Some Identities Involving Fibonacci Numbers and Sequences with Similar Recursions

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(joint work with Tomislav Došlić)

4^{th} Croatian Combinatorial Days

22–23.9.2022.

Fibonacci, tribonacci and tetranacci numbers

$F_0 = 0$ and $F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$

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$F_n = F_{n-1} + F_{n-2}$

$T_0 = 0,\ T_1 = 1$ and $T_2 = 1$

$T_n = T_{n-1} + T_{n-2} + T_{n-3}$

$Q_0 = 0,\ Q_1 = 0,\ Q_2 = 0$ and $Q_3 = 1$

$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4}$

Honeycomb strip

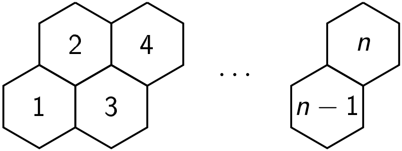
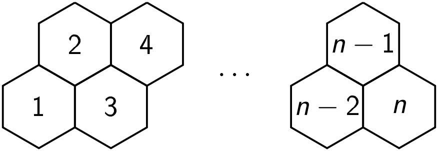


Figure: Honeycomb strip.



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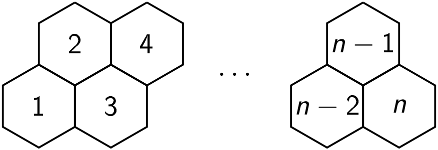

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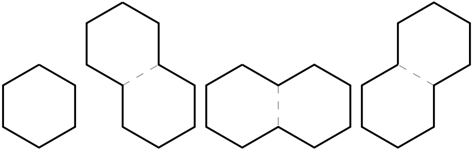

Figure: Monomer and three possible positions of a dimer tile.

$h_0 = 1,\ h_1 = 1,\ h_2 = 2$ and $h_3 = 4$.

$$h_n = h_{n-1} + h_{n-2} + h_{n-3} + h_{n-4}$$ [1]

[1] G. Dresden, Z. Jin, Tetranacci Identities via Hexagonal Tilings,Fibonacci Quart. (2022)

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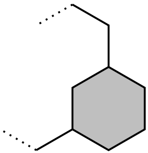

Figure: All possible endings of a tiled honeycomb strip.

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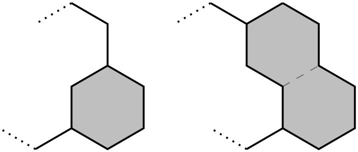

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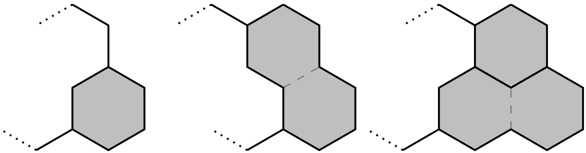

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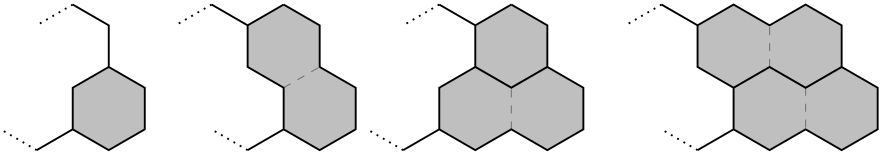

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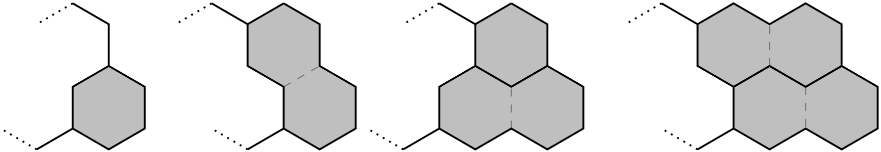

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$$h_n = Q_{n+3}$$

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- What is the number $c_{n,k}$ of tilings when number of dimers is fixed?
- What is the number $h_n^{a,b}$ of tilings when we allow colored tiles?

Tiling a honeycomb strip with exactly k dimers

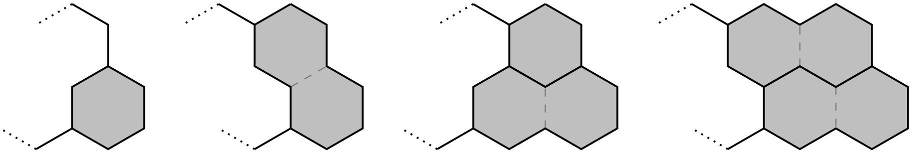

Figure: All possible endings of a tiled honeycomb strip.

$$c_{n,k} = c_{n-1,k} + c_{n-2,k-1} + c_{n-3,k-1} + c_{n-4,k-2}$$

Theorem

The number of ways to tile a honeycomb strip of length n using k dimers and $n-2k$ monomers is equal to

$$c_{n,k} = \sum_{m=0}^{k} \binom{n-k-m}{m}\binom{n-k-m}{k-m}.$$

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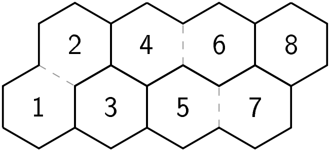

Figure: Breakability of a tiling

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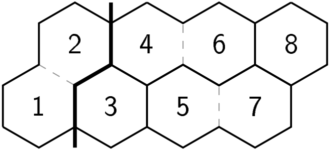

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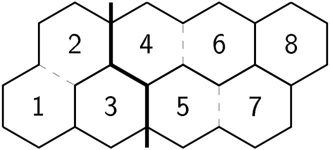

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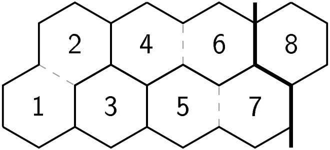

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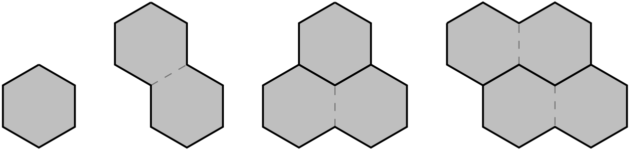

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$$c_{n,k} = \sum_{m=0}^{k} \binom{n-k-m}{m} \binom{n-k-m}{k-m}.$$

$$Q_{n+3} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} \binom{n-k-m}{m} \binom{n-k-m}{k-m}.$$

Tilings of honeycomb strip with colored dimers and monomers

Theorem

The number $h_n^{a,b}$ of all possible tilings of the honeycomb strip of length n with monomers of a different colors and dimers of b different colors is given by

$$h_n^{a,b} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k} a^{n-2k} b^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} \binom{n-k-m}{m} \binom{n-k-m}{k-m} a^{k-m} b^k.$$

Tiling of a honeycomb strip and tribonacci numbers

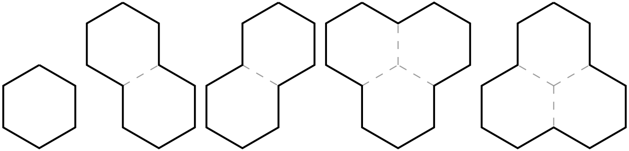

Figure: The allowed types of tiles.

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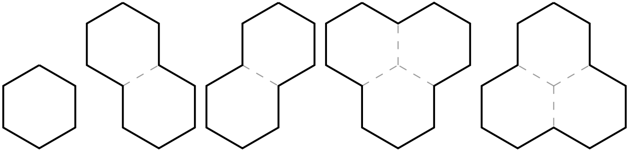

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Narayana's cows sequence - $N_0 = N_1 = N_2 = 1$ defined by

$$N_n = N_{n-1} + N_{n-3}.$$

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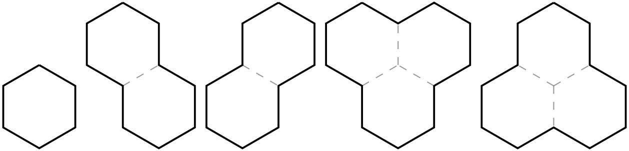

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Narayana's cows sequence - $N_0 = N_1 = N_2 = 1$ defined by

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Padovan sequence - $P_0 = 1,\ P_1 = P_2 = 0$ defined by

$$P_n = P_{n-2} + P_{n-3}.$$

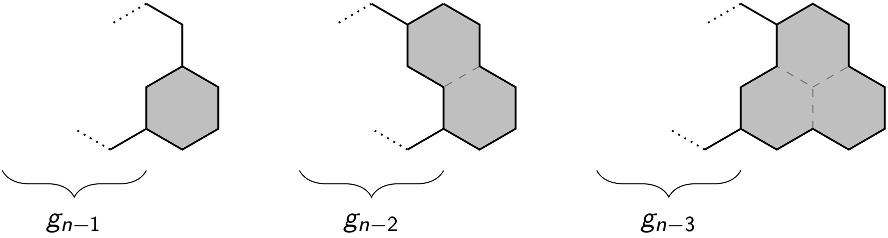


Figure: All possible endings of a tiling of a strip with n hexagons.

Theorem

Let g_n denote the number of all ways to tile a honeycomb strip of length by using only the allowed tiles. Then

$$g_n = T_{n+2},$$

where T_n denotes n-th tribonacci number.

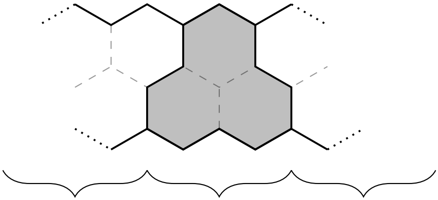
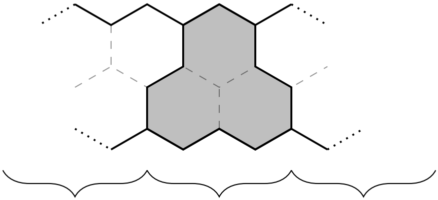


Figure: Section of a tiling.

Theorem

For $n \geq 0$, the number of ways to tile a strip with n hexagons using exactly k trimers is

$$t_{n,k} = \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k = n-2k+1}} F_{i_0} \cdots F_{i_k}.$$



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Theorem

For $n \geq 0$, the number of ways to tile a strip with n hexagons using exactly k monomers is

$$v_{n,k} = \sum_{\substack{i_0,\ldots,i_k > 0 \\ i_0+\cdots+i_k=n+2k+3}} P_{i_0} \cdots P_{i_k}. \tag{0.2}$$

Some combinatorial identities involving tribonacci numbers

Theorem

For $n \geq 4$, $T_n + T_{n-4} = 2T_{n-1}$.

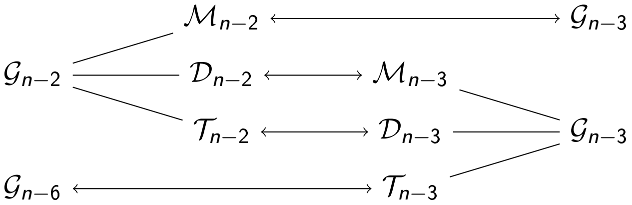

Figure: 1-1 correspondence between sets $\mathcal{G}_{n-2} \cup \mathcal{G}_{n-6}$ and $\mathcal{G}_{n-3} \times \{0, 1\}$.

Theorem

For any integer $n \geq 0$ we have the identity

$$T_{n+2} = \sum_{k=0}^{n+1} F_k T_{n-k}.$$

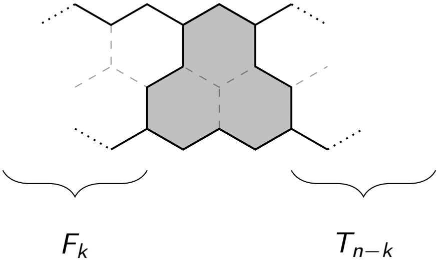

Figure: The trimer first occurs at position k.

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For any integer $n \geq 0$ *we have the identity*

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Theorem

For any integers $m, n \geq 1$ *we have the identity*

$$T_{m+n} = T_m T_n + T_{m+1} T_{n+1} + T_{m-1} T_n + T_m T_{n-1}.$$

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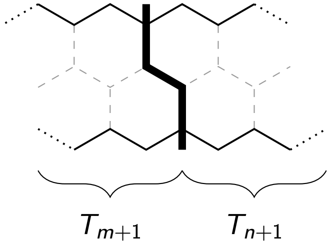

Figure: Layouts that can occur at position m.

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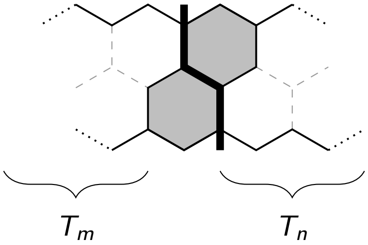

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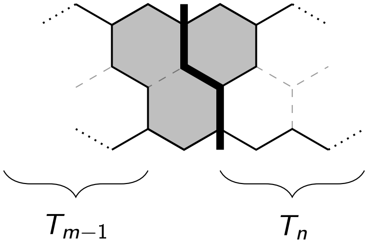

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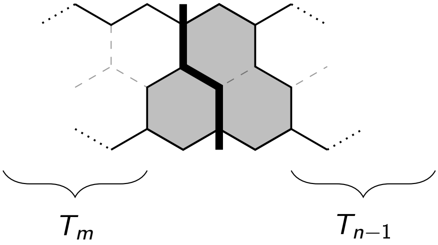

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