On the Maximum Value of W(L(G))/W(G)

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4th Croatian Combinatorial Days,
Zagreb, Croatia

September 22-23, 2022

Introduction

The **Wiener index** of the graph G is defined by

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v).$$

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- vertices of $L(G)$ correspond to edges of G;
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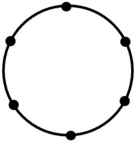

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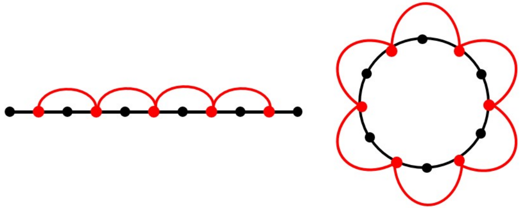

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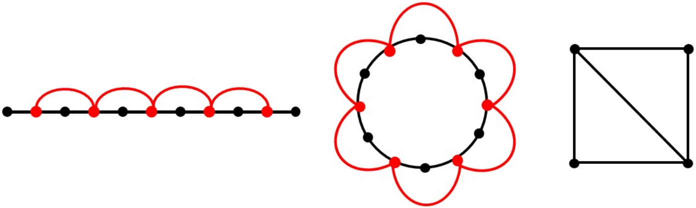

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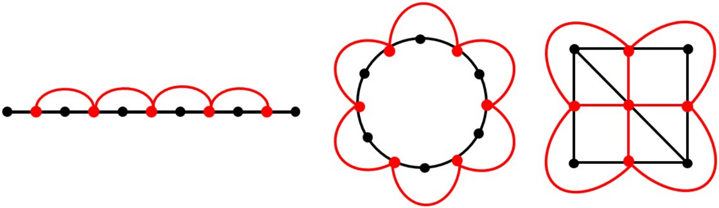

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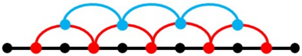

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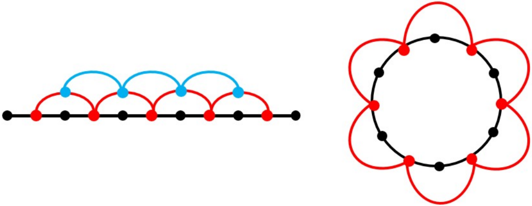

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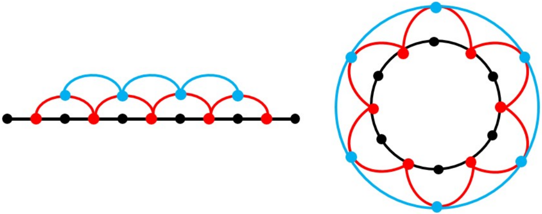

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Problem [1]. Establish the extremal values and graphs for the ratio $W(L^i(G))/W(G)$ for $i \geq 1$.

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and notice that from G to $L(G)$:

- distances change;
- number of pairs of vertices **also** changes!

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Partial solution [2]. Among all connected graphs on n vertices, the fraction $W(L(G))/W(G)$ is minimum for the star S_n.

[2] M. Knor, R. Škrekovski, A. Tepeh, An inequality between the edge-Wiener index and the Wiener index of a graph, Appl. Math. Comput. 269 (2015) 714–721.

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Open problem. Solve the initial problem for $i > 1$.

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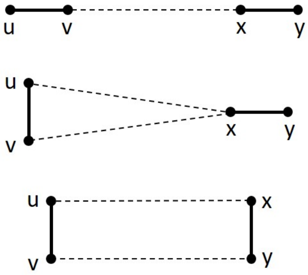

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The tightening of the bound on $W(L(G))$. For $e = uv$ and $f = xy$ from $E(G)$, the bound

$$d_{L(G)}(e, f) \leq D(e, f) + 1$$

is not tight for $v = x$ where

$$D(e, f) = \begin{cases} 1 & \text{if } uy \notin E(G), \\ \frac{3}{4} & \text{if } uy \in E(G). \end{cases}$$

thus

$$d_{L(G)}(e, f) = 1 \leq D(e, f) + 1 - \frac{3}{4}$$

Main results

Without tightening: we had

$$\begin{aligned} W(L(G)) &\leq \sum_{\{e,f\}\subseteq E(G)} (D(e,f)+1) = \sum_{\{e,f\}\subseteq E(G)} D(e,f) + \binom{m}{2} \\ &= \ldots = \frac{1}{4}(\mathrm{Gut}(G)-1) + \binom{m}{2} \end{aligned}$$

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With tightening: we have

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Main results

Lemma. For a graph G on n vertices with maximum degree Δ and minimum degree δ it holds that

$$\frac{W(L(G))}{W(G)} \leq \frac{1}{2n(n-1)}\left((n-1)^2\Delta^2 - \delta^2\right)$$

with equality if and only if $G = K_n$.

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Theorem. For a graph G on n vertices it holds that

$$\frac{W(L(G))}{W(G)} \leq \binom{n-1}{2}$$

with equality if and only if $G = K_n$.

Thank you...

...for the attention!