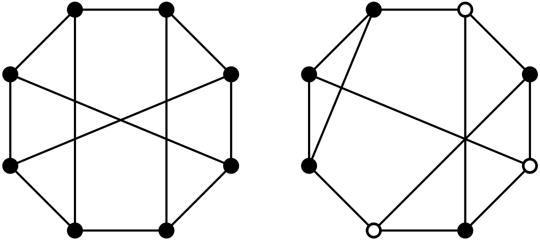
$$\chi'_{2\mathrm{s}}(G) \leq 5\,.$$

2-strong edge-coloring

- Why at least 9 vertices?

2-strong edge-coloring

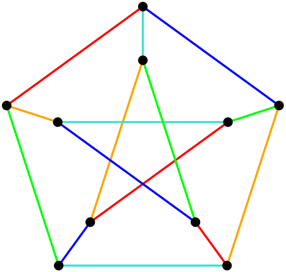
- Why at least 9 vertices?
- There are two cubic graphs on 8 vertices of diameter 2:



- There are $\binom{5}{3} = 10$ possible triples of colors; every color appears in 6 of them; every edge contributes its color twice; we need every color to appear on three edges; we need 8 distinct triples; but at least 4 distinct colors are missing; four colors appear only on two edges (and one on three); we need to color 12 edges; a contradiction.

2-strong edge-coloring

- Verified by computer that the conjecture holds for small instances;
- For the Petersen graph there is only one coloring:



- Open even for bipartite cubic graphs with large girth;

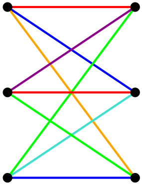
Star edge-coloring

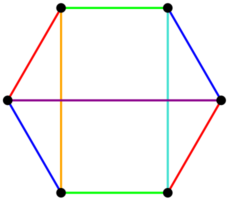
- Proper edge-coloring in which **there are no bichromatic 4-cycles nor 4-paths**;

Conjecture 6 (Dvořák, Mohar & Šámal [7])

Every subcubic graph admits a star edge-coloring with at most 6 colors.

- Only known to be tight for three simple bridgeless graphs $\rightarrow$ two cubic;





Star edge-coloring

- Dvořák, Mohar & Šámal (2013) [7] proved 7 colors suffice;

Question 7

Does every bridgeless cubic graph, distinct from $K_{3,3}$ and $\overline{C_6}$, admit a star 5-edge-coloring?

- True for outerplanar subcubic graphs [4];
- Implies result on 2-distance vertex-distinguishing edge-coloring since $\chi'_{\mathrm{2dd}}(G) \leq \chi'_{\mathrm{st}}(G)$;

To conclude...

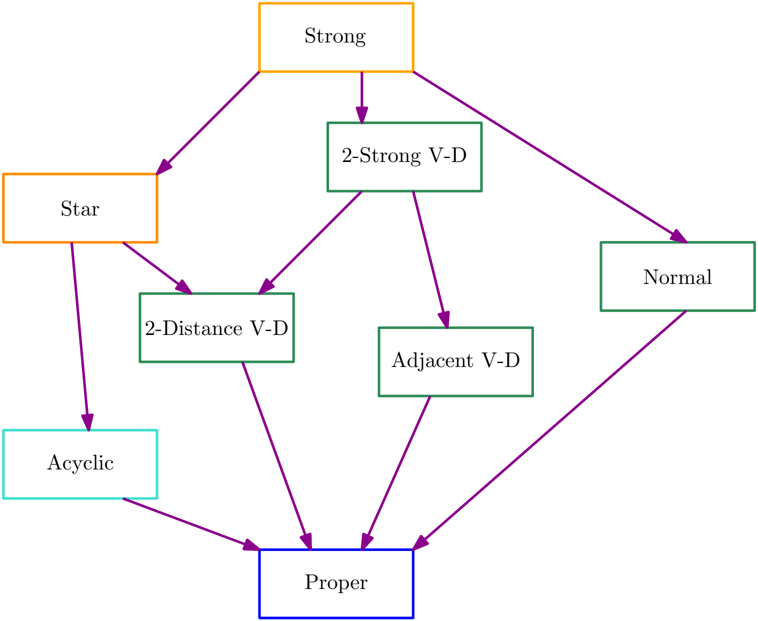
- A set of edges is a **k-packing** if every pair of edges is at distance at least $k+1$;
- For a non-decreasing sequence of positive integers, $S = (s_1, \ldots, s_\ell)$, Gastineau and Togni (2019) [8], defined an **S-packing edge-coloring** of G as a partition of the edge set of G into ℓ subsets $\{X_1, \ldots, X_\ell\}$ such that each X_i is an s_i-packing;

Conjecture 8 (Hocquard, Lajou & BL (2020^+))

Every subcubic planar graph is $(1,1,2,2,2)$-packing edge-colorable.

Question 9 (Hocquard, Lajou & BL (2020^+))

Is it true that every bipartite subcubic graph of large enough girth admits a $(1,2,2,2,2)$-packing edge-coloring?


Strong
2-Strong V-D
Star
Normal
2-Distance V-D
Adjacent V-D
Acyclic
Proper

Thank you!

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