On Divisibility Properties of Some Binomial Sums Connected with the Catalan and Fibonacci Numbers

Mr Jovan Mikić, Faculty of Technology,

University of Banja Luka,

Email address: jovan.mikic@tf.unibl.org

Introduction

- Let us consider the following sum:

$$S(n,m) = \sum_{k=0}^{n} \binom{n}{k}^{m} F(n,k), \qquad (Eq{:}\,1)$$

- The function F(n,k) is an integer-valued function.
- The number n is a non-negative number and m is a natural number.

New Class of Binomial Sums: the definition

- Our goal is to investigate some divisibility properties of the sum $S(n, m)$.
- In order to do so, we introduce a new class of binomial sums which we call "M-sums".
- Let $\boldsymbol{M_S(n, j, t)}$ denote the following sum:

$$\binom{\boldsymbol{n-j}}{\boldsymbol{j}} \sum_{\boldsymbol{k=0}}^{\boldsymbol{n-2j}} \binom{\boldsymbol{n-2j}}{\boldsymbol{k}} \binom{\boldsymbol{n}}{\boldsymbol{j+k}}^{\boldsymbol{t}} \boldsymbol{F(n, j+k)}, \qquad (Eq: 2)$$

- Numbers n, j, t are non-negative integers such that j $\leq \left\lfloor \frac{n}{2} \right\rfloor$.

The connection between sums

- Obviously, by setting $j \coloneqq 0$ in the (Eq:2), it follows that

$$\boldsymbol{S}(\boldsymbol{n}, \boldsymbol{t}+\mathbf{1}) = \boldsymbol{M_S}(\mathbf{n}, \mathbf{0}, \mathbf{t}), \qquad (Eq{:}\,3)$$

- Due to $(\mathrm{Eq}{:}\,3)$, we can see $M_S(n, j, t)$ sum as a generalization of $S(n, m)$.

Main recurrence for new sums

- The following recurrence holds:

$$\boldsymbol{M_S}(\mathbf{n}, \mathbf{j}, \mathbf{t}+\mathbf{1}) = \binom{\boldsymbol{n}}{\boldsymbol{j}} \sum_{\boldsymbol{u}=\mathbf{0}}^{\left\lfloor \frac{\boldsymbol{n}-\mathbf{2}\boldsymbol{j}}{\mathbf{2}} \right\rfloor} \binom{\boldsymbol{n}-\boldsymbol{j}}{\boldsymbol{u}} \boldsymbol{M_S}(\boldsymbol{n}, \boldsymbol{j}+\mathbf{u}, \boldsymbol{t}), \; (Eq: 4)$$

A simple consequence of $(Eq: 3)$ and $(Eq: 4)$

- Let us suppose that an integer $q(n)$ divides $M_S(n, j, t_0)$ for all $0 \leq$ j $\leq \left\lfloor \frac{n}{2} \right\rfloor$; where t_0 is a fixed non-negative number.

- By using the $(Eq: 4)$, and the induction principle, it can be shown that $q(n)$ must divide $M_S(n, j, t)$, for all $t \geq t_0$.

- By $(Eq: 3)$, it follows that $q(n)$ divides $S(n, t)$, for all $t \geq t_0$*+1.*

A curious formula

- It is well-known that:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{y}=(-1)^n\binom{x}{y-n}, \qquad (Eq:5)$$

- The (*Eq:5*) can be proved by using Chu-Vandermonde formula. There is also a nice combinatorial proof of the (Eq:5).

The first sum

- Let $S_1(n, m)$ denote the following sum:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{2n+k}{2n+1}, \qquad \text{(Eq: 6)}$$

- By the (Eq:5), it follows that:

$$S_1(\mathrm{n}, 1) = (-1)^n n C_n, \quad \text{(Eq: 7)}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the $n-th$ **Catalan number.**

The first application of new sums

- **Theorem 1.**

For all natural numbers n and m, the following congruence is true:

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}^m \binom{2n+k}{2n+1} \equiv 0(\mathrm{mod}\ \mathrm{n})$$

- In other words, $\boldsymbol{S_1(\mathrm{n},\mathrm{m}) \equiv 0(\mathrm{mod}\ \mathrm{n})}$.

A Proof of Theorem 1

- The sum $S_1(\mathrm{n}, \mathrm{m})$ is an instance of the $(Eq{:}\,1)$, where

$$F_1(\mathrm{n,k}) = (-1)^k \binom{2n+k}{2n+1}.$$

- Let us calculate $\mathrm{M}_{S_1}(\mathrm{n}, \mathrm{j}, 0)$ sum.

- By the $(Eq{:}\,2)$, it follows that

$$\mathrm{M}_{S_1}(\mathrm{n}, \mathrm{j}, 0) = \binom{\mathrm{n}-\mathrm{j}}{\mathrm{j}} \sum_{\mathrm{k}=0}^{\mathrm{n}-2\mathrm{j}} \binom{\mathrm{n}-2\mathrm{j}}{\mathrm{k}} F_1(\mathrm{n}, \mathrm{j}+\mathrm{k}).$$

A Proof of Theorem 1

- By the (Eq:5), it can be shown that

$$M_{S_1}(\mathrm{n},\mathrm{j},0)=(-1)^{n-j}\binom{2n+j}{j}\binom{2n+1}{n-2j}\frac{n-j}{2n+1}, \quad (Eq: 8)$$

A Proof of Theorem 1

- By using main recurrence $(Eq: 4)$ and $(Eq: 8)$, it can be shown that

$$\boldsymbol{M_{S_1}}(\mathbf{n}, \mathbf{j}, \mathbf{1}) = \mathbf{n}(-\mathbf{1})^{n-j} \sum_{u=0}^{\left\lfloor \frac{n-2j}{2} \right\rfloor} \frac{(-\mathbf{1})^{u}}{\mathbf{2}n+\mathbf{1}} \binom{n-\mathbf{1}}{j+u} \binom{j+u}{u} \binom{\mathbf{2}n+j+u}{j+u} \binom{\mathbf{2}n+\mathbf{1}}{n-\mathbf{2}j-\mathbf{2}u},$$

$$(\boldsymbol{Eq}: \mathbf{9})$$

A Proof of Theorem 1

- We assert that an integer

$$\binom{n-1}{j+u}\binom{j+u}{u}\binom{2n+j+u}{j+u}\binom{2n+1}{n-2j-2u}$$

is divisible by $2n+1$. Note that gcd $(n, 2\,n+1)=1$.

- By the $(Eq: 9)$, it follows that $M_{S_1}(n, j, 1)$ is divisible by n for **all** $0 \leq \mathrm{j} \leq \left\lfloor \frac{n}{2} \right\rfloor$ and for all natural numbers n.

- Therefore, we can take $q_1(n) = n$.

A Proof of Theorem 1

- By using the main recurrence $(Eq: 4)$ and the induction principle, it can be shown that $M_{S_1}(n, j, t)$ is divisible by n, for all $t \geq 1$.
- By $(Eq: 3)$, it follows that $S_1(n, t)$ is divisible by n, for all $t \geq 2$.
- *By* $(Eq: 6)$, we know that $S_1(n, 1)$ is divisible by n. This completes the proof of Theorem 1.

A generalization of the first sum

- Let $S_1(\mathrm{n}, \mathrm{m}; \mathrm{a})$ denote the following sum:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{an+k}{an+1}, \qquad (\mathrm{Eq}: 10)$$

where a is a natural number and n, m are non-negative integer.

A generalization of the first sum

- By the $(\mathrm{Eq}: 5)$, we obtain an interesting formula:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{an+k}{an+1} = (-1)^n \frac{n}{(a-1)n+1} \binom{an}{n}, (Eq: 11)$$

- Note that $gcd(n,(a-1)n+1)=1$.

- Therefore, by the $(Eq: 11)$, it follows that the number $\frac{1}{(a-1)n+1}\binom{an}{n}$ is always an integer

A generalized Catalan number

- The number $\frac{\mathbf{1}}{(\boldsymbol{a}-\mathbf{1})\boldsymbol{n}+\mathbf{1}}\binom{\boldsymbol{an}}{\boldsymbol{n}}$ is called a generalized Catalan number.

- For $a = 2$, generalized Catalan number reduces to Catalan number.

- The number $\frac{1}{(a-1)n+1}\binom{an}{n}$ is also known as Fuss-Catalan number.

The first application of new sums: a generalization

- **Theorem 1'.**

For all natural numbers n, m, and a the following congruence is true:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{an+k}{an+1} \equiv 0 (\mathrm{mod}\ n)$$

- In other words, $S_1(\mathrm{n}, \mathrm{m}; a) \equiv 0 (\mathrm{mod}\ \mathrm{n})$.

A generalization of new sums

- Let us consider the following sum:

$$S(n,m;a) = \sum_{k=0}^{n} \binom{n}{k}^m F(n,k,a), \qquad (Eq: 1')$$

- The function F(n,k,a) is an integer-valued function.

A generalization of new sums

- We give a slight generalization of $\mathrm{M_S}$ sums.

- $$\boldsymbol{M_S(n, j, t; a) = \binom{n-j}{j} \sum_{k=0}^{n-2j} \binom{n-2j}{k} \binom{n}{j+k}^t F(n, j+k, a)}, \quad (Eq{:}\,2')$$

- Numbers n, j, t are non-negative integers such that j $\leq \left\lfloor \frac{n}{2} \right\rfloor$; and

a is a natural number.

A generalization of new sums

- Obviously, by setting $j := 0$ in the (Eq:2'), it follows that

$$\boldsymbol{S}(\boldsymbol{n}, \boldsymbol{t}+\mathbf{1}; \boldsymbol{a}) = \boldsymbol{M_S}(\mathbf{n}, \mathbf{0}, \mathbf{t}; \boldsymbol{a}), \qquad (Eq: 3')$$

- The $(Eq: 3')$ is similar with the $(Eq: 3)$.
- The recurrence $(Eq: 4)$ also holds:
- $\boldsymbol{M_S}(\mathbf{n}, \mathbf{j}, \mathbf{t}+\mathbf{1}; \mathbf{a}) = \binom{\boldsymbol{n}}{\boldsymbol{j}} \sum_{\boldsymbol{u}=\mathbf{0}}^{\left\lfloor \frac{\boldsymbol{n}-2\boldsymbol{j}}{2} \right\rfloor} \binom{\boldsymbol{n}-\boldsymbol{j}}{\boldsymbol{u}} \boldsymbol{M_S}(\boldsymbol{n}, \boldsymbol{j}+\mathbf{u}, \boldsymbol{t}; \boldsymbol{a}),\ (Eq: 4')$

A Proof of Theorem 1'

- The sum $S_1(\mathrm{n}, \mathrm{m}; \mathrm{a})$ is an instance of the $(Eq: 1')$, where

$$F_1(\mathrm{n},\mathrm{k},\mathrm{a}) = (-1)^k \binom{an+k}{an+1}.$$

- Let us calculate $\mathrm{M}_{S_1}(\mathrm{n}, \mathrm{j}, 0; a)$ sum.

- By the $(Eq: 2')$, it follows that

$$\mathrm{M}_{S_1}(\mathrm{n}, \mathrm{j}, 0; a) = \binom{\mathrm{n}-\mathrm{j}}{\mathrm{j}} \sum_{\mathrm{k}=0}^{\mathrm{n}-2\mathrm{j}} \binom{\mathrm{n}-2\mathrm{j}}{\mathrm{k}} F_1(\mathrm{n}, \mathrm{j}+\mathrm{k}, a).$$

A Proof of Theorem 1’

- By the (Eq:5), it can be shown that

$$M_{S_1}(\mathbf{n},\mathbf{j},\mathbf{0};a)=(-1)^{n-j}\binom{an+j}{j}\binom{an+1}{n-2j}\frac{n-j}{an+1}, \qquad (Eq:5')$$

A Proof of Theorem 1′

- By using recurrence $(Eq\!:4')$ and $(Eq\!:5')$, it can be shown that

$$\boldsymbol{M_{S_1}}(\mathbf{n},\mathbf{j},\mathbf{1};\boldsymbol{a}) = \mathbf{n}(-\mathbf{1})^{\boldsymbol{n-j}} \sum_{\boldsymbol{u=0}}^{\left\lfloor \frac{\boldsymbol{n-2j}}{\mathbf{2}} \right\rfloor} \frac{(-\mathbf{1})^{\boldsymbol{u}}}{\boldsymbol{an}+\mathbf{1}} \binom{\boldsymbol{n}-\mathbf{1}}{\boldsymbol{j}+\boldsymbol{u}} \binom{\boldsymbol{j}+\boldsymbol{u}}{\boldsymbol{u}} \binom{\boldsymbol{an}+\boldsymbol{j}+\boldsymbol{u}}{\boldsymbol{j}+\boldsymbol{u}} \binom{\boldsymbol{an}+\mathbf{1}}{\boldsymbol{n}-\mathbf{2}\boldsymbol{j}-\mathbf{2}\boldsymbol{u}}, \qquad (\boldsymbol{Eq}\!:\mathbf{6}')$$

A Proof of Theorem 1'

- We assert that an integer

$$\binom{n-1}{j+u}\binom{j+u}{u}\binom{an+j+u}{j+u}\binom{an+1}{n-2j-2u}$$

is divisible by $an+1$. Note that gcd $(n, an+1)=1$.

- By the $(Eq: 6')$, it follows that $M_{S_1}(n, j, 1; a)$ is divisible by n for **all** $0 \leq \mathrm{j} \leq \left\lfloor \frac{n}{2} \right\rfloor$ and for all natural numbers n and a .

- Therefore, we can take $q_1(n, a) = n$.

A Proof of Theorem 1’

- By using the recurrence $(Eq: 4')$ and the induction principle, it can be shown that $M_{S_1}(n, j, t; a)$ is divisible by n, for all $t \geq 1$.

- By the $(Eq: 3')$, it follows that $S_1(n, t; a)$ is divisible by n, for all $t \geq 2$.

- By the $(Eq: 11)$, we know that $S_1(n, 1; a)$ is divisible by n. This completes the proof of Theorem 1’.

The second sum

- Let $S_2(n,m)$ denote the following sum:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^m\left(\binom{2n+k}{2n}+2\binom{2n+k}{2n+1}\right), (Eq: 12)$$

- By the (Eq:5), it can be shown that:

$$S_2(n,1)=(-1)^n(3n+1)C_n. (Eq: 13)$$

The second application of new sums

- **Theorem 2.**

For all natural numbers n and m, the following congruence is true:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \left(\binom{2n+k}{2n} + 2\binom{2n+k}{2n+1} \right) \equiv 0 (\bmod 3n+1)$$

- In other words, $S_2(\mathbf{n}, \mathbf{m}) \equiv \mathbf{0} (\bmod 3n + 1)$.

A Proof of Theorem 2

- The sum $S_2(\mathrm{n},\mathrm{m})$ is an instance of the $(Eq{:}\,1)$, where

$$F_2(\text{n,k})=(-1)^k\left(\binom{2n+k}{2n}+2\binom{2n+k}{2n+1}\right).$$

- Let us calculate $\mathrm{M}_{S_2}(\mathrm{n},\mathrm{j},0)$ sum.

- By the $(Eq{:}\,2)$, it follows that

$$\mathrm{M}_{S_1}(\mathrm{n},\mathrm{j},0)=\binom{\mathrm{n}-\mathrm{j}}{\mathrm{j}}\sum_{\mathrm{k}=0}^{\mathrm{n}-2\mathrm{j}}\binom{\mathrm{n}-2\mathrm{j}}{\mathrm{k}}F_2(\mathrm{n},\mathrm{j}+\mathrm{k}).$$

A Proof of Theorem 2

- By the (Eq:5), it can be shown that

$$M_{S_2}(\mathbf{n},\mathbf{j},\mathbf{0})=(-1)^{n-j}\binom{2n+j}{j}\binom{2n+1}{n-2j}\frac{3n+1}{2n+1}, \quad (Eq:14)$$

A Proof of Theorem 2

- Clearly, an integer

$$\binom{2n+j}{j}\binom{2n+1}{n-2j}$$

is divisible by $2n+1$. Note that gcd $(3n+1, 2n+1)=1$.

- By the $(Eq: 14)$, it follows that $M_{S_2}(n, j, 0)$ is divisible by $3n+1$ for **all** $0 \leq \mathrm{j} \leq \left\lfloor \frac{n}{2} \right\rfloor$ and for all natural numbers n.

- Therefore, we can take $q_2(n) = 3n+1$.

A Proof of Theorem 2

- By using the recurrence $(Eq{:}\,4)$ and the induction principle, it can be shown that $M_{S_2}(n, j, t)$ is divisible by $3n + 1$, for all $t \geq 0$.

- By $(Eq{:}\,3)$, it follows that $S_2(n, t)$ is divisible by $3n + 1$, for all $t \geq 1$.

- This completes the proof of Theorem 2.

A generalization of the second sum

- Let $S_2(\mathrm{n}, \mathrm{m}; \mathrm{a})$ denote the following sum:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \left(\binom{an+k}{an} + 2\binom{an+k}{an+1} \right), \qquad \text{(Eq: 15)}$$

where a is a natural number and n, m are non-negative integer.

The second application of new sums: a generalization

- **Theorem 2'.**

For all natural numbers n, m, and a the following congruence is true:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \left(\binom{an+k}{an} + 2\binom{an+k}{an+1} \right) \equiv 0 (\mathrm{mod}\ (a+1)n+1)$$

- In other words, $S_2(\mathrm{n}, \mathrm{m}; a) \equiv 0 (\mathrm{mod}\ (a+1)n+1)$.

A Proof of Theorem 2'

- The sum $S_2(\mathrm{n,m;a})$ is an instance of the $(Eq\colon 1')$, where

$$F_2(\mathrm{n,k,a}) = (-1)^k \left(\binom{an+k}{an} + 2\binom{an+k}{an+1} \right).$$

- Let us calculate $\mathrm{M}_{S_2}(\mathrm{n,j,0};a)$ sum.

- By the $(Eq\colon 2')$, it follows that

$$\mathrm{M}_{S_2}(\mathrm{n,j,0};a) = \binom{\mathrm{n-j}}{\mathrm{j}} \sum_{\mathrm{k=0}}^{\mathrm{n-2j}} \binom{\mathrm{n-2j}}{\mathrm{k}} F_2(\mathrm{n,j+k},a).$$

A Proof of Theorem 2’

- By the (Eq:5), it can be shown that

$$\boldsymbol{M_{S_2}(\mathbf{n}, \mathbf{j}, \mathbf{0}; a)} = (-\mathbf{1})^{n-j}\binom{an+j}{j}\binom{an+1}{n-2j}\frac{(a+1)n+1}{an+1},$$

$$(\boldsymbol{Eq}: \mathbf{16})$$

A Proof of Theorem 2'

- Clearly, an integer

$$\binom{an+j}{j}\binom{an+1}{n-2j}$$

is divisible by $an+1$. Note that gcd $((a+1)n+1, an+1)$=1.

- By the $(Eq: 16)$, it follows that $M_{S_2}(n,j,0;a)$ is divisible by $(a+1)n+1$ for **all** $0 \leq \mathrm{j} \leq \left\lfloor\frac{n}{2}\right\rfloor$ and for all natural numbers n and a.

- Therefore, we can take $q_2(n,a) = (a+1)n+1$.

A Proof of Theorem 2'

- By using the recurrence $(Eq: 4')$ and the induction principle, it can be shown that $M_{S_2}(n, j, t; a)$ is divisible by $(a + 1)n + 1$, for all $t \geq 0$.

- By $(Eq: 3')$, it follows that $S_2(n, t)$ is divisible by $(a + 1)n + 1$, for all $t \geq 1$.

- This completes the proof of Theorem 2'.

The Fibonacci numbers

- Let F_n denote the n-th Fibonacci number.
- The first two terms are $F_0 = 0$ and $F_1 = 1$.
- The other terms are defined by the recurrence relation: $F_n = F_{n-1} + F_{n-2}$ (for n≥ 2).
- It is well-known that (the Binet formula) that:

$$F_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$$

The Lucas numbers

- Let L_n denote the n-th Lucas number.
- The first two terms are $L_0 = 1$ and $L_1 = 3$.
- The other terms are defined by the recurrence relation: L_n= L_{n-1}+L_{n-2} (for n≥ 2).
- It is known that:

$$L_n = \frac{\left(1+\sqrt{5}\right)^n+\left(1-\sqrt{5}\right)^n}{2^n}, \quad (Eq\text{: }17)$$

Two binomial formulas with Fibonacci and Lucas numbers

- The Binet formula is equivalent with the following identity:

$$\sum_{k=0}^{n} \binom{n}{2k+1} 5^k = 2^{n-1} F_{n,} \quad (Eq: 18)$$

- The $(Eq: 17)$ is equivalent with the following identity:

$$\sum_{k=0}^{n} \binom{n}{2k} 5^k = 2^{n-1} L_{n,} \quad (Eq: 19)$$

An interesting formula with Fibonacci numbers

By using $(Eq: 18)$ and $(Eq: 19)$, it can be shown that:

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\left\lfloor \frac{k}{2} \right\rfloor} = 2^n F_{n+1}, \quad (Eq: 20)$$

The third application: The sum connected with the Fibonacci numbers

- Let the sum $S_3(n,m)$ denote the following sum:

$$\sum_{k=0}^{n} \binom{n}{k}^m 5^{\left\lfloor \frac{k}{2} \right\rfloor}$$

- By the $(Eq: 20)$, it follows that:

$$\boldsymbol{S_3(\mathrm{n}, 1) = 2^{\mathrm{n}} F_{n+1}}.$$

- The sum $S_3(\mathrm{n}, \mathrm{m})$ is an instance of the $(Eq: 1)$, where

$$F_3(\mathrm{n,k}) = 5^{\left\lfloor \frac{k}{2} \right\rfloor}.$$

The third application: The sum connected with the Fibonacci numbers

- Let us calculate $\mathrm{M}_{S_3}(\mathrm{n}, \mathrm{j}, 0)$ sum.

- By the $(Eq: 2)$, it follows that

$$\mathrm{M}_{S_3}(\mathrm{n}, \mathrm{j}, 0) = \binom{\mathrm{n}-\mathrm{j}}{\mathrm{j}} \sum_{\mathrm{k}=0}^{\mathrm{n}-2\mathrm{j}} \binom{\mathrm{n}-2\mathrm{j}}{\mathrm{k}} F_3(\mathrm{n}, \mathrm{j}+\mathrm{k}).$$

The third application: The sum connected with the Fibonacci numbers

- We have the **two cases**:

If j is **even** , then it can be shown that:

- $\boldsymbol{M_{S_3}(\mathbf{n}, \mathbf{j}, \mathbf{0}) = \binom{\mathbf{n} - j}{j} 5^{\frac{j}{2}} \, 2^{n-2j} F_{n-2j+1.}}$ (***Eq*:21**)

If j is odd , then it can be shown that:

- $\boldsymbol{M_{S_3}(\mathbf{n}, \mathbf{j}, \mathbf{0}) = \binom{\mathbf{n} - j}{j} 5^{\frac{j-1}{2}} \, 2^{n-2j} L_{n-2j+1.}}$ (***Eq*:22**)

The End

- Thanks for your attention.
  ☺