On a Linear Recurrence Relation

- Definition of the six-parameter $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ linear recurrence relation
- Special cases:
  1. The Josephus Problem
  2. Number of 1's in the binary expression of $n \in \mathbb{N}$
  3. The Tower of Hanoi (Gros sequence)
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  6. The "infinity sequence" of Per Nørgård
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Linear Recurrence Relation

$$\begin{cases} f(1) = \zeta \\ f(2n) = \alpha f(n) + \beta \quad , \quad f(2n+1) = \gamma f(n) + \delta f(n+1) + \varepsilon, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geq 1 \end{cases}$$

with $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{Z}$

Equivalent form

$$(*) \begin{cases} f(1) = \zeta \\ f(n) = a(n) f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + b(n) f\left(\left\lceil \frac{n}{2} \right\rceil\right) + c(n) \quad , \quad n \geq 2 \end{cases}$$

with $a(n) := \frac{\alpha+\gamma}{2} + (-1)^n \frac{\alpha-\gamma}{2}$, $b(n) := \frac{1-(-1)^n}{2} \delta$

$c(n) := \frac{\beta+\varepsilon}{2} + (-1)^n \frac{\beta-\varepsilon}{2}$

Recurrence relations of the form $(*)$ appear often in computer science, because algorithms based on the technique "divide et impera" (divide and conquer) often reduce a problem of size $n \in \mathbb{N}$ to the solution of similar problems of sizes $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lceil \frac{n}{2} \right\rceil$.

Note that $n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil$.

Def.: A formal series $\varphi(s)$ is of the <u>divide-and-conquer (DC) type</u>, if it satisfies a functional equation of the form

$$c_0(s)\varphi(s) + c_1(s)\varphi(s^2) + \dots + c_n(s)\varphi(s^{2^n}) = b(s),$$

where $b(s)$ is a formal series and $c_k(s)$, $k = 0, 1, \dots, n$ are polynomials not all zero.

A sequence is of the <u>divide-and-conquer type</u> if its generating function $\varphi(s)$ is of the DC-type.

<u>Special case</u>: If $b(s) = 0$, then the series $\varphi(s)$ is said to be <u>Mahlerian</u>.

<u>Proposition</u> The generating function $\varphi(s)$ of $(f(n))_{n\in\mathbb{N}_0}$ satisfies the functional equation

$$s\varphi(s) = (\delta + \alpha s + \gamma s^2)\varphi(s^2) + (1-\delta)\left\{ s^2 + \frac{s^3(\beta+\varepsilon s)}{1-s^2} + \right.$$
$$\left. + f(0)\cdot\left[-\delta + (1-\alpha)s - \gamma s^2\right]\right.$$

or

$$s(1-s^2)\varphi(s) - (1-s^2)(\delta + \alpha s + \gamma s^2)\varphi(s^2)$$
$$= (1-\delta)\{ s^2(1-s^2) + s^3(\beta + \varepsilon s) + f(0)\cdot[(1-\alpha)s$$
$$-\gamma s^2 - \delta](1-s^2)),$$

that is $(f(n))_{n\in\mathbb{N}_0}$ is of the DC-type.

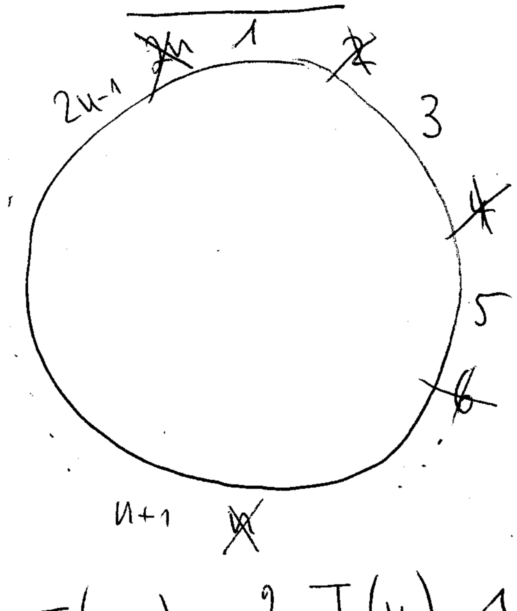
Some special cases

E1. The Josephus Problem

Let n people numbered 1 to n around a circle and eliminate every second remaining person until one survives. (Original problem: eliminate every third remaining person).
Determine the survivor number $J(n)$

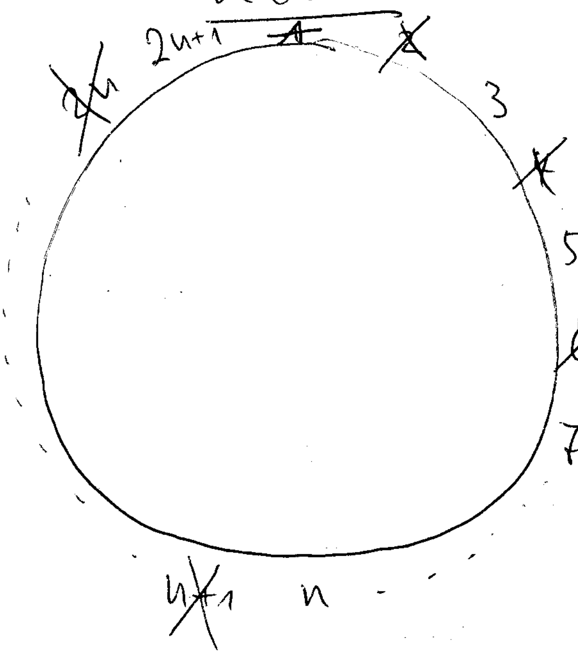
Recurrence relation, $J(1) := 1$

n even



$$J(2n) = 2 \cdot J(n) - 1$$

n odd



$$J(2n+1) = 2 \cdot J(n) + 1$$

Special case. $\alpha = 2, \beta = -1, \gamma = 2, \delta = 0, \varepsilon = 1, \zeta = 1$

Solution Let $n = (b_m b_{m-1} \dots b_1 b_0)_2 = \sum_{k=0}^{m} b_k \cdot 2^k$

$$J(n) = (b_{m-1} b_{m-2} \dots b_1 b_0 b_m)_2 = 2n + 1 - 2^{m+1} = \sum_{k=0}^{m} (-1)^{1-b_k} \cdot 2^k$$

E2) Number of 1's in the binary expansion of $n \in \mathbb{N}$

For $n = (b_m \dots b_1 b_0)_2$ let
$$S_2(n) := \sum_{k=0}^{m} b_k$$

Then
$$\begin{cases} S_2(1) = 1 \\ S_2(2n) = S_2(n), \quad S_2(2n+1) = S_2(n) + 1, \quad n \geq 1 \end{cases}$$

Special case: $\alpha = 1, \beta = 0, \gamma = 1, \delta = 0, \varepsilon = 1, \zeta = 1$

Solution: $S_2(n) = \sum_{k=0}^{m} b_k$

or $S_2(2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}) = \ell + 1, \quad a_0 > a_1 > \dots > a_\ell \geq 0$

First few values:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$S_2(n)$	1	1	2	1	2	2	3	1	2	2	3	2	3	3	4	1

A000120 in the OEIS

The generating function $\varphi(s)$ satisfies the functional equation
$$\varphi(s) = (1+s)\varphi(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}},$$

since $(1+s)\varphi(s^2) + \frac{s}{1-s^2} = (1+s)\frac{1}{1-s^2} \sum_{k=0}^{\infty} \frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1-s^2}$

$$= \frac{1}{1-s} \left(\sum_{k=0}^{\infty} \frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1+s} \right)$$

$$= \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}} = \varphi(s).$$

E3) The Gros sequence (Tower of Hanoi with 3 pegs H_3^n and n discs)

Let $d_2(n)$ be the number of the disc to be moved at the nth step of the optimal solution of H_3^n, then

$$\begin{cases} d_2(1) = 1 \\ d_2(2n) = d_2(n) + 1, \quad d_2(2n+1) = 1, \quad n \geq 1 \end{cases}$$

Special case: $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$

Solution: $d_2(n) = r + 1$, where $n = 2^r(2k+1)$, $k, r \in \mathbb{N}_0$

or $d_2(n) = a_\ell + 1$, $n = 2^{a_0} + \dots + 2^{a_\ell}$, $a_0 > a_1 > \dots > a_\ell \geq 0$, $\ell \geq 0$

First few values

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_2(n)$	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1	5

A001511 in the OEIS

Other interpretations

- $d_2(n)$ gives the number of 2's that divides $2n$
- $d_2(n)$ gives the position of the first 1 (counting

from right to left and beginning with 1) of the binary representation of n.

- $d_2(n)$ gives the position of the bit (counting from right to left and beginning with 1) to be changed in the Gray-Code.

<u>Gray-Code</u> (~ 1953)

decimal	binary	Gray-Code	$d_2(n)$
0	0	0000	
1	1	0001	1
2	10	0011	2
3	11	0010	1
4	100	0110	3
5	101	0111	1
6	110	0101	2
7	111	0100	1
8	1000	1100	4
9	1001	1101	1
10	1010	1111	2
11	1011	1110	1
12	1100	1010	3
13	1101	1011	1
14	1110	1001	2
15	1111	1000	1

The sequence of the changed bit is the so-called <u>regular paper-folding</u> (or <u>dragon curve</u> sequence)
(1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, ...) <u>A014577</u>

<u>Proposition</u> For all $n \in \mathbb{N}$ it is

$$d_2(n) = 2 - S_2(n) + S_2(n-1) \qquad (1)$$

and, conversely,

$$S_2(n) = 2n - \sum_{k=1}^{n} d_2(k)$$

<u>Corollary</u> $\forall n \in \mathbb{N}$: $\sum_{k=1}^{n} d_2(k) \le 2n$

The generating function $\varphi(s)$ satisfies the funct. equation

$$\varphi(s) = \varphi(s^2) + \frac{s}{1-s}$$

and it is given by

$$\varphi(s) = \sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^k}} = \sum_{n=1}^{\infty} \varrho(n) \frac{s^n}{1-s^n}$$

where $\varrho(n) = (n = 2^m) = \begin{cases} 1 & \text{if } n = 2^m, m \in \mathbb{N}_0 \\ 0 & \text{, otherwise} \end{cases}$

$\varphi(s)$ is a <u>Lambert series</u>

Since $d_2(n)$ is the number of 2's that divides $2n$, or, equivalently, the number of the positive powers of 2 which divides n, we have

$$d_2(n) = \sum_{t|n} \varrho(t) \quad , n \in \mathbb{N}$$

E7) <u>Number of odd binomial coefficients in the nth row of Pascal's triangle (Gould's sequence)</u>

n	Pascal's triangle	Pascal's triangle modulo 2	$\gamma(n)$
0	1	1	1
1	1 1	1 1	2
2	1 2 1	1 0 1	2
3	1 3 3 1	1 1 1 1	4
4	1 4 6 4 1	1 0 0 0 1	2
5	1 5 10 10 5 1	1 1 0 0 1 1	4
$\vdots$	$\cdots$	$\cdots$	4
			8
			2
			4
			4
			8
			$\vdots$

Recurrence relation: $\gamma(0)=1,\ \gamma(1)=2$

$$\gamma(2n)=\gamma(n),\quad \gamma(2n+1)=2\gamma(n),\quad n\geq 1$$

<u>Special case</u>: $\alpha=1,\ \beta=0,\ \gamma=2,\ \delta=0,\ \varepsilon=0,\ \zeta=2$

- Generating function is Mahlerian

$$\varphi(s)=(1+2s)\,\varphi(s^2)$$

$$\Rightarrow\quad \varphi(s)=\prod_{k=0}^{\infty}\left(1+2s^{2^k}\right)$$

<u>Solution</u> $n=(b_m\ldots b_1b_0)_2$

$$\Rightarrow\ \gamma(n)=2^{S_2(n)}=2^{\sum_{k=0}^{m}b_k}$$

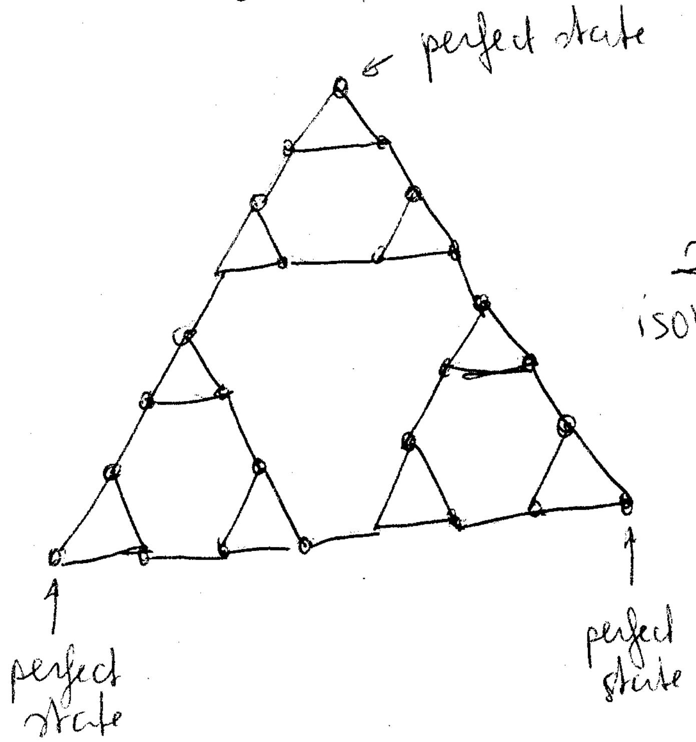
or $\gamma\left(2^{a_0}+2^{a_1}+\cdots+2^{a_\ell}\right)=2^{\ell+1},\quad n=2^{a_0}+2^{a_1}+\cdots+2^{a_\ell}$

$$a_0>a_1>\cdots>a_\ell\geq 0,\quad \ell\geq 0$$

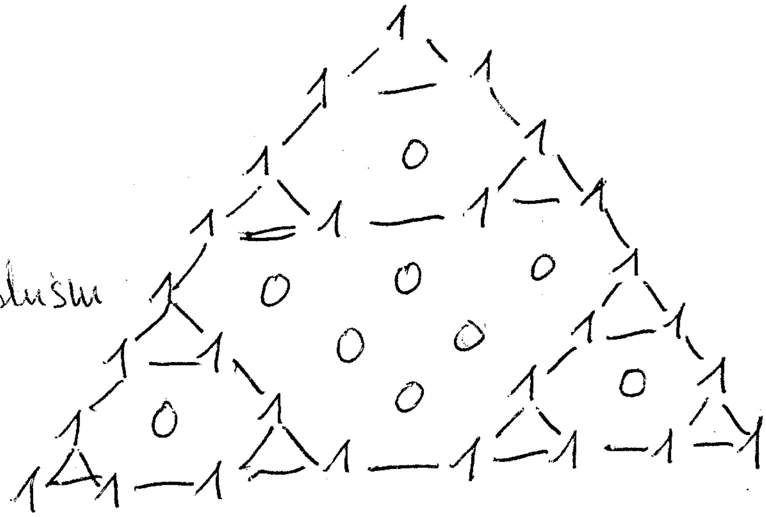
Other interpretation

Graph of the Tower of Hanoi with 3 pegs and 3 discs

Pascal's triangle modulo 2



$\simeq$ isomorphism



E5) The Prouhet-Morse-Thue sequence $m(n)$ on $\{0,1\}$

It can be defined in several ways:

- Sequence of the natural numbers $0,1,2,3,4,5,6,\dots$ in binary form

  $0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, \dots$

  Sum of the digits modulo 2:

  $0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, \dots$

- Recursively, starting with $m(0)=0$ and for all $k \in \mathbb{N}_0$, $m(0), m(1), \dots, m(2^k-1)$ is followed by its <u>one's complement</u>, i.e., $m(2^k) = \overline{m(0)}, \dots, m(2^{k+1}) = \overline{m(2^k-1)}$

  k=0 0

  k=1 0 ¦ 1

  k=2 0 1 ¦ 1 0

  k=3 0 1 1 0 ¦ 1 0 0 1

  k=4 0 1 1 0 1 0 0 1 ¦ 1 0 0 1 0 1 1 0

  - - - - - -

- By the logical function XOR (<u>exclusive or</u>)

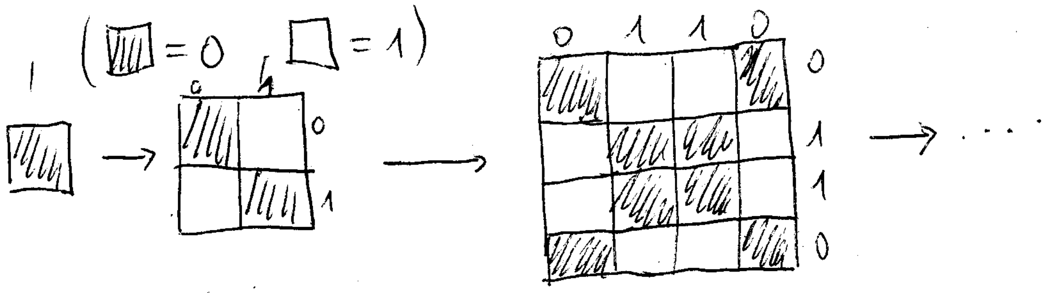
x	y	x XOR y
0	0	0
0	1	1
1	0	1
1	1	0

x	y	z	x XOR y	$(x$ XOR $y)$ XOR z
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	0	1

- By iteration according to the rules

$$0 \longrightarrow 01 \quad , \quad 1 \longrightarrow 10$$

or, graphically, (▨ = 0, □ = 1)



Recurrence relation

$$\begin{cases} m(1) = 1 \qquad (m(0)=0) \\ m(2n) = m(n), \quad m(2n+1) = -m(n)+1, \quad n \geq 1 \end{cases}$$

Special case: $\alpha = 1, \beta = 0, \gamma = -1, \delta = 0, \varepsilon = 1, \zeta = 1$

Solution: $m(n) = \frac{1}{2}\left(1 - (-1)^{s_2(n)}\right)$

or $m((b_m \dots b_1 b_0)_2) = \sum_{k=0}^{m} b_k \cdot (-1)^{\sum_{j=0}^{k-1} b_j}$

or $m(2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}) = \sum_{k=0}^{\ell} (-1)^k = \frac{1}{2}\left(1 + (-1)^\ell\right), \ \ell \geq 0$, $\quad a_0 > a_1 > \dots > a_\ell \geq 0$

Since $m(2n) = m(n)$ the sequence $(m(n))_{n\in\mathbb{N}_0}$ is self-similar. This can be seen also by setting

$01 =: 0$ and $10 =: 1$. Then

01 | 10 | 10 | 01 | 10 | 01 | 01 | 10 | ...
0 1 1 0 1 0 0 1 ...

First few values:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$m(n)$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1	...

A010060 in the OEIS

Problem of Prouhet (1851), rediscovered ~50 years later by Tarry and Escott

Given $N \geq 1$ and $A = \{1, 2, 3, \ldots, 2^N\}$

Choose a partition I, J so that

$I \neq \emptyset,\ J \neq \emptyset,\ I \cap J = \emptyset,\ I \cup J = A$

$$\mathrm{card}(I) = \mathrm{card}(J)$$

and $\bigwedge k = 0, 1, \ldots, N-1$: $\displaystyle\sum_{i\in I} i^k = \sum_{j\in J} j^k$

Example: $N = 3$, $A = \{1, 2, \ldots, 8\}$

Choose $I = \{1, 4, 6, 7\}$ and $J = \{2, 3, 5, 8\}$

$\Rightarrow \mathrm{card}(I) = \mathrm{card}(J) = 4$, $I \cup J = A$, $I \cap J = \emptyset$

and $1 + 4 + 6 + 7 = 2 + 3 + 5 + 8 = 18$

$1^2 + 4^2 + 6^2 + 7^2 = 2^2 + 3^2 + 5^2 + 8^2 = 102$

Prouhet-Morse-Thue

0	1	1	0	1	0	0	1
1	2	3	4	5	6	7	8

E6) The "*infinity sequence*" of Per Nørgård

It satisfies the recurrence relation

$$a(1) = 1 \qquad (\Rightarrow a(0) = 0)$$

$$a(2n) = -a(n), \quad a(2n+1) = a(n) + 1, \quad n \geq 1$$

Special case: $\alpha = -1, \beta = 0, \gamma = 1, \delta = 0, \varepsilon = 1, \zeta = 1$

This sequence has been invented by the Danish composer Per Nørgård in an attempt to unify in a perfect way repetition and variation.

Solution (*) $$a(2^{a_0} + \dots + 2^{a_l}) = \sum_{k=0}^{l} (-1)^k (-1)^{a_{l-k}}$$

$$= (-1)^{a_l} - (-1)^{a_{l-1}} + (-1)^{a_{l-2}} - + \dots + (-1)^l (-1)^{a_0}$$

First few values

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a(n)$	0	1	-1	2	1	0	-2	3	-1	2	0	1	2	-1	-3	4	1

A004718 in the OEIS

Generation by a procedure by J. Mortensen:
Write n in binary form and read from left to right starting with 0 and interpreting 1 as "add 1" and 0 as "change sign". For example: $23 = (10111)_2$ gives $0 \to 1 \to -1 \to 0 \to 1 \to 2$, that is $a(23) = 2$.

From (*) we obtain

$$\bigwedge_{n \in \mathbb{N}} : \ |a(n)| \leq l + 1 = s_2(n).$$

The generating function $\varphi(s)$ satisfies the functional equation

$$\varphi(s) = -(1-s)\,\varphi(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi(s) = \sum_{k=0}^{\infty} (-1)^k \, \frac{s^{2^k} \cdot \prod_{j=0}^{k-1} \left(1 - s^{2^j}\right)}{1 - s^{2^{k+1}}}$$

E7) Coefficients of the power series expansion of the Euler infinite product $\varphi(s) = \prod_{k=0}^{\infty} (1 - s^{2^k})$

Functional equation

$$(1-s)\varphi(s^2) = \varphi(s) \qquad (*)$$

$\varphi(s)$ is Mahlerian

Inserting $\varphi(s) = \sum_{n=0}^{\infty} e(n) \cdot s^n$ into $(*)$ and comparing the coefficients we obtain

$$\begin{cases} e(1) = -1 \quad (e(0)=1) \\ e(2n) = e(n), \quad e(2n+1) = -e(n), \quad n \geq 1 \end{cases}$$

Special case: $\alpha = 1, \beta = 0, \gamma = -1, \delta = 0, \varepsilon = 0, \zeta = -1$
$e(0) = 1$

Solution $\quad e(n) = (-1)^{s_2(n)}$

or $\quad e((b_m \dots b_1 b_0)_2) = (-1)^{\sum_{k=0}^{m} b_k}$

or $\quad e(2^{a_0} + \dots + 2^{a_\ell}) = (-1)^{\ell+1}, \quad a_0 > a_1 > \dots > a_\ell \geq 0, \ \ell \geq 0$

First few values

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$e(n)$	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	-1

A106400 in the OEIS

Note that $e(n) = (-1)^{m(n)}$

and $\underline{e(n) = 1 - 2 \cdot m(n)}$

Therefore,

$$\prod_{k=0}^{\infty} \left(1 - s^{2^k}\right) = \sum_{n=0}^{\infty} e(n)\, s^n = \sum_{n=0}^{\infty} \left(1 - 2 \cdot m(n)\right) s^n$$

$$= \sum_{n=0}^{\infty} s^n - 2 \cdot \sum_{n=0}^{\infty} m(n)\, s^n$$

$$= \frac{1}{1-s} - 2 \sum_{n=0}^{\infty} m(n)\, s^n$$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} m(n)\, s^n = \frac{1}{2} \left(\frac{1}{1-s} - \prod_{k=0}^{\infty} \left(1 - s^{2^k}\right) \right)}$$

E8) Arithmetic sequence of first order

It is defined by ($a_0, d \in \mathbb{Z}$)

$$\begin{cases} a(0) = a_0 & (\Rightarrow a(1) = a_0 + d) \\ a(n+1) = a(n) + d & , n \geq 0 \end{cases}$$

Solution: $a(n) = a_0 + nd$, $n \in \mathbb{N}_0$

This solution satisfies also the recurrence relation

$$(*) \begin{cases} a(1) = a_0 + d & (\Rightarrow a(0) = a_0) \\ a(2n) = 2a(n) - a_0 \\ a(2n+1) = a(n) + a(n+1) - a_0 \end{cases}$$

Special case: $\alpha = 2, \beta = -a_0, \gamma = 1, \delta = 1, \varepsilon = -a_0, \xi = a_0 + d$

Conversely, the difference between two consecutive terms of (*) is a constant.

Proof Let $d(n) := a(n+1) - a(n)$

$$\begin{cases} \Rightarrow d(0) = a(1) - a(0) = a_0 + d - a_0 = d & \left(\Rightarrow d(1) = a(2) - a(1) = a_0 + 2d - a_0 - d = d\right) \\ d(2n) = d(n) \\ d(2n+1) = d(n) \end{cases}$$

Special case $\alpha = 1, \beta = 0, \gamma = 1, \delta = 0, \varepsilon = 0, \xi = d$

Solution: $d(n) = d$ the constant sequence.

Solution of (*): $a(n) = \underbrace{-\beta}_{=a_0} + \underbrace{(\beta + \xi)}_{=d} n$, $n \in \mathbb{N}_0$.

E9) The Stern's diatomic sequence

The first 2^n+1, $n \in \mathbb{N}_0$, terms of this sequence form the nth row in the Stern's diatomic array

$$\begin{array}{ccccccccccccccccc}
0 & & & & & & & & & & & & & & & & 1\\
0 & & & & & & & & 1 & & & & & & & & 1\\
0 & & & & 1 & & & & 1 & & & & 2 & & & & 1\\
0 & & 1 & & 1 & & 2 & & 1 & & 3 & & 2 & & 3 & & 1\\
0 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 1\\
\cdots & & & & & & & & & & & & & & & & \cdots
\end{array}$$

0 and 1 are called the atoms (or seeds) of the array. (Note that the array can be built with arbitrary seeds a and b, $a, b \in \mathbb{N}_0$).

The sequence $(0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, \ldots)$ (A002487 in the OEIS) is called the Stern's diatomic sequence (or Stern sequence)

The array has been studied in 1858 by M. Stern and in 1860 (in a different form) by the French clockmaker A. Brocot. Later has been studied among others by É. Lucas, G. Halphen, D.H. Lehmer and many others.

As a sequence it has been defined for the first time by G. de Rham

It satisfies the recurrence relation

$$\begin{cases} s(0)=0, \quad s(1)=1 \\ s(2n)=s(n), \quad s(2n+1)=s(n)+s(n+1), \quad n\geq 1 \end{cases}$$

Special case : $\alpha=1,\ \beta=0,\ \gamma=1,\ \delta=1,\ \varepsilon=0,\ \zeta=1$

Its generating function satisfies the functional equation

$$s\varphi(s)=(1+s+s^2)\varphi(s^2)$$

which shows that $\varphi(s)$ is Mahlerian. It can be given by an infinite product (L. Carlitz)

$$\varphi(s)=s\cdot\prod_{k=0}^{\infty}\left(1+s^{2^k}+s^{2^{k+1}}\right).$$

It satisfies the linear recurrence relation of second order $\quad s(2^{a_0})=1,\quad s(2^{a_0}+2^{a_1})=a_0-a_1+1,$

$$(*)\begin{cases} s(2^{a_0}+2^{a_1}+\dots+2^{a_\ell})=(a_{\ell-1}-a_\ell+1)\,s(2^{a_0}+2^{a_1}+\dots+2^{a_{\ell-1}}) \\ \qquad -\,s(2^{a_0}+2^{a_1}+\dots+2^{a_{\ell-2}}), \quad \ell\geq 2 \end{cases}$$

for a strictly decreasing sequence of integers $a_0>a_1>\dots>a_\ell\geq 0$, $\ell\geq 0$.

The solution of $(*)$ is given by $(n=2^{a_0}+2^{a_1}+\dots+2^{a_\ell})$

$$s(n)=A_\ell$$

where A_ℓ is the numerator of the semi-regular continued fraction

$$\begin{cases} \dfrac{A_\ell}{B_\ell} = a_0 - a_1 + 1 - \dfrac{1|}{|a_1 - a_2 + 1} - \dfrac{1|}{|a_2 - a_3 + 1} - \dots - \dfrac{1|}{|a_{\ell-1} - a_\ell + 1}, & \ell > 0 \\ \text{and} \quad s(2^{a_0}) = 1 & \text{for } \ell = 0 \end{cases}$$

Ex.: $n = 13 = 2^3 + 2^2 + 2^0$, then $\ell = 2$, $a_0 = 3$, $a_1 = 2$, $a_2 = 0$

$$\frac{A_2}{B_2} = 3 - 2 + 1 - \frac{1}{2 - 0 + 1} = 2 - \frac{1}{3} = \frac{5}{3}$$

$$\Rightarrow s(13) = \underline{\underline{5}}.$$

Another solution is given by

$$s(n) = C_k,$$

where C_k is the numerator of the simple continued fraction (Knuth, Graham, Patashnik)

$$\frac{C_k}{D_k} = b_1 + \frac{1|}{|b_2} + \dots + \frac{1|}{|b_k}$$

for $n = 2^r (1^{b_1} 0^{b_2} \dots 1^{b_k})_2$, $1^{b_1} := \underbrace{1 \dots 1}_{b_1 \text{ times}}$

Ex.: $n = 13 = (1^2 0^1 1^1)_2$, then $b_1 = 2$, $b_2 = 1$, $b_3 = 1$

$$2 + \frac{1|}{|1} + \frac{1|}{|1} = 2 + \frac{1}{1 + \frac{1}{1}} = 2 + \frac{1}{2} = \frac{5}{2} \Rightarrow s(13) = 5 //$$

Other interpretations of $s(n)$

1) $s(n+1)$ is the third binary partition function $b(3;n)$, i.e. $s(n+1)$ is the number of representations of $n = \sum_{i=0}^{\infty} \xi_i 2^i$, $\xi_i \in \{0,1,2\}$

Ex: $n = 12 = 2^3 + 2^2$, then $s(13) = 5$,

since
$$\begin{aligned} 12 &= 1\cdot 2^3 + 1\cdot 2^2 &&= (1,1,0,0)_2 \\ &= 1\cdot 2^3 + 2\cdot 2^1 &&= (1,0,2,0)_2 \\ &= 1\cdot 2^3 + 1\cdot 2^1 + 2\cdot 2^0 &&= (1,0,1,2)_2 \\ &= 2\cdot 2^2 + 1\cdot 2^1 + 2\cdot 2^0 &&= (0,2,1,2)_2 \\ &= 1\cdot 2^2 + 2\cdot 2^1 &&= (0,1,2,0)_2 \end{aligned}$$

2) $s(n)$ is the number of odd Stirling numbers of second kind $\left\{ {n \atop k} \right\}$ with odd $k \le n$. (L. Carlitz) 1964

n\k	0	1	2	3	4	5	6	7	8	9, 10 …	s(n)
0	1										0
1	0	~~1~~									1
2	0	~~1~~	1								1
3	0	~~1~~	3	~~1~~							2
4	0	~~1~~	7	6	1						1
5	0	~~1~~	15	~~25~~	10	~~1~~					3
6	0	~~1~~	31	90	65	~~15~~	1				2
7	0	~~1~~	63	~~301~~	350	140	21	~~1~~			3
8	0	~~1~~	127	966	1701	1050	266	28	1		1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮

3) $s(n)$ is the number of odd binomial coefficients in the nth subdiagonal, i.e., $\binom{l}{k}$ with $l+k=n$, $l \geq k \geq 0$, of Pascal's triangle.

1 1 2 3 5 8 ... Fibonacci sequence

$$\begin{array}{llllll}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & \\
1 & 5 & 10 & 10 & 5 & 1 \\
\ldots & & & & &
\end{array}
\xrightarrow{\text{modulo } 2}
\begin{array}{llllll}
1 & & & & & \\
1 & 1 & & & & \\
1 & 0 & 1 & & & \\
1 & 1 & 1 & 1 & & \\
1 & 0 & 0 & 0 & 1 & \\
1 & 1 & 0 & 0 & 1 & 1 \\
\ldots & & & & &
\end{array}$$

Stern's diatomic sequence 1 1 2 1 3 2 ...

in formula:

$$s(n+1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}_2 \, , \quad n \geq 0$$

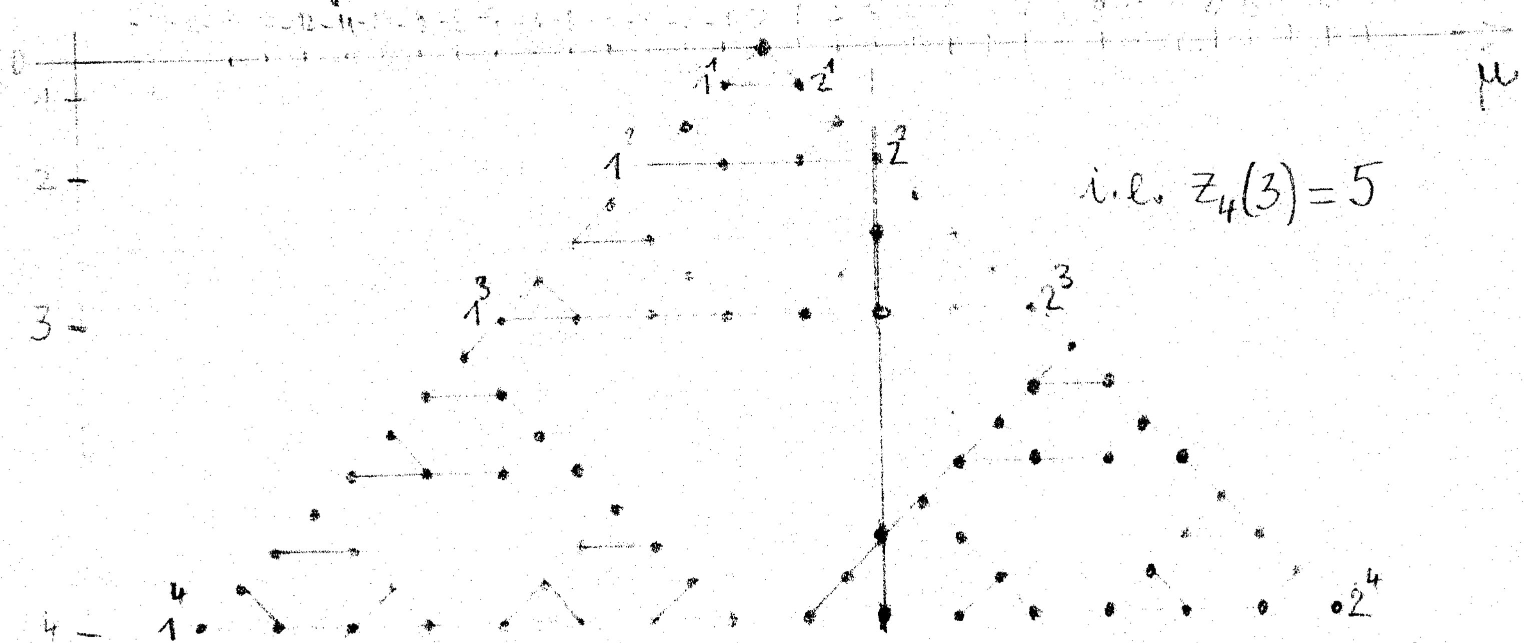
where $0 \leq \binom{n-k}{k}_2 < 2$ denotes the remainder of $\binom{n-k}{k}$ when divided by 2.

$\bigwedge n \in \mathbb{N}_0 :$

4) Tower of Hanoi

$s(\mu) = z_n(2^n - \mu)$, $\mu = 0, 1, \dots, 2^n$

$z_n(\mu)$ is the number of regular states in H_3^n for which the difference to two distinct perfect states is equal to μ.



i.e. $z_4(3) = 5$

$n \,\nabla$

$$\boxed{z_n(\mu) := \left|\left\{ r \in H_3^n \mid d(s, 1^n) - d(s, 2^n) = \mu \right\}\right|}$$

n	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0																0	1	0															
1															0	1	1	1	0														
2													0	1	1	2	1	2	1	1	0												
3									0	1	1	2	1	3	2	3	1	3	2	3	1	2	1	1	0								
4	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	4	3	5	2	5	3	4	1	3	2	3	1	2	1	1	0
5																																	
⋮																																	

$s(\mu) = z_{n+1}(\mu) - z_n(\mu)$, $\mu = 0, 1, \dots, 2^n$

$(z_n(\mu))_{\mu \in \mathbb{Z}}$ is the nth row of the STERN-BROCOT array $\left(((0,1,0))_n\right)_{n \in \mathbb{N}_0}$

0 1 0

0 1 1 1 0

0 1 1 2 1 2 1 1 0

0 1 1 2 1 3 2 3 1 3 2 3 1 2 1 1 0

E10) Some variations of the Stern sequence

a) The Reznick sequence $r(n)$ (1984)

Recurrence relation:

$$\begin{cases} r(0)=0, \quad r(1)=1 \\ r(2n)=r(n), \quad r(2n+1)=-r(n)+r(n+1) \end{cases}$$

Special case: $\alpha=1,\ \beta=0,\ \gamma=-1,\ \delta=1,\ \varepsilon=0,\ \zeta=1$

The generating function $\varphi(s)$ is Mahlerian:

$$s\varphi(s)=(1+s-s^2)\varphi(s^2)$$

It is given by the infinite product

$$\varphi(s)=s\cdot\prod_{k=0}^{\infty}\left(1+s^{2^k}-s^{2^{k+1}}\right)$$

The first few values of $r(n)$ are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$r(n)$	0	1	1	0	1	-1	0	1	1	-2	-1	1	0	1	1	0	1

A005590 in the OEIS

b) <u>The twisted Stern sequence $t(n)$</u> (Bacher, 2010)

Recurrence relation:

$$\begin{cases} t(0)=0, \; t(1)=1 \\ t(2n) = -t(n), \quad t(2n+1) = -t(n) - t(n+1) \end{cases}$$

Special case: $\alpha = -1$, $\beta = 0$, $\gamma = -1$, $\delta = -1$, $\varepsilon = 0$, $\zeta = 1$

The generating function $\varphi(s)$ satisfies the functional equation

$$s\varphi(s) = -(1+s+s^2)\varphi(s^2) + 2s^2$$

The first few values of $t(n)$ are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$t(n)$	0	1	-1	0	1	1	0	-1	-1	-2	-1	-1	0	1	1	2	1

<u>A213369</u> in the OEIS

Solution of the special case $\delta = 0$

$$\begin{cases} f(1) = \zeta \\ f(2n) = \alpha f(n) + \beta, \quad f(2n+1) = \gamma f(n) + \varepsilon, \quad n \geq 1 \end{cases}$$

Let $n = (b_m \ldots b_0)_2$, then

$$f((b_m \ldots b_0)_2) = \zeta \cdot \prod_{k=0}^{m-1} \left(\gamma b_k + \alpha(1-b_k)\right)$$
$$+ \sum_{k=0}^{m-1} \left(\prod_{j=0}^{k-1} \left(\gamma b_j + \alpha(1-b_j)\right)\right) \cdot \left(\varepsilon b_k + \beta(1-b_k)\right)$$

Proof

$$f((b_m \ldots b_0)_2) = b_0 \cdot \left(\gamma f((b_m \ldots b_1)_2) + \varepsilon\right) +$$
$$+ (1-b_0)\left(\alpha \cdot f((b_m \ldots b_1)_2) + \beta\right)$$
$$= \left(\gamma b_0 + \alpha(1-b_0)\right) \cdot f((b_m \ldots b_1)_2) + \left(\varepsilon b_0 + \beta(1-b_0)\right)$$
$$= \left(\gamma b_0 + \alpha(1-b_0)\right) \cdot \left[\left(\gamma b_1 + \alpha(1-b_1)\right) \cdot f((b_m \ldots b_2)_2)\right.$$
$$\left. + \varepsilon b_1 + \beta(1-b_1)\right] + \left(\varepsilon b_0 + \beta(1-b_0)\right)$$
$$= \left(\gamma b_0 + \alpha(1-b_0)\right) \cdot \left(\gamma b_1 + \alpha(1-b_1)\right) \cdot f((b_m \ldots b_2)_2)$$
$$+ \left(\gamma b_0 + \alpha(1-b_0)\right) \cdot \left(\varepsilon b_1 + \beta(1-b_1)\right) + \left(\varepsilon b_0 + \beta(1-b_0)\right)$$
$$= \ldots \quad \text{by iteration}$$
$$\Rightarrow \text{Assertion}$$

Examples:

Let $\gamma = \alpha$, then

$$f((b_m \ldots b_0)_2) = \delta \alpha^m + \sum_{k=0}^{m-1} \alpha^k \left(\varepsilon b_k + \beta (1 - b_k) \right)$$

$$= \delta \alpha^m + (\varepsilon - \beta) \sum_{k=0}^{m-1} \alpha^k \cdot b_k + \beta \sum_{k=0}^{m-1} \alpha^k$$

Special cases: (E1), (E2)

If, in addition, $\varepsilon = \beta$, then

$$f((b_m \ldots b_0)_2) = \delta \alpha^m + \beta \sum_{k=0}^{m-1} \alpha^k$$

depends only on $m = \lfloor \log_2 n \rfloor$ (= number of bits of the binary expression of $n - 1$.)

General First Order Linear Difference Equation with Variable Coefficients

Let $a(n+1) = q_n \cdot a(n) + r_n$, $n \in \mathbb{N}_0$
with initial value a_0. Then

$$\boxed{\bigwedge_{n\in\mathbb{N}_0}: \quad a(n) = a_0\left(\prod_{k=0}^{n-1} q_k\right) + \sum_{k=0}^{n-1}\left(\prod_{j=0}^{k-1} q_{n-j}\right)\cdot r_{n-1-k}}$$

Special cases:

i) $q_n = q = \text{const}$: $\quad a(n) = a_0 \cdot q^n + \sum_{k=0}^{n-1} q^k \cdot r_{n-1-k}$

ii) $r_n = r = \text{const}$: $\quad a(n) = a_0\left(\prod_{k=0}^{n-1} q_k\right) + r \cdot \sum_{k=0}^{n-1}\left(\prod_{j=0}^{k-1} q_{n-j}\right)$

iii) $q_n = q = \text{const}$ and $r_n = r = \text{const}$:

$$\boxed{\underset{n\geq 0}{a(n)} = a_0 \cdot q^n + r \cdot \sum_{k=0}^{n-1} q^k = \begin{cases} a_0 + n \cdot r, & q = 1 \\ a_0 q^n + r\dfrac{q^n - 1}{q-1}, & q \neq 1 \end{cases}}$$

$q=1$: arithmetic sequence of first order

$r=0$: geometric sequence

Special values

i) $$\boxed{f(2^m) = \xi\alpha^m + \beta \sum_{k=0}^{m-1} \alpha^k \quad , m \geq 0}$$ $\left[2^m = (10\ldots0)_2\right]$

ii) $$\boxed{f(2^m+1) = \gamma\xi\alpha^{m-1} + \beta\gamma \sum_{k=0}^{m-2} \alpha^k + \varepsilon \quad , m \geq 1}$$ $\left[2^m+1 = (10\ldots01)_2\right]$

iii) $$\boxed{f(2^{m+1}-1) = \xi\gamma^m + \varepsilon \sum_{k=0}^{m-1} \gamma^k \quad , m \geq 0}$$ $\left[2^{m+1}-1 = (11\ldots1)_2\right]$

Proof i) Let $a_m := f(2^m)$. Then $a_{m+1} = \alpha a_m + \beta$ and $a_0 = \xi$ $\Rightarrow$ Assertion

ii) Let $a_m := f(2^m+1)$. Then $a_{m+1} = \gamma f(2^m) + \varepsilon$ $\Rightarrow$ Assertion

iii) Let $a_m := f(2^{m+1}-1)$. Then
$$a_{m+1} = \gamma a_m + \varepsilon \quad , a_0 = \xi \quad \Rightarrow \text{Assertion}$$

Alternative solution

Let $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$ a strictly decreasing sequence of integers. Then

i) $\underline{\alpha \neq 0}$:

$$f(2^{a_0} + \cdots + 2^{a_\ell}) = \left(\frac{\gamma}{\alpha}\right)^\ell \cdot \left\{ \zeta \alpha^{a_0} + \beta \cdot \sum_{k=0}^{a_0 - 1} \alpha^k \right\}$$
$$+ \sum_{k=0}^{\ell-1} \left(\frac{\gamma}{\alpha}\right)^k \cdot \left\{ (\varepsilon - \beta)\, \alpha^{a_{\ell-k}} + \beta \left(1 - \frac{\gamma}{\alpha}\right) \sum_{j=0}^{a_{\ell-k}} \alpha^j \right\}$$

ii) $\underline{\alpha = 0}$:

$$f(2^{a_0} + \cdots + 2^{a_\ell}) = \beta + \gamma^\ell (\zeta - \beta) \prod_{k=0}^{\ell} (a_{\ell-k} = k)$$
$$+ (\beta\gamma + \varepsilon - \beta) \sum_{k=0}^{\ell} \gamma^k \left(\prod_{j=0}^{k-1} (a_{\ell-j} = j) \right)$$

Example **Binary search**

$$\begin{cases} B(1) = 1 \\ B(n) = B\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 \quad , n \geq 2 \end{cases}$$

that is
$$\begin{cases} B(1) = 1 \\ B(2n) = B(n)+1 \quad , \quad B(2n+1) = B(n)+1 \quad , n \geq 1 \end{cases}$$

Special case: $\alpha = 1, \beta = 1, \gamma = 1, \delta = 0, \varepsilon = 1, \zeta = 1$

Solution: $B(n) = 1 + a_0 = 1 + \lfloor \log_2 n \rfloor$

(= number of bits in the binary expansion of n)

First few values

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$B(n)$	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4	5

"n appears 2^{n-1} times" A113473 in the OEIS

Generating function

$B(n)$: $\boxed{\varphi(s) = (1+s)\,\varphi(s^2) + \dfrac{s}{1-s}}$

The general case $\delta \neq 0$

Special values

i) $$f(2^m) = \xi \cdot \alpha^m + \beta \sum_{k=0}^{m-1} \alpha^k \quad , m \geq 0$$

ii) $$f(2^m+1) = (\alpha\xi+\beta)\delta^m + \gamma\xi \sum_{k=0}^{m-1} \delta^k \alpha^{m-1-k} + \beta\gamma \sum_{k=0}^{m-1} \delta^k \left(\sum_{j=0}^{m-2-k} \alpha^j\right) + \varepsilon \cdot \sum_{k=0}^{m-1} \delta^k \quad , m \geq 0$$

iii) $$f(2^{m+1}-1) = \xi\gamma^m + \delta\xi \sum_{k=0}^{m-1} \gamma^k \alpha^{m-k} + \beta\delta \sum_{k=0}^{m-1} \delta^k \left(\sum_{j=0}^{m-1-k} \alpha^j\right) + \varepsilon \cdot \sum_{k=0}^{m-1} \gamma^k \quad , m \geq 0$$

Proof i) Let $a_m := f(2^m)$, $a_0 = \xi$. Then
$a_{m+1} = \alpha a_m + \beta \Rightarrow$ Assertion

ii) Let $b_m := f(2^m+1)$, $b_0 = \alpha\xi + \beta$. Then
$b_{m+1} = \delta b_m + \gamma a_m + \varepsilon \Rightarrow$ Assertion

iii) Let $c_m := f(2^{m+1}-1)$, $c_0 = \xi$. Then
$c_{m+1} = \gamma \cdot c_m + \delta \cdot a_{m+1} + \varepsilon \Rightarrow$ Assertion

Solution of the general case $\delta \neq 0$

$$\begin{cases} f(1) = \xi \\ f(2n) = \alpha f(n) + \beta,\ f(2n+1) = \gamma f(n) + \delta f(n+1) + \varepsilon \\ \qquad n \geq 1 \end{cases}$$

Let $\vec{f}(n) := \begin{pmatrix} f(n) \\ f(n+1) \end{pmatrix}$, then $\vec{f}(1) = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix} = \begin{pmatrix} \xi \\ \alpha\xi + \beta \end{pmatrix}$

$$\vec{f}(2n) = \begin{pmatrix} f(2n) \\ f(2n+1) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \vec{f}(n) + \begin{pmatrix} \beta \\ \varepsilon \end{pmatrix}$$

and $\vec{f}(2n+1) = \begin{pmatrix} f(2n+1) \\ f(2n+2) \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ 0 & \alpha \end{pmatrix} \vec{f}(n) + \begin{pmatrix} \varepsilon \\ \beta \end{pmatrix}$

$$\Longleftrightarrow \quad (*) \quad \boxed{\begin{array}{c} \vec{f}(1) = \begin{pmatrix} \xi \\ \alpha\xi + \beta \end{pmatrix} \\ \vec{f}(2n) = A \cdot \vec{f}(n) + \vec{b},\quad \vec{f}(2n+1) = B \cdot \vec{f}(n) + C \cdot \vec{b} \end{array}}$$

where $A := \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$, $B := \begin{pmatrix} \gamma & \delta \\ 0 & \alpha \end{pmatrix}$, $\vec{b} := \begin{pmatrix} \beta \\ \varepsilon \end{pmatrix}$,
$C := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\alpha, \beta, \gamma, \delta, \varepsilon, \xi \in \mathbb{Z}$.

Let $n=(b_m\dots b_0)_2$. Then the solution of $(*)$ is given by

$$\boxed{\vec{f}\left((b_m\dots b_0)_2\right)=\chi(b_0)\cdot\chi(b_1)\cdots\chi(b_{m-1})\cdot\begin{pmatrix}\xi\\ \alpha\xi+\beta\end{pmatrix} + \left(\chi(b_0)\cdots\chi(b_{m-1})\cdot C^{b_{m-1}}+\dots+\chi(b_0)C^{b_1}+C^{b_0}\right)\begin{pmatrix}\beta\\ \varepsilon\end{pmatrix}}$$

where $\chi(b_i)=A^{1-b_i}B^{b_i}=(1-b_i)A+b_iB$, $i=0,1,\dots,m-1$,

i.e., $\chi(0)=A$, $\chi(1)=B$

<u>Proof</u> Let $n=(b_m\dots b_0)_2$. If $b_0=0$, then

$$\vec{f}\left((b_m\dots b_0)_2\right)=A\vec{f}\left((b_m\dots b_1)_2\right)+\vec{b}$$

and if $b_0=1$:

$$\vec{f}\left((b_m\dots b_0)_2\right)=B\cdot\vec{f}\left((b_m\dots b_1)_2\right)+C\cdot\vec{b}.$$

Hence,

$$\vec{f}\left((b_m\dots b_0)_2\right)=(1-b_0)\left\{A\cdot\vec{f}\left((b_m\dots b_1)_2\right)+\vec{b}\right\}+$$
$$+b_0\cdot\left\{B\cdot\vec{f}\left((b_m\dots b_1)_2\right)+C\cdot\vec{b}\right\}$$
$$=\left[(1-b_0)A+b_0B\right]\cdot\vec{f}\left((b_m\dots b_1)_2\right)+\left[(1-b_0)\vec{b}+b_0C\vec{b}\right]$$
$$=\chi(b_0)\cdot\vec{f}\left((b_m\dots b_1)_2\right)+C^{b_0}\cdot\vec{b},$$ where $C^0=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is the unit matrix

$\Longrightarrow$ Iteration

$$\vec{f}\left((b_m\dots b_0)_2\right)=\chi(b_0)\cdots\chi(b_{m-1})\cdot\vec{f}(1)+\sum_{k=0}^{m-1}\left(\chi(b_0)\cdots\chi(b_{k-1})\right)C^{b_k}\begin{pmatrix}\beta\\ \varepsilon\end{pmatrix}$$

$\Longrightarrow$ Assertion.

Examples

1) <u>E7</u> : $\alpha=1, \beta=0, \gamma=-1, \delta=0, \varepsilon=0, \zeta=-1$

$$\Rightarrow A=\begin{pmatrix}1&0\\-1&0\end{pmatrix}, \quad B=\begin{pmatrix}-1&0\\0&1\end{pmatrix}, \quad \vec{b}=\begin{pmatrix}0\\0\end{pmatrix}, \quad \begin{pmatrix}\zeta\\ \alpha\zeta+\beta\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}$$

$$(1-b_i)A+b_iB=\begin{pmatrix}1-2b_i&0\\b_i-1&b_i\end{pmatrix}, \quad i=0,1,\dots,m$$

<u>Solution</u> $\vec{e}\left((b_m\dots b_0)_2\right)=\begin{pmatrix}1-2b_0&0\\b_0-1&b_0\end{pmatrix}\begin{pmatrix}1-2b_1&0\\b_1-1&b_1\end{pmatrix}\cdots\begin{pmatrix}1-2b_{m-1}&0\\b_{m-1}-1&b_{m-1}\end{pmatrix}\begin{pmatrix}-1\\-1\end{pmatrix}$

<u>$n=21$</u> $=(10101)_2 \Rightarrow m=4, b_0=1, b_1=0, b_2=1, b_3=0$
$b_4=1$

$$\vec{e}(21)=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&0\end{pmatrix}\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&0\end{pmatrix}\begin{pmatrix}-1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\-1\end{pmatrix}, \text{ that is } \underline{\underline{e(21)=-1}}, \underline{\underline{e(22)=-1}}.$$

2) <u>E9</u> : $\alpha=1, \beta=0, \gamma=1, \delta=1, \varepsilon=0, \zeta=1$

$$\Rightarrow A=\begin{pmatrix}1&0\\1&1\end{pmatrix}, \quad B=\begin{pmatrix}1&1\\0&1\end{pmatrix}, \quad \vec{b}=\begin{pmatrix}0\\0\end{pmatrix}, \quad \begin{pmatrix}\zeta\\ \alpha\zeta+\beta\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$(1-b_i)A+b_iB=\begin{pmatrix}1&b_i\\1-b_i&1\end{pmatrix}, \quad i=0,1,\dots,m$$

<u>Solution</u>

$$\vec{s}\left((b_m\dots b_0)_2\right)=\begin{pmatrix}1&b_0\\1-b_0&1\end{pmatrix}\begin{pmatrix}1&b_1\\1-b_1&1\end{pmatrix}\cdots\begin{pmatrix}1&b_{m-1}\\1-b_{m-1}&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$n = 21 = (10101)_2$

$$\vec{s}(21) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

that is $\underline{\underline{s(21) = 8}}$, $\underline{\underline{s(22) = 5}}$.

Note that $CA = BC = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = Q$, Q-matrix with $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

3) $\underline{E10a)}$ $\alpha_1 = 1$, $\beta = 0$, $\gamma = -1$, $\delta = 1$, $\varepsilon = 0$, $\zeta = 1$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \zeta \\ \alpha\zeta + \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(1-b_i)A + b_i B = \begin{pmatrix} 1-2b_i & b_i \\ b_i - 1 & 1 \end{pmatrix}, \quad i = 0, 1, \ldots, m$$

<u>Solution</u>

$$\vec{r}((b_m \ldots b_0)_2) = \begin{pmatrix} 1-2b_0 & b_0 \\ b_0 - 1 & 1 \end{pmatrix} \begin{pmatrix} 1-2b_1 & b_1 \\ b_1 - 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1-2b_{m-1} & b_{m-1} \\ b_{m-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$n = 21 = (10101)_2$

$$\vec{r}(21) = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{\begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}} \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{\begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$, that is $r(21) = 2$, $r(22) = 1$.

Alternative solution (for $\varepsilon=\beta=0,\ \alpha\neq 0$)

Let
$$\begin{cases} f(1)=\zeta \\ f(2n)=\alpha f(n), \quad f(2n+1)=\gamma f(n)+\delta f(n+1), \\ \qquad n\geq 1 \end{cases}$$

then ~~the~~ $f(n)$ satisfies the recurrence relation of second order:

$$f(2^{a_0}+\dots+2^{a_\ell})=\left\{\frac{\gamma}{\alpha}\cdot\sum_{k=0}^{a_{\ell-1}-a_\ell-1}\left(\frac{\delta}{\alpha}\right)^k+\left(\frac{\delta}{\alpha}\right)^{a_{\ell-1}-a_\ell}\right\}\cdot$$
$$\cdot f(2^{a_0}+\dots+2^{a_{\ell-1}})+\left(\frac{\delta}{\alpha}\right)^{a_{\ell-2}-a_\ell-1}\cdot\left(\alpha-\delta-\frac{\gamma}{\alpha}\right)\cdot f(2^{a_0}+\dots+2^{a_{\ell-2}}),$$
$$\ell\geq 0$$

with
$$f(2^{a_0})=\zeta\,\alpha^{a_0}$$
$$f(2^{a_0}+2^{a_1})=\zeta\,\alpha^{a_0}\left\{\alpha\left(\frac{\delta}{\alpha}\right)^{a_0-a_1}+\frac{\gamma}{\alpha}\sum_{k=0}^{a_0-a_1-1}\left(\frac{\delta}{\alpha}\right)^k\right\}$$

and the solution can be given as the numerator of a continued fraction.

<u>Example</u>: <u>E9</u>: $\alpha=1,\ \beta=0,\ \gamma=1,\ \delta=1,\ \varepsilon=0,\ \zeta=1$

$$\Rightarrow f(2^{a_0}+\dots+2^{a_\ell})\overset{f(2^{a_0})=1}{=}(a_{\ell-1}-a_\ell+1)\cdot f(2^{a_0}+\dots+2^{a_{\ell-1}})-f(2^{a_0}+\dots+2^{a_{\ell-2}})$$

continued fraction $\quad a_0-a_1+1-\dfrac{1|}{|a_1-a_2+1}-\dots-\dfrac{1|}{|a_{\ell-1}-a_\ell+1}=\dfrac{A_\ell}{B_\ell}$

$$S(2^{a_0}+\dots+2^{a_\ell})=A_\ell.$$

Generalization

$$\begin{cases} f(1) = \xi & (f(0)) \\ f(2n) = \alpha f(n) + g(n) & , n \geq 1 \\ f(2n+1) = \gamma f(n) + \delta f(n+1) + h(n) & \end{cases}$$

where $g, h : \mathbb{N} \longrightarrow \mathbb{Z}$

Example 1 : $\delta = 0$

The sequence $(D_2(n))_{n \in \mathbb{N}}$ = partial sum of $d_2(n)$ (E3)

$$D_2(n) := \sum_{k=1}^{n} d_2(k)$$

We have $D_2(1) = 1$

$$(*) \begin{cases} D_2(2n) = D_2(n) + n \quad , \quad D_2(2n+1) = D_2(n) + 2n+1, \; n \geq 1 \end{cases}$$

Special case : $\alpha = 1, \gamma = 1, \delta = 0, g(n) = 2n, h(n) = 2n+1, \xi = 1$

(*) can be written as

$$\begin{cases} D_2(1) = 1 \\ D_2(n) = D_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n \quad , \quad n \geq 2 \end{cases}$$

First few values

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$D_2(n)$	0	1	3	4	7	8	10	11	15	16	18	19	22	23	25	26	31

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Solution of (*)

$$\boxed{D_2(n) = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor}$$

Proof : We have

$$D_2(n) = \sum_{k=1}^{n} d_2(k) = \sum_{k=1}^{n} \left(\sum_{t \mid k} \rho(t) \right), \quad \rho(n) = \begin{cases} 1, & n = 2^m, \ m \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

$$= \sum_{j=1}^{n} \rho(j) \left\lfloor \frac{n}{j} \right\rfloor = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor. \quad \square$$

This formule means:
$D_2(n)$ is the number of positive powers of 2 which divides all numbers $\leq n$.

Example : $n = 21$

$$D_2(21) = \sum_{k=0}^{\lfloor \log_2 21 \rfloor} \left\lfloor \frac{21}{2^k} \right\rfloor = \left\lfloor \frac{21}{1} \right\rfloor + \left\lfloor \frac{21}{2} \right\rfloor + \left\lfloor \frac{21}{4} \right\rfloor + \left\lfloor \frac{21}{8} \right\rfloor + \left\lfloor \frac{21}{16} \right\rfloor = 21 + 10 + 5 + 2 + 1 = \underline{\underline{39}}$$

The generating function $\varphi(s) := \sum_{n=1}^{\infty} D_2(n)\, s^n$ satisfies the functional equation

$$\varphi(s) = (1+s)\,\varphi(s^2) + \frac{s}{(1-s)^2}, \qquad (*)$$

$(D_2(n))_{n\in\mathbb{N}}$ is of the DC-type.

The solution of $(*)$ is given either by

1) $$\varphi(s) = \frac{1}{1-s} \cdot \underbrace{\sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^k}}}_{\text{gen. function of } d_2(n)}$$

$\uparrow$ gen. function of the partial sums of $d_2(n)$

or by a Lambert series

2) $$\varphi(s) = \sum_{n=1}^{\infty} a_n \cdot \frac{s^n}{1-s^n}$$

where (a_n) is given by $D_2(n) = \sum_{t|n} a_n$

$\Longrightarrow$ Möbius inversion formula

$$a_n = \sum_{t|n} \mu\left(\tfrac{n}{t}\right) \cdot \underbrace{D_2(t)}_{= 2t - s_2(t)}$$

where $$\mu(n) := \begin{cases} 1, & n = 0 \\ (-1)^k, & \text{if } n = p_1 \cdots p_k \text{ different primes} \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow a_n = 2 \sum_{t|n} \mu\left(\tfrac{n}{t}\right) t - \sum_{t|n} \mu\left(\tfrac{n}{t}\right) s_2(t)$$

$$\boxed{a_n = 2\,\varphi(n) - \sum_{t|n} \mu\left(\tfrac{n}{t}\right) \cdot s_2(t)},$$

$\varphi(n)$ Euler-totient function giving the number of positive integers not exceeding n which are relatively prime to n

The first few values are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	1	2	3	4	7	4	10	8	12	8	18	8	22	12	15	16

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Additional properties

a) $D_2(n) - n = n - S_2(n)$ is the highest power of 2 dividing $n!$, that is

$$\boxed{n! = 2^{n-S_2(n)} \cdot (2k+1) \quad , k \in \mathbb{N}_0}$$

First few values

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$D_2(n)-n$	0	0	1	1	3	3	4	4	7	7	8	8	10	10	11	11	15

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b) $\lim\limits_{n\to\infty} \frac{D_2(n)}{n} = 2$, that is

$$\lim_{n\to\infty} \frac{S_2(n)}{n} = \lim_{n\to\infty} \left(2 - \frac{D_2(n)}{n}\right) = 0$$

c) Generating function of $S_2(n)$ (other representation)

$$\varphi(s) = \sum_{n=1}^{\infty} S_2(n) s^n = \sum_{n=1}^{\infty} (2n - D_2(n)) s^n$$

$$= 2\sum_{n=1}^{\infty} n s^n - \sum_{n=1}^{\infty} D_2(n) s^n = 2 \cdot \frac{s}{(1-s)^2} - \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^k}}$$

Corollary
$$\boxed{\sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^{k+1}}} = \frac{s}{1-s}}$$

Proof
$$\frac{2s}{(1-s)^2} - \frac{1}{1-s}\sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^{k+1}}} = \frac{1}{1-s}\sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}}$$

$$\frac{2s}{1-s} = \sum_{k=0}^{\infty}\left(\frac{s^{2^k}}{1-s^{2^k}} + \frac{s^{2^k}}{1+s^{2^k}}\right) = \sum_{k=0}^{\infty} \frac{s^{2^k}\left(1+\cancel{s^{2^k}}+1-\cancel{s^{2^k}}\right)}{1-s^{2^{k+1}}}$$

$\Rightarrow$ Assertion

d) Generating function of $d_2(n)$ and $D_2(n)$ (other representation)

$$\varphi(s) = \sum_{n=1}^{\infty} d_2(n)\, s^n = \sum_{n=1}^{\infty} \left(2 - \Delta s_2(n)\right) s^n$$

$$= 2\sum_{n=1}^{\infty} s^n - \sum_{n=1}^{\infty} \Delta s_2(n)\, s^n$$

$$= 2\frac{s}{1-s} - (1-s)\underbrace{\sum_{n=1}^{\infty} s_2(n)\, s^n}$$

$$\boxed{\varphi(s) = \frac{2s}{1-s} - \sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}}} \quad : d_2(n)$$

$$\varphi(s) = \sum_{n=1}^{\infty} D_2(n)\, s^n = \frac{1}{1-s}\underbrace{\sum_{n=1}^{\infty} d_2(n)\, s^n}$$

or
$$\boxed{\varphi(s) = \frac{2s}{(1-s)^2} - \frac{1}{1-s}\sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}}} \quad : D_2(n)$$

l) The sequence $e_0(n)$ (= number of 0's in the binary expansion of n)

$$e_0(n) = B(n) - S_2(n)$$

Recurrence relation

$$\begin{cases} e_0(1) = 0 \\ e_0(2n) = e_0(n) + 1 \quad , \quad e_0(2n+1) = e_0(n) \quad , \quad n \geq 1 \end{cases}$$

Special case $\alpha = 1, \beta = 1, \gamma = 1, \delta = 0, \varepsilon = 0, \zeta = 0$

First few values

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$e_0(n)$	0	1	0	2	1	1	0	3	2	2	1	2	1	1	0	4
$S_2(n)$	1	1	2	1	2	2	3	1	2	2	3					

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Generating function

$$\varphi(s) = (1+s)\,\varphi(s^2) + \frac{s^2}{1-s^2}$$

$$\Rightarrow \boxed{\varphi(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{s^{2^{k+1}}}{1+s^{2^k}}}$$

β) Generating function of B(n)

$$\sum_{n=1}^{\infty} B(n)\, s^n = \sum_{n=1} \left(\ell_0(n) + s_2(n)\right) s^n$$

$$= \frac{1}{1-s} \sum_{k=0}^{\infty} \left(\frac{s^{2^{k+1}}}{1+s^{2^k}} + \frac{s^{2^k}}{1+s^{2^k}} \right)$$

$$\boxed{\varphi(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} s^{2^k}}$$

Cauchy-Product $= \left(\sum_{n=0}^{\infty} s^n\right) \cdot \left(\sum_{k=0}^{\infty} s^{2^k}\right)$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \rho(k) \cdot 1 \right) s^n$$

$$\Rightarrow \boxed{B(n) = \sum_{k=0}^{n} \rho(k)}$$

partial sum of $\rho(n)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\rho(n)$	0	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1
$B(n)$	0	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4	5

Example 2 : $\delta \neq 0$

Mergesort (see Graham, Knuth, Patashnik Exercise 3.34 p. 98 and p. 496)

In order to compare n records, $n > 1$, divide them into two approximately equal parts,

$$n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil .$$

Sort each part separately (by the same method applied recursively). Then merge the records into their final order by doing at most $n-1$ further comparisons. Thus the total number of comparisons performed is at most $f(n)$, where $f(n)$ satisfies the recurrence relation (worst case)

$$\begin{cases} f(1) = 0 \\ f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + n - 1, & n \geq 2 \end{cases}$$

or, equivalently,

$$\begin{cases} f(1) = 0 \\ f(2n) = 2f(n) + 2n - 1 & , n \geq 1 \\ f(2n+1) = f(n) + f(n+1) + 2n \end{cases}$$

Special case: $\alpha = 2$, $\gamma = 1$, $\delta = 1$, $\zeta = 0$, $g(n) = 2n-1$
$h(n) = 2n$.

The solution is given by

$$f(n) = \sum_{k=1}^{n} \lceil \log_2 k \rceil = n \lceil \log_2 n \rceil - 2^{\lceil \log_2 n \rceil} + 1, \quad n \geq 1$$

Complexity: $O(n \log_2 n)$.