On some constructions of strongly regular graphs

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joint work with Dean Crnković

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Introduction

Behbahani, Lam

Introduction

Behbahani, Lam ↔ Construction of SRG with prime order automorphism group

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Behbahani, Lam ↔ Construction of SRG with prime order automorphism group

Crnković, MM

Introduction

Behbahani, Lam ⟺ Construction of SRG with prime order automorphism group

Crnković, MM ⟺ Construction of SRG with composite order automorphism group

Introduction

- M. Behbahani, C. Lam, Strongly regular graphs with non-trivial automorphisms, Discrete Math. 311 (2011), 132-144.
- D. Crnković, M. Maksimović, Construction of strongly regular graphs having an automorphism group of composite order, Contributions to Discrete Mathematics(1715-0868) 15 (2020), 1; 22-41

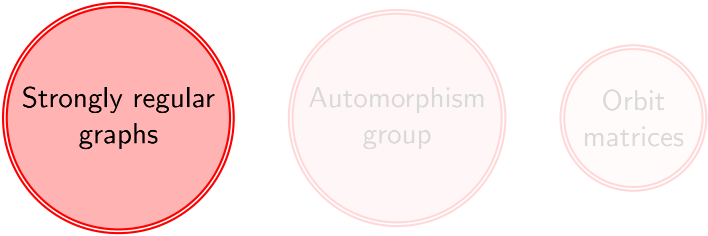
Definitions

Strongly regular graphs

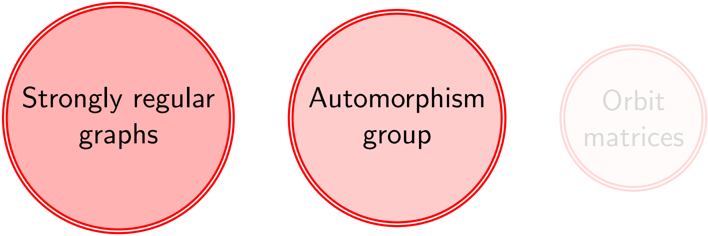
Automorphism group

Orbit matrices

Definitions



Definitions



Definitions

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Automorphism group

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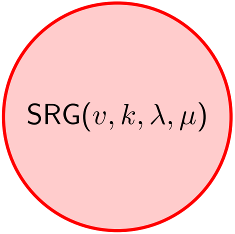
Strongly regular graphs



Strongly regular graphs

SRG(v, k, λ, μ)

v-number of vertices

Strongly regular graphs

SRG(v, k, λ, μ)

v-number of vertices

k-valency

Strongly regular graphs

SRG(v, k, λ, μ)

v-number of vertices

k-valency

neighbours have λ common neighbours

Strongly regular graphs

SRG(v, k, λ, μ)

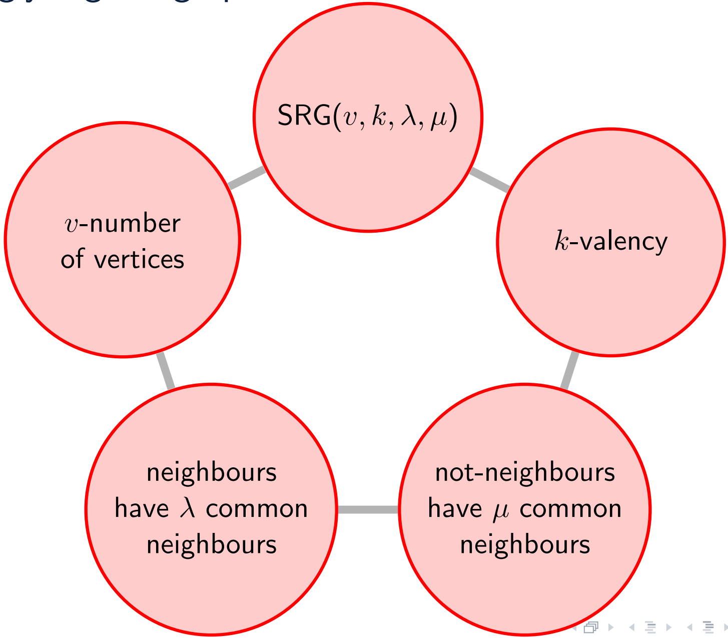
v-number of vertices

k-valency

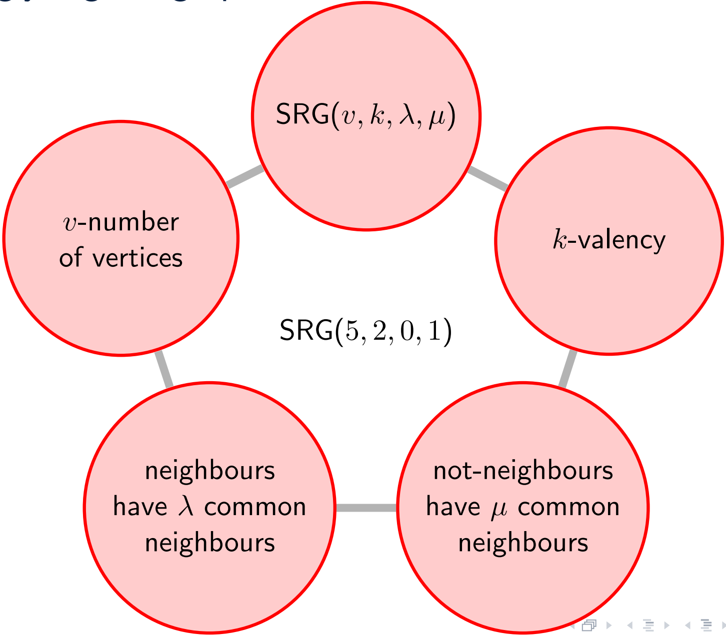
neighbours have λ common neighbours

not-neighbours have μ common neighbours

Strongly regular graphs

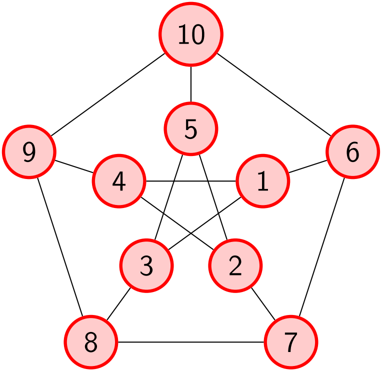


Strongly regular graphs



Adjacency matrix

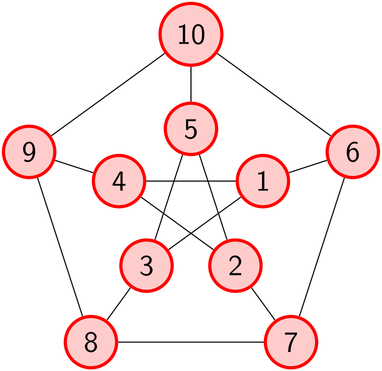
SRG(10,3,0,1)



$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Adjacency matrix

SRG(10,3,0,1)



$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Automorphism group

An automorphism ρ of strongly regular graph Γ is a permutation on the vertices of a graph Γ such that for any two vertices of Γ u and v follows that: u and v are adjacent in Γ if and only if ρu and ρv are adjacent in Γ. Set of all automorphisms of strongly regular graph under the composition of functions forms a group that we call full automorphism group and denote $\mathrm{Aut}(\Gamma)$.

Orbit matrix

Example

Let an automorphism group G generated with element $\rho = (1)(3,4,6)(2,7,8,9,10,5)$ partitions the set of vertices of Petersen graph into orbits $O_1 = \{1\}, O_2 = \{3,4,6\}, O_3 = \{2,5,7,8,9,10\}$.

Orbit matrix

$O_1 = \{1\}, O_2 = \{3,4,6\}, O_3 = \{2,5,7,8,9,10\}, n_1 = 1, n_2 = 3, n_3 = 6$

	1	2	3	4	5	6	7	8	9	10
1	0	0	1	1	0	1	0	0	0	0
2	0	0	0	1	1	0	1	0	0	0
3	1	0	0	0	1	0	0	1	0	0
4	1	1	0	0	0	0	0	0	1	0
5	0	1	1	0	0	0	0	0	0	1
6	1	0	0	0	0	0	1	0	0	1
7	0	1	0	0	0	1	0	1	0	0
8	0	0	1	0	0	0	1	0	1	0
9	0	0	0	1	0	0	0	1	0	1
10	0	0	0	0	1	1	0	0	1	0

	1	3	4	6	2	5	7	8	9	10
1	0	1	1	1	0	0	0	0	0	0
3	1	0	0	0	0	1	0	1	0	0
4	1	0	0	0	1	0	0	0	1	0
6	1	0	0	0	0	0	1	0	0	1
2	0	0	1	0	0	1	1	0	0	0
5	0	1	0	0	1	0	0	0	0	1
7	0	0	0	1	1	0	0	1	0	0
8	0	1	0	0	0	0	1	0	1	0
9	0	0	1	0	0	0	0	1	0	1
10	0	0	0	1	0	1	0	0	1	0

Orbit matrix

$O_1 = \{1\}, O_2 = \{3,4,6\}, O_3 = \{2,5,7,8,9,10\}, n_1 = 1, n_2 = 3, n_3 = 6$

	1	2	3	4	5	6	7	8	9	10
1	0	0	1	1	0	1	0	0	0	0
2	0	0	0	1	1	0	1	0	0	0
3	1	0	0	0	1	0	0	1	0	0
4	1	1	0	0	0	0	0	0	1	0
5	0	1	1	0	0	0	0	0	0	1
6	1	0	0	0	0	0	1	0	0	1
7	0	1	0	0	0	1	0	1	0	0
8	0	0	1	0	0	0	1	0	1	0
9	0	0	0	1	0	0	0	1	0	1
10	0	0	0	0	1	1	0	0	1	0

	1	3	4	6	2	5	7	8	9	10
1	0	1	1	1	0	0	0	0	0	0
3	1	0	0	0	0	1	0	1	0	0
4	1	0	0	0	1	0	0	0	1	0
6	1	0	0	0	0	0	1	0	0	1
2	0	0	1	0	0	1	1	0	0	0
5	0	1	0	0	1	0	0	0	0	1
7	0	0	0	1	1	0	0	1	0	0
8	0	1	0	0	0	0	1	0	1	0
9	0	0	1	0	0	0	0	1	0	1
10	0	0	0	1	0	1	0	0	1	0

Column orbit matrix

	1	3	4	6	2	5	7	8	9	10
1	0	1	1	1	0	0	0	0	0	0
3	1	0	0	0	0	1	0	1	0	0
4	1	0	0	0	1	0	0	0	1	0
6	1	0	0	0	0	0	1	0	0	1
2	0	0	1	0	0	1	1	0	0	0
5	0	1	0	0	1	0	0	0	0	1
7	0	0	0	1	1	0	0	1	0	0
8	0	1	0	0	0	0	1	0	1	0
9	0	0	1	0	0	0	0	1	0	1
10	0	0	0	1	0	1	0	0	1	0

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

Column orbit matrices

Definition

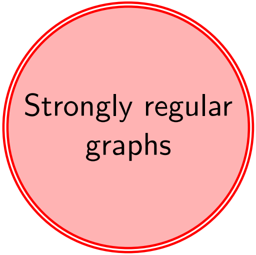
A $(b \times b)$-matrix $C = [c_{ij}]$ with entries satisfying conditions:

$$\sum_{i=1}^{b} c_{ij} = \sum_{j=1}^{b} \frac{n_j}{n_i} c_{ij} = k \tag{1}$$

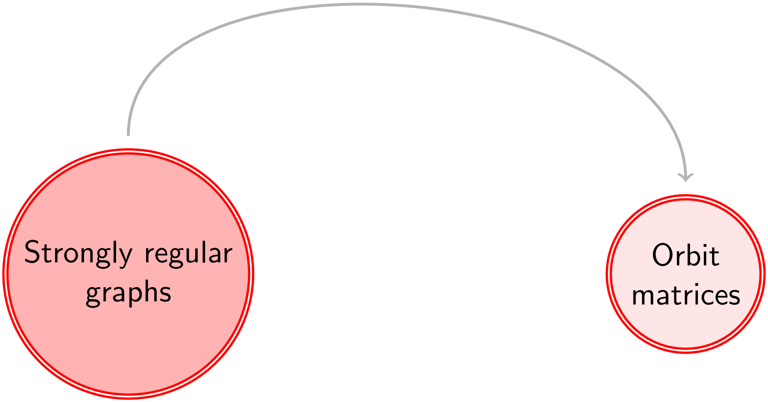
$$\sum_{s=1}^{b} \frac{n_s}{n_j} c_{is} c_{js} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda - \mu) c_{ij} \tag{2}$$

where $0 \leq c_{ij} \leq n_i$, $0 \leq c_{ii} \leq n_i - 1$ and $\sum_{i=1}^{b} n_i = v$, is called a **column orbit matrix** for a strongly regular graph with parameters (v, k, λ, μ) and the orbit lengths distribution $(n_1, \ldots, n_b)$.

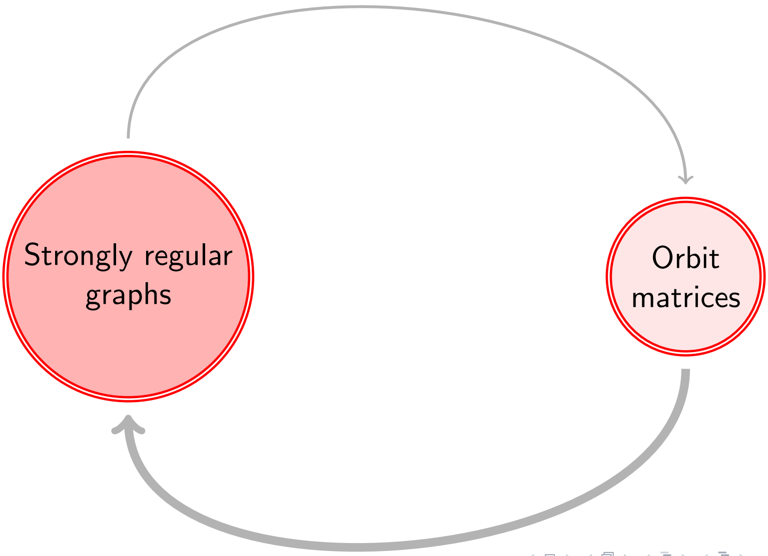
Construction



Construction



Construction



Construction

Check parameters

Construction

Check parameters

↓

Assume aut. group G
having a composition series
$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$

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Find orbit length
distribution and prototypes

Construction

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Find orbit length
distribution and prototypes

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Construct OM
for the group G

Construction

Check parameters

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Find orbit length
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Construct OM
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Construct OM
for H_{n-1}

Construction

Check parameters

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Assume aut. group G
having a composition series
$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$

↓

Find orbit length
distribution and prototypes

↓

Construct OM
for the group G

Construct OM
for H_{n-1}

⋮

Construct OM
for H_1

Construction

Check parameters

↓

Assume aut. group G having a composition series $\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$

↓

Find orbit length distribution and prototypes

↓

Construct OM for the group G

Construct OM for H_{n-1}

⋮

Construct OM for H_1

↓

Construct OM for $H_0 = \{1\}$, i.e. AM of SRG

Construction

Check parameters

↓

Assume aut. group G having a composition series $\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$

↓

Find orbit length distribution and prototypes

↓

Construct OM for the group G

Construct OM for H_{n-1}

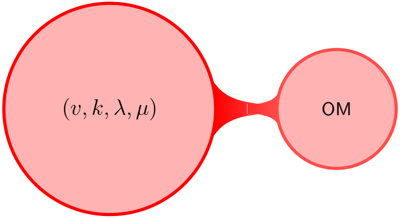
$\vdots$

Construct OM for H_1

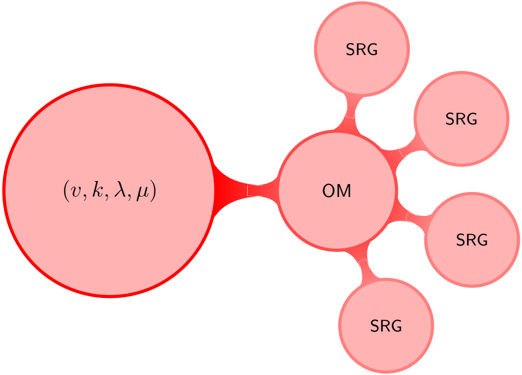
↓

Construct OM for $H_0 = \{1\}$, i.e. AM of SRG

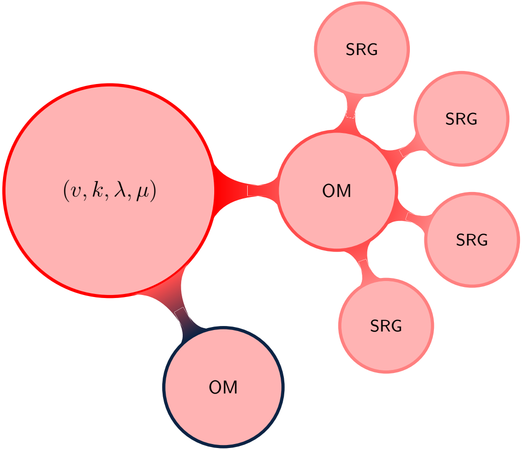
Orbit matrices



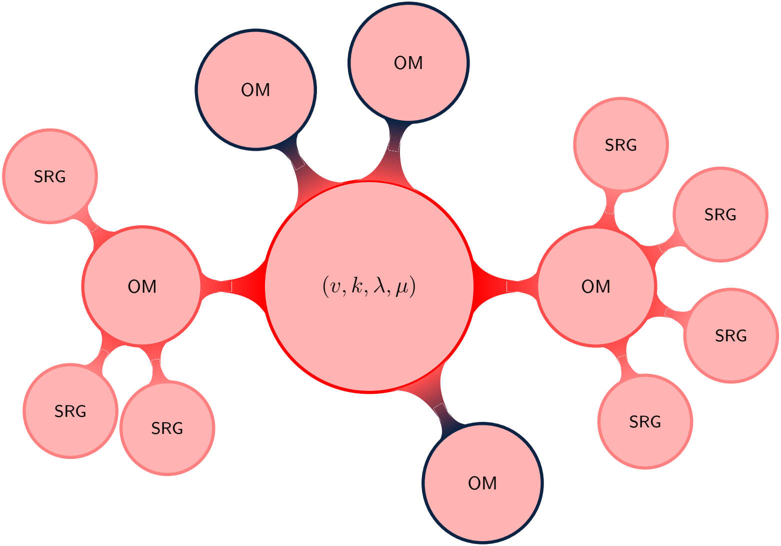
Orbit matrices



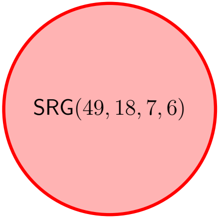
Orbit matrices



Orbit matrices



SRG$(49, 18, 7, 6)$



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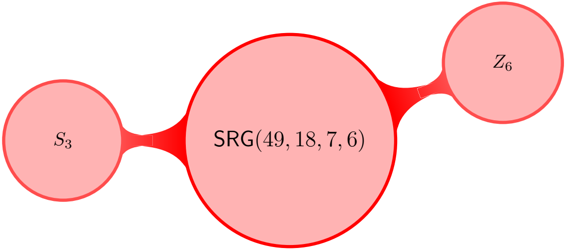


SRG$(49, 18, 7, 6)$-Z_6

distribution	#OM-Z_6	#OM-Z_3	#SRGs	distribution	#OM-Z_6	#OM-Z_3	#SRGs
(0, 2, 3, 6)	5	6	4	(3, 2, 0, 7)	2	3	0
(0, 2, 5, 5)	2	2	0	(3, 2, 2, 6)	3	5	6
(0, 2, 7, 4)	3	6	0	(3, 2, 4, 5)	3	6	0
(1, 0, 0, 8)	4	10	2	(3, 2, 6, 4)	2	4	0
(1, 0, 2, 7)	23	11	5	(4, 0, 3, 6)	4	9	0
(1, 0, 4, 6)	37	66	16	(4, 0, 5, 5)	9	16	0
(1, 0, 6, 5)	63	128	0	(5, 1, 0, 7)	1	1	0
(1, 3, 0, 7)	3	2	1	(5, 1, 2, 6)	2	2	0
(1, 3, 2, 6)	2	1	0	(5, 1, 4, 5)	2	2	0
(1, 3, 4, 5)	1	1	0	(5, 1, 6, 4)	1	1	0
(1, 3, 6, 4)	1	1	0	(7, 0, 0, 7)	1	1	0
(2, 1, 3, 6)	19	35	0	(7, 0, 2, 6)	1	1	0
(2, 1, 5, 5)	19	31	0	(7, 0, 4, 5)	1	1	0
(2, 1, 7, 4)	7	7	0				

Table: Number of orbit matrices and SRGs(49,18,7,6) for the automorphism group Z_6

SRG$(49, 18, 7, 6)$-S_3

distribution	#OM-S_3	#OM-Z_3	#SRGs	distribution	#OM-S_3	#OM-Z_3	#SRGs
(0,2,3,6)	5	6	0	(3,2,4,5)	3	6	5
(0,2,5,5)	2	2	0	(3,2,6,4)	2	4	0
(0,2,7,4)	3	6	4	(4,0,3,6)	4	9	4
(1,0,0,8)	4	10	1	(4,0,5,5)	9	16	0
(1,0,2,7)	23	11	0	(4,0,7,4)	11	11	0
(1,0,4,6)	37	66	0	(4,0,9,3)	11	7	1
(1,0,6,5)	63	128	20	(4,0,11,2)	22	22	0
(1,0,8,4)	127	117	2	(4,0,13,1)	74	73	0
(1,0,10,3)	133	39	0	(5,1,0,7)	1	1	0
(1,0,12,2)	191	170	0	(5,1,2,6)	2	2	3
(1,3,0,7)	3	2	0	(5,1,4,5)	2	2	0
(1,3,2,6)	2	1	0	(5,1,6,4)	1	1	0
(1,3,4,5)	1	1	0	(7,0,0,7)	1	1	4
(1,3,6,4)	1	1	3	(7,0,2,6)	1	1	0
(2,1,3,6)	19	35	0	(7,0,4,5)	1	1	0
(2,1,5,5)	19	31	11	(7,0,6,4)	2	2	0
(2,1,7,4)	7	7	0	(7,0,8,3)	3	3	0
(3,2,0,7)	2	3	0	(7,0,10,2)	2	2	0
(3,2,2,6)	3	5	0	(7,0,12,1)	3	3	0

Table: Number of orbit matrices and SRGs(49,18,7,6) for the automorphism group S_3

SRG$(49, 18, 7, 6)$-Results

Theorem

Up to isomorphism there exists exactly 34 strongly regular graphs with parameters $(49, 18, 7, 6)$ having a cyclic automorphism group of order 6.

Theorem

Up to isomorphism there exists exactly 36 strongly regular graphs with parameters $(49, 18, 7, 6)$ having an automorphism group isomorphic to the symmetric group S_3.

Theorem

Up to isomorphism there exists exactly 55 strongly regular graphs with parameters $(49, 18, 7, 6)$ having an automorphism group of order six.

SRG$(49, 18, 7, 6)$-Results

Theorem

Up to isomorphism there exists exactly 34 *strongly regular graphs with parameters* $(49, 18, 7, 6)$ *having a cyclic automorphism group of order* 6.

Theorem

Up to isomorphism there exists exactly 36 *strongly regular graphs with parameters* $(49, 18, 7, 6)$ *having an automorphism group isomorphic to the symmetric group* S_3.

Theorem

Up to isomorphism there exists exactly 55 *strongly regular graphs with parameters* $(49, 18, 7, 6)$ *having an automorphism group of order six.*

SRG$(49, 18, 7, 6)$-Results

Theorem

Up to isomorphism there exists exactly 34 strongly regular graphs with parameters $(49, 18, 7, 6)$ having a cyclic automorphism group of order 6.

Theorem

Up to isomorphism there exists exactly 36 strongly regular graphs with parameters $(49, 18, 7, 6)$ having an automorphism group isomorphic to the symmetric group S_3.

Theorem

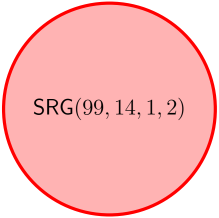
Up to isomorphism there exists exactly 55 strongly regular graphs with parameters $(49, 18, 7, 6)$ having an automorphism group of order six.

SRG$(49, 18, 7, 6)$-Results

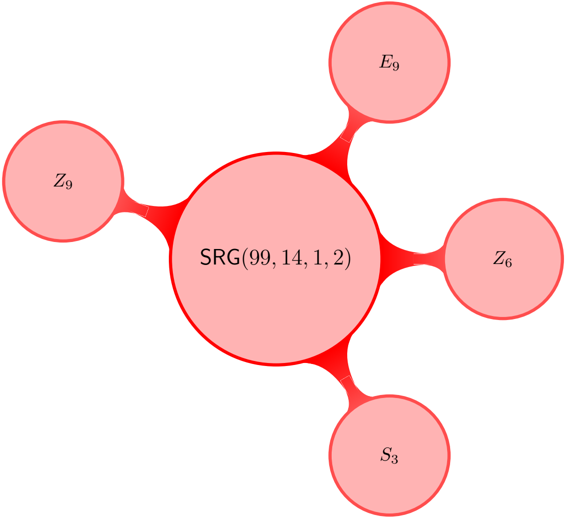
$\lvert Aut(\Gamma_i)\rvert$	#SRGs	$\lvert Aut(\Gamma_i)\rvert$	#SRGs
6	8+26	72	4
12	2+2	126	1
18	1+1	144	2
24	4	1008	1
30	1	1764	1
48	1		

Table: SRG(49,18,7,6) having an automorphism group of order 6

SRG$(99, 14, 1, 2)$



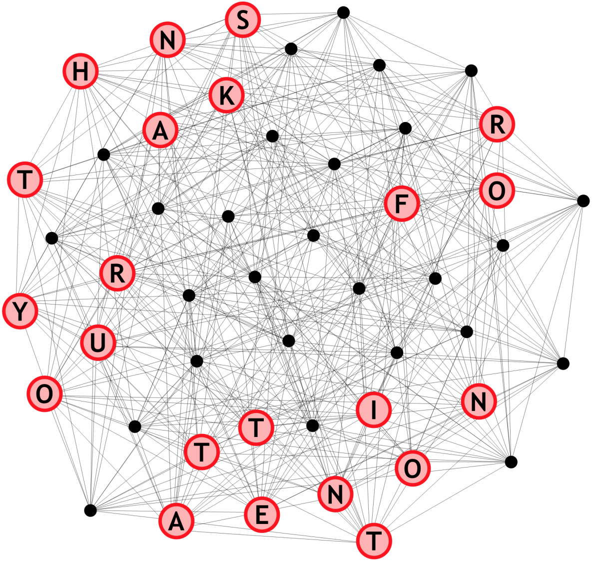
SRG$(99, 14, 1, 2)$



SRG$(99, 14, 1, 2)$-Results

Theorem

If there exists a SRG$(99, 14, 1, 2)$, then the order of its full automorphism group is $2^a 3^b$, and $b \in \{0, 1\}$. If a SRG$(99, 14, 1, 2)$ has an automorphism ϕ of order 3, then ϕ has no fixed points. Further, there is no SRG$(99, 14, 1, 2)$ having an automorphism group of order six.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