Visibility in restricted involutions

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joint work with M. Barnabei, F. Bonetti, and N. Castronuovo (Bologna)

Visibile pairs in words

Let $w = x_1x_2\dots x_n$ be a word whose letters are pairwise distinct integers. A visible pair in w is a pair (x_i, x_{i+r}), $r \geq 2$, such that $x_j < \min\{x_i, x_{i+r}\}$, for all $i < j < i + r$.

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Visibile pairs on the bar-diagram of a permutation

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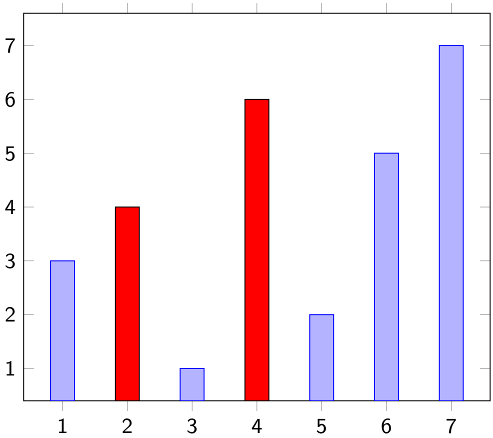

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For more details

T. Mansour and M. Shattuck, *Visibility in pattern-restricted permutations*, J. Differ. Equ. Appl. **26** (2020), pp 657-675

This paper contains a list of references concerning the cited applications.

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$S(\tau) = \bigcup_{n\geq 0} S_n(\tau)$ and $I(\tau) = \bigcup_{n\geq 0} I_n(\tau)$

The statistic "number of visible pairs"

$$\text{vis}(\sigma) = \text{number of visible pairs in } \sigma$$

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Our contribution (2021)

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Useful definitions

Definition of left-to-right maximum

Given a permutation

$$\pi = \pi_1\pi_2\dots\pi_n$$

we say that π_i is a left-to-right maximum of π if and only if

$$\pi_j < \pi_i \quad \forall j < i$$

We denote by $\text{ltrmax}(\pi)$ the number of left-to-right maxima of π.

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The notions of left-to-right minimum, right-to-left maximum, and right-to-left minimum can be defined analogously.

Useful results

Useful Lemma

Let $w = x_1 x_2 \dots x_n$ be a sequence of pairwise distinct positive integers such that (x_1, x_n) is a visible pair. Then:

$$vis(w) = n - 2$$

Useful Corollary

Let $\pi \in S_n$ be a permutation such that $\pi(n) = n$. Then:

$$vis(\pi) = n - ltrmax(\pi)$$

Involutions avoiding 231 and 312

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$d_{n,k}$= number of involutions in $I_n(231, 312)$ with k visible pairs

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$\sigma \in I_n(231, 312) \iff \sigma = \tau \; n \; n-1 \; \ldots \; h$ where $\tau \in I_{h-1}(231, 312)$

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Distribution of vis over $I_n(231, 312)$

- if $k > 0$, then $d_{n,k} = \binom{n-1}{k+1}$ (by Useful Corollary)
- $d_{n,0} = n$

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$\mathcal{M}_n$ = set of Motzkin paths of length n

$$\beta : I_n(4321) \to \mathcal{M}_n$$

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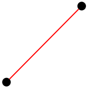

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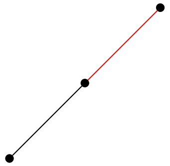

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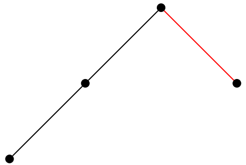

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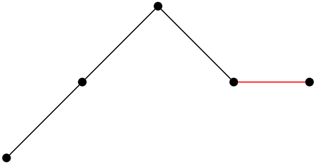

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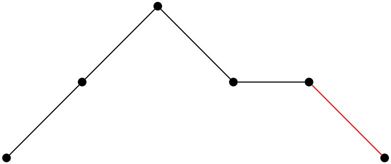

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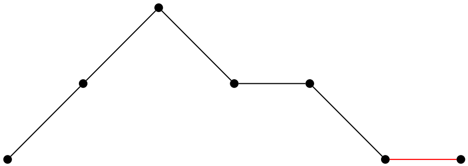

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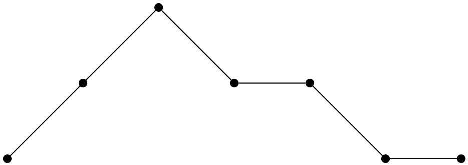

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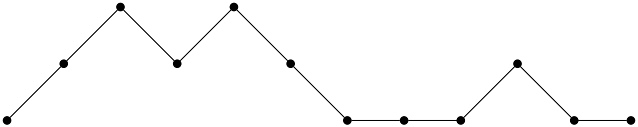
Dis_n = set of Motzkin paths of length n with no horizontal steps at positive height



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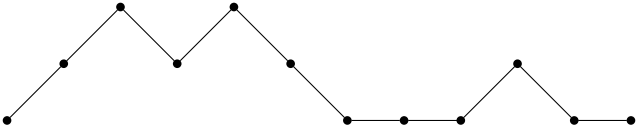

Image of $I_n(321)$ under β

$\beta(I_n(321)) = Dis_n$.

Involutions avoiding 321

We can deduce the value of $vis(\pi)$ from the associated Motzkin path:

Visible pairs read on the corresponding path

Let $\pi \in I_n(321)$. Denote by s the number of D steps of the Motzkin path $\beta(\pi)$. Then

$$vis(\pi) = \begin{cases} s & \text{if } \pi(n) = n \\ s-1 & \text{otherwise.} \end{cases}$$

Involutions avoiding 321

Distribution of vis over $I_n(321)$

Let

$$F(x,y) = \sum_{\pi \in I(321)} x^{|\pi|} y^{vis(\pi)}$$

Then

$$F(x,y) = \frac{1 + x^2(1-y)C(x^2y)}{1 - x^2yC(x^2y) - x}$$

where

$$C(t) = \frac{1-\sqrt{1-4t}}{2t}$$

is the generating function of the sequence of Catalan numbers.

Involutions avoiding 321

Sketch of the proof

- The statistics **vis** is equidistributed with the following statistic on Motzkin paths:

$$s(d) = \begin{cases} \text{number of } D \text{ steps of } d & \text{if } d \text{ ends by } H \\ \text{number of } D \text{ steps of } d \; -1 & \text{if } d \text{ ends by } D. \end{cases}$$

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- We can decompose $\beta(\pi) = aUbD$
- $s(\beta(\pi)) = \begin{cases} s(a) + s(UbD) & \text{if } a \text{ ends by } H \\ s(a) + 1 + s(UbD) & \text{if } a \text{ ends by } D. \end{cases}$

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- $M_i \to M_i - M_{i-1}$ up steps (with $M_0 = 0$)
- $w_i \to |w_i| + 1$ down steps

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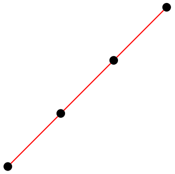

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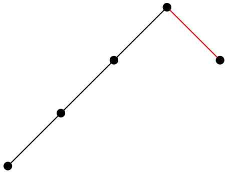

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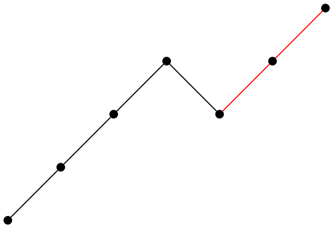

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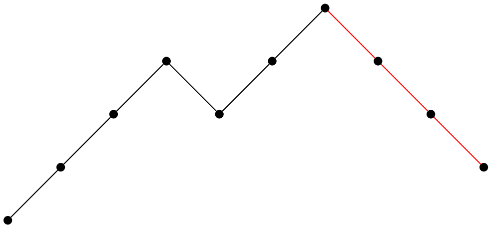

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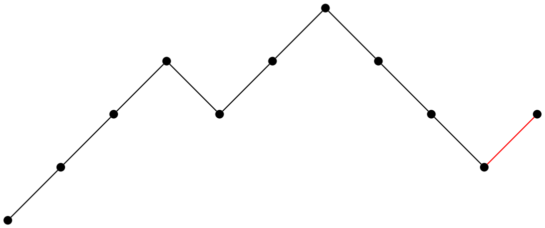

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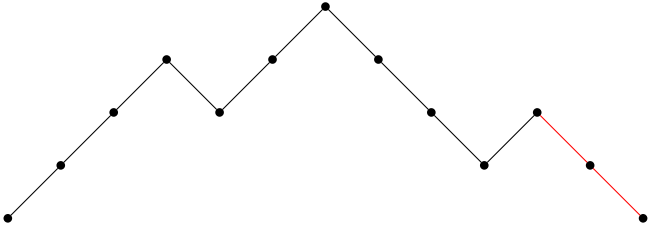

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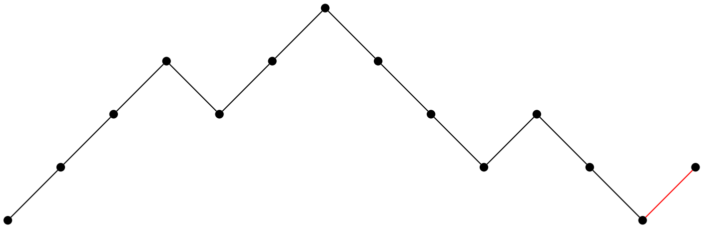

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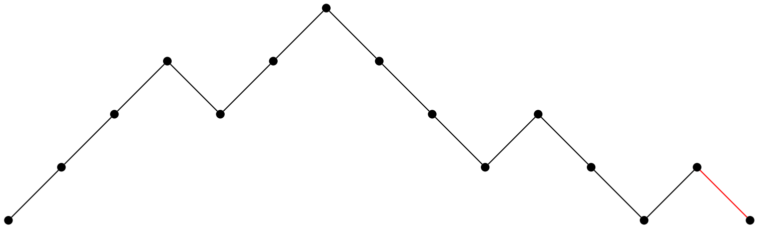

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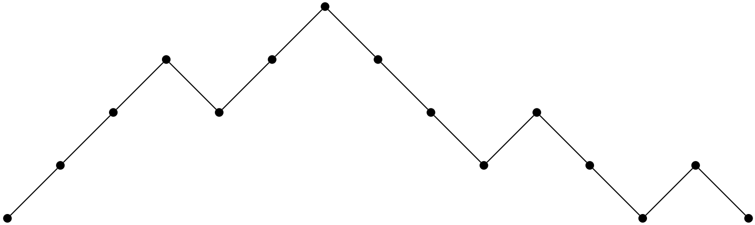

Involutions avoiding 123

Visible pairs read on the corresponding path

Let $\pi \in S_n(123)$ and let

$$\Psi(\pi) = U^{h_1} D^{k_1} \dots U^{h_s} D^{k_s},$$

Then

$$vis(\pi) = \sum_{i=1}^{s-1} (k_i - 1) + \max\{0, k_s - 2\}.$$

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Visible pairs read on the corresponding prefix

Let $\pi \in I_n(123)$ and let

$$\hat{\Psi}(\pi) = U^{h_1} D^{k_1} \dots U^{h_t} D^{k_t},$$

(k_t may be 0). Then:

$$vis(\pi) = \sum_{i=2}^{t} (h_i - 1) + \sum_{i=1}^{t-1} (k_i - 1) + \max\{h_1 - 2, 0\} + \max\{k_t - 1, 0\}.$$

Involutions avoiding 123

Distribution of vis over $I_n(123)$

Let

$$F(x,y) = \sum_{\pi \in I(123)} x^{|\pi|} y^{vis(\pi)}$$

Then

$$F(x,y) = \frac{H(x,y) + (x^2 - x^2y - 1)G(x,y)}{2(x^2y^3 + x^2y^2 - xy^2)},$$

where

$$G(x,y) = \sqrt{x^4y^4 - 2x^4y^2 + x^4 - 2x^2y^2 - 2x^2 + 1},$$

and

$$H(x,y) = (x^4+2x^3+2x^2)y^3-(x^4+2x^3+x^2+2x)y^2-(x^4-x^2)y+x^4-2x^2+1.$$

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- $\pi = v_1\, n\, v_2\, v_3\, h$, where:

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- $\pi = v_1\, n\, v_2\, v_3\, h$, where:
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Involutions avoiding 132

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  - let x be the leftmost element in v_2. Then, in the word $x\, v_3\, h$ the pair (x, h) is a visible pair and we have $\mathrm{vis}(x\, v_3\, h) = h - 1$ (by Useful Lemma)

Involutions avoiding 132

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- $w_i \to |w_i| + 1$ down steps

Involutions avoiding 132

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Hence $I_n(132)$ corresponds bijectively to P_n.

Involutions avoiding 132

ltrmax over $S_n(132)$

Let $\pi \in S_n(132)$ and set $t = \pi(1)$.

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Involutions avoiding 132

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- $\text{trmax}(\pi) = n + 1 - \pi(1) =$ length of the first ascending run of $\Phi(\pi)$

The number of Dyck paths of semilength n whose first ascending run has length s is

$$b_{n,s} = \frac{s}{n}\binom{2n-s-1}{n-s}$$

Involutions avoiding 132

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As before, we study the distribution of the **length of the first ascending run** over Dyck prefixes.

Involutions avoiding 132

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The number of Dyck prefixes of length n whose first ascending run has length s is

$$c_{n,s} = \sum_{i=1}^{\lceil \frac{s}{2} \rceil} (-1)^{i-1} \binom{n-2i}{\lfloor \frac{n-2i}{2} \rfloor} \binom{s-i}{i-1} \text{ if } s < n$$

and $c_{n,n} = 1$.

Involutions avoiding 132

rtlmax over $I_n(132)$

$$\pi = u_s M_s \dots u_2 M_2 u_1 M_1,$$

where the M_i's are the right-to-left maxima of π and $u_1, \dots, u_s$ are possibly empty words. We can show that:

Involutions avoiding 132

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- for every i, the portion of $\Phi(\pi)$ corresponding to $u_s M_s \dots u_i M_i$ ends on the x-axis;

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$f_{n,s}$ = number of Dyck path of length n with s returns:

Involutions avoiding 132

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$f_{n,s}$ = number of Dyck path of length n with s returns:

$$f_{2m,2t} = b_{m,t} \quad \text{and} \quad f_{2m+1,2t} = 0$$

$$f_{n,2t+1} = \sum_{i \geq t}^{\lfloor \frac{n-1}{2} \rfloor} b_{i,t} \binom{n-2i-1}{\lfloor \frac{n-2i-1}{2} \rfloor}.$$

Involutions avoiding 132

Distribution of vis over $I_n(132)$

The number of involutions in $I_n(132)$ with k visible pair is:

$$d_{n,k} = c_{n-1,n-1-k} + f_{n-2,n-2-k} + \sum_{j=2}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{j-1} b_{j-1,i} f_{n-2j,n-2-i-k}$$

Involutions avoiding 213

Definition of co-visible pair

Let $w = x_1 x_2 \ldots x_n$ be a word whose letters are pairwise distinct integers. A co-visible pair in w is a pair (x_i, x_{i+r}), $r \geq 2$, such that $x_j > \max\{x_i, x_{i+r}\}$, for all $i < j < i + r$.

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For example, the co-visible pairs of the word

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are $(2, 4)$,$(3, 2)$, and $(1, 2)$

Involutions avoiding 213

Definition of rc operator

The **reverse-complement** (or rc) of a permutation $\pi \in S_n$ is the permutation π^{rc} defined by

$$\pi_i^{rc} = n + 1 - \pi_{n+1-i}$$

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The statistic number of visibile pair over $I_n(213)$ is equidistributed with the statistic number of co-visibile pair over $I_n(132)$ (via reverse-complement, which also maps involutions to involutions)

Involutions avoiding 213

Useful Lemma (2)

Let $w = x_1x_2 \ldots x_n$ be a sequence of pairwise distinct positive integers such that (x_1, x_n) is a co-visible pair. Then the number of co-visible pairs in w is $|w| - 2$

Involutions avoiding 213

Useful Lemma (2)

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Every involution $\pi \in I_n(132)$ corresponds to the symmetric Dyck path

$$d = \Phi(\pi) = U^{j_1} D^{l_1} U^{j_2} D^{l_2} \dots U^{j_s} D^{l_s}.$$

Then, by Useful Lemma (2)

$$covis(\pi) = l_1 - 1 + l_2 - 1 + \dots + l_{s-1} - 1$$

Involutions avoiding 213

Every $\sigma \in I_n(213)$ corresponds to the Dyck prefix

$$p = U^{h_1} D^{k_1} U^{h_2} D^{k_2} \dots U^{h_t} D^{k_t}$$

consisting of the first n steps in $\Phi(\sigma^{rc})$. Then:

$$vis(\sigma) = \sum_{j=2}^{t} (h_j - 1) + \sum_{i=1}^{t-1} (k_i - 1) + \max(0, k_t - 1)$$

Involutions avoiding 213

Distribution of vis over $I_n(213)$

Let

$$Q(x,y) = \sum_{\pi \in I(213)} x^{|\pi|} y^{vis(\pi)} = \sum_{\pi \in I(132)} x^{|\pi|} y^{covis(\pi)}.$$

Then

$$Q(x,y) = \frac{1 - x + x^2 - x^2 y + (x^2 y - xy)R}{(1-x)(1-x-xy)}$$

where

$$R = 1 + \frac{2x^2}{1 - 2x^2 y - x^2 + x^2 y^2 + \sqrt{(x^2 + x^2 y^2 - 1)^2 - 4x^4 y^2}}.$$

Thank you

Our paper

M. Barnabei, F. Bonetti, N. Castronuovo, and M.Silimbani, *Visibility on restricted involutions*, J. Differ. Equ. Appl. **27**(4) (2021), pp. 481-496

Thank you for your attention!

For more questions

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