On automorphisms of a Fano plane 2-analog design

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A 2-analog of a Fano plane is a collection of 3-dimensional blocks from $\mathbb{F}_{2^7}$ such that any 2-dimensional subspace of $\mathbb{F}_{2^7}$ is contained in one block from a collection of blocks

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in general $|E_{2^k}[E_{2^n}]| = \begin{bmatrix} n \\ k \end{bmatrix}_2$, where $\begin{bmatrix} n \\ k \end{bmatrix}_2$ is a gaussian 2-coefficient.

if $T \in E_{2^t}[E_{2^7}]$, then $|E_{2^k}[T]^{-1}| = |E_{2^{k-t}}[E_{2^7}/T]| = |E_{2^{k-t}}[E_{2^{7-t}}]| = \begin{bmatrix} 7-t \\ k-t \end{bmatrix}_2$.

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The estimated time for involution is 8×10^{12} CPU-years

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Using the formula of inclusion and exclusion we get $127 = X_1 - X_2 + X_3 - X_4 + \cdots$, where $X_j = \sum_{P \in \binom{[127]}{j}} |\bigcap_{s \in P} (A^*)^{\alpha^s}|$, $j \geq 1$, where $\binom{[127]}{j}$ is a collection of j-element subsets of $[127] = \{1, 2, \ldots, 127\}$. Thus, we get $127 = 7 \cdot 127 - X_2 + X_3 - X_4 + \cdots$.

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Let us assume the opposite. Let $\langle\alpha\rangle \hookrightarrow \mathcal{H}$, where α is of order 31. Let $\mathcal{H}_g = \{H \in \mathcal{H} \mid g \in H\}$. One can see that $|\mathcal{H}_g| = 21$. Also, from $|\{\mathcal{H}_g\}_{g\neq 1}| = 127$, we get $|Fix(\alpha, \{\mathcal{H}_g\}_{g\neq 1})| \equiv 3 \ (mod\ 31)$.

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Lemma: Let $\langle \alpha \rangle \hookrightarrow \mathcal{H}$ be of order 7, then $|Fix(\alpha)| \in \{1, 15\}$. Furthermore, $|Fix(\alpha, E_{2^3}[E_{2^7}])| \equiv 2(mod\, 7)$ and $|Fix(\alpha, \mathcal{H})| \equiv 3(mod\, 7)$.

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Lemma: Let $\langle\alpha\rangle \hookrightarrow \mathcal{H}$ be of order 7, then $|Fix(\alpha)| \in \{1, 15\}$. Furthermore, $|Fix(\alpha, E_{2^3}[E_{2^7}])| \equiv 2 (mod\, 7)$ and $|Fix(\alpha, \mathcal{H})| \equiv 3 (mod\, 7)$.

Lemma: Let $\langle\alpha\rangle \hookrightarrow \mathcal{H}$ be of order 7 and let $|Fix(\alpha)| = 1$, where $Fix(\alpha) = \{c\}$. Then, there are $\widetilde{X}_i$, $\widetilde{Y}_i \in E_2[E_{2^7}/\langle c\rangle]$ and $\widetilde{\alpha} \in Aut(E_{2^7}/\langle c\rangle)$, given by a rule $(g\langle c\rangle)^{\widetilde{\alpha}} = g^{\alpha}\langle c\rangle$, such that all $\widetilde{X}_i\widetilde{Y}_i \cong E_{2^2}$ are mutually disjoint. Furthermore, the following holds: $\sum_{i=1}^{3} (\widetilde{X}_i\widetilde{Y}_i)^{\langle\widetilde{\alpha}\rangle} = E_{2^7}/\langle c\rangle + 20\langle c\rangle$ and $|Fix(\widetilde{\alpha})| = 0$ and $\widetilde{\alpha}$ is of order 7.

Automorphism of order 7

Difficult case, since order of Singer automorphism of E_8 is SEVEN

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Lemma: Let $\langle\alpha\rangle \hookrightarrow \mathcal{H}$ be of order 7. Then, $|Fix(\alpha)| = 1$.

Theorem: Let $\alpha \in Aut(E_{2^6})$ be of order 7 and $Fix(\alpha) = \phi$. If $\langle g^{\langle\alpha\rangle}\rangle < E_{2^6}$, then, $\langle g^{\langle\alpha\rangle}\rangle \cong E_{2^3}$. Furthermore, $Fix(\alpha, E_{2^3}[E_{2^6}]) = \{A, B\}$ and $E_{2^6} = A \times B$.

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Theorem: Let $\alpha \in Aut(E_{2^6})$ be of order 7 and $Fix(\alpha) = \phi$. If $\langle g^{\langle \alpha \rangle} \rangle < E_{2^6}$, then, $\langle g^{\langle \alpha \rangle} \rangle \cong E_{2^3}$. Furthermore, $Fix(\alpha, E_{2^3}[E_{2^6}]) = \{A, B\}$ and $E_{2^6} = A \times B$.

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Lemma: Let $\langle \alpha \rangle \hookrightarrow \mathcal{H}$, where α is of order 5. Then $|Fix(\alpha)| = 7$.

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Sketch:

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Sketch:

Let's assume the opposite. Then, $|Fix(\alpha)| = 7$, $Fix(\alpha, E_{2^3}[E_{2^7}]) = Fix(\alpha, \mathcal{H}) = H_0$.

Also, $Fix(\alpha) = H_0^*$. Let $c \in H_0^*$. For $\mathcal{H}_c = \sum_{c \in H \in \mathcal{H}} H$, the following holds: $\mathcal{H}_c^\alpha = \mathcal{H}_{c^\alpha} = \mathcal{H}_c$.

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Then, there are orbit representatives $H_i \in \mathcal{H}_c$ such that $\mathcal{H}_c = \sum_{i=1}^{4} H_i^{\langle \alpha \rangle} + H_0 = E_{2^7} + 20\langle c \rangle$.

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We can expand the natural epimorphism $E_{2^7} \to E_{2^7}/H_0$ to a group ring by $\varphi : \mathbb{Z}[E_{2^7}] \to \mathbb{Z}[E_{2^7}/H_0]$, where $\varphi(H_i^{\alpha^j}) = 2H_i^{\alpha^j}/H_0,\ i \in [4],\ j \in [5]$.

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Automorphism of order 3

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Lemma: Let $\langle\alpha\rangle \hookrightarrow \mathcal{H}$, where α is of order 3, then $|Fix(\alpha)| \in \{1, 7, 31\}$.

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$E_{2^2}[E_{2^6}],\ j \in [7-m],\ \beta \in Aut(E_{2^6})$ of order 3, such that

$$\sum_{i=1}^{3m} A_i + \sum_{j=1}^{7-m} B_j^{\langle \alpha \rangle} = E_{2^6} + 20$$

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Automorphism of order 4

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Automorphism of order 4

Lemma: Let $\beta \in Aut(E_{2^n})$ be of order 2. Let $F = 1 + Fix(\beta)$. Then $|F| \geq 2^{n/2}$.

where $A_i^\beta = A_i$, $B_j \cap B_j^\beta = 1$, and $A_i \cong B_j \cong E_{2^2}$.

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Lemma: Let $\alpha \in Aut(\mathcal{H})$ is of order 4. Then there are 28 α-orbits on E_{2^7} of a size 4. Furthermore, $Fix(\alpha^2) = Fix(\alpha) + \sum_{i=1}^{a_2} x_i^{\langle\alpha\rangle}$, where a_2 is the number of α-orbits on E_{2^7} of a size 2.

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Thank You!