Combinatorial settlement model - a variant of Flory model

4th Croatian Combinatorial Days

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Combinatorial settlement planning

The structure of the talk

1. Problem description

2. Riviera Model

3. Other interesting questions and further research

Problem setting

- **A tract of land** is divided into $m \times n$ unit squares (**lots**).
- Each lot can be either **occupied** (by a house) or left vacant.
- The arrangement of houses on such a tract of land is called a **configuration**.

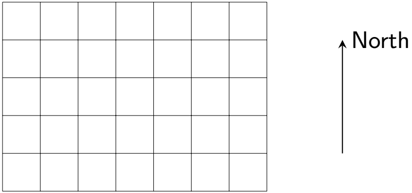


Figure: Tract of land with dimensions 5×7.

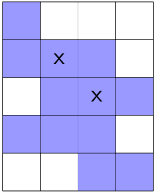
Configuration is encoded as an $m \times n$ matrix C, where $c_{i,j} = 1$ iff the lot (i, j) is occupied.

Problem setting

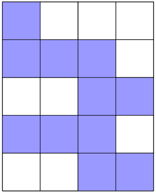
- An occupied lot on the position (i, j) is **completely shaded** if it borders with three other occupied lots to its east, south and west.
- The lots on the eastern, southern and western border are never completely shaded.

Configurations of interest

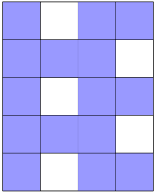
- A configuration C is **permissible** if no occupied lot is completely shaded.
- A permissible configuration is **maximal** if no additional lots can be made occupied without making it impermissible.



(a) Impermissible

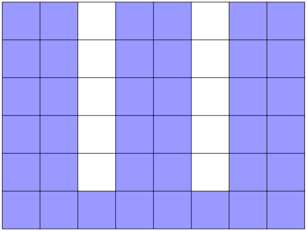


(b) Permissible

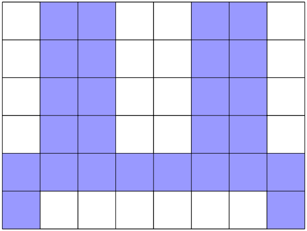


(c) Maximal

What can be the occupancy of maximal configurations?



(a) Example of an efficient configuration



(b) Example of an inefficient configuration

Two natural problems

- (a) To maximise the profit of an investor (maximise the occupancy) while assuring the permissibility. **Efficient maximal configurations**
- (b) To maximise the quality of living (minimise the occupancy) while assuring the maximality. **Inefficient maximal configurations**

Peculiar patterns of efficient configurations

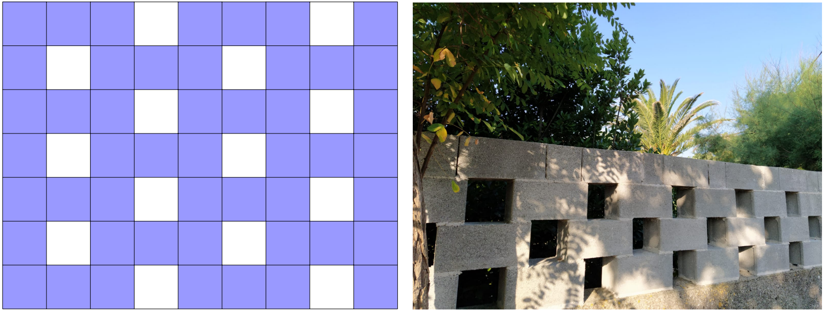


Figure: "Brick pattern" formed on one of efficient maximal configurations

Known results

In the paper M. Puljiz, S. Šebek i J. Ž, *Combinatorial settlement planning*, accepted in Contributions to Discrete Mathematics (2021), `https://arxiv.org/abs/2107.07555`, the authors have proven:

Proposition

Inefficient maximal configurations on $m \times n$ grid have the following occupancy:

$$I_{m,n} = \begin{cases} \frac{mn}{2} + 2, & \text{if } n \equiv 0 \pmod 4, \\ \frac{m(n+2)}{2}, & \text{if } n \equiv 2 \pmod 4, \\ \frac{m(n+1)}{2} + 1, & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Proposition

The occupancy of efficient maximal configurations on $m \times n$ grid, $m, n \geqslant 2$, satisfies:

$$E_{m,n} \leqslant \begin{cases} mn - \left\lfloor \frac{n}{4} \right\rfloor \cdot (m-1), & \text{if } n \not\equiv 3 \pmod 4, \\ mn - \left\lfloor \frac{n}{4} \right\rfloor \cdot (m-1) - \left\lfloor \frac{m}{2} \right\rfloor, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

But what about randomness?

Question

If the lots are occupied at random, what is the probability of obtaining a particular occupancy?

Caution

The answer depends on the way we model the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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Uniform choosing of a maximal configuration

$$\mathbb{P}(C) = \frac{1}{|\mathcal{M}_{m,n}|}, \qquad \mathbb{P}(|C| = k) = \frac{\#\{C : |C| = k\}}{|\mathcal{M}_{m,n}|}$$

Sequential building

$P_n =$ the set of n-permutations; $G : P_{mn} \to \mathcal{M}_{m,n}$ - sequential building

$$\mathbb{P}(\mathcal{K}) = \frac{\#\{\sigma \in P_{mn}, G(\sigma) = \mathcal{K}\}}{(mn)!}$$

$$\mathbb{P}(|\mathcal{K}| = k) = \frac{\#\{\sigma \in P_{mn} : |G(\sigma)| = k\}}{(mn)!}$$

Uniform $\neq$ Sequential

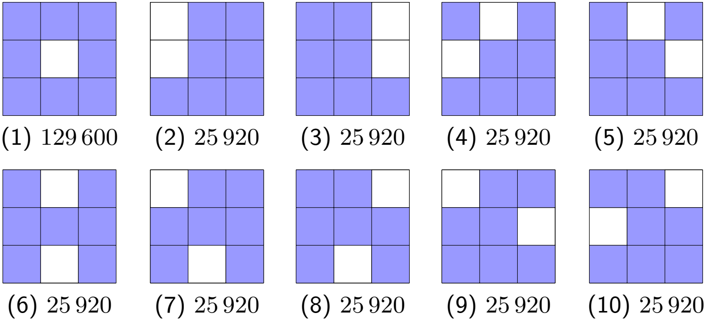


Figure: All the elements of $\mathcal{M}_{3,3}$ together with the number of permutations in P_9 that are mapped to each.

$$X^u_{3,3} \sim \begin{pmatrix} 7 & 8 \\ \frac{9}{10} & \frac{1}{10} \end{pmatrix}, \qquad X^s_{3,3} \sim \begin{pmatrix} 7 & 8 \\ \frac{9}{14} & \frac{5}{14} \end{pmatrix}.$$

Counting the maximal configurations?

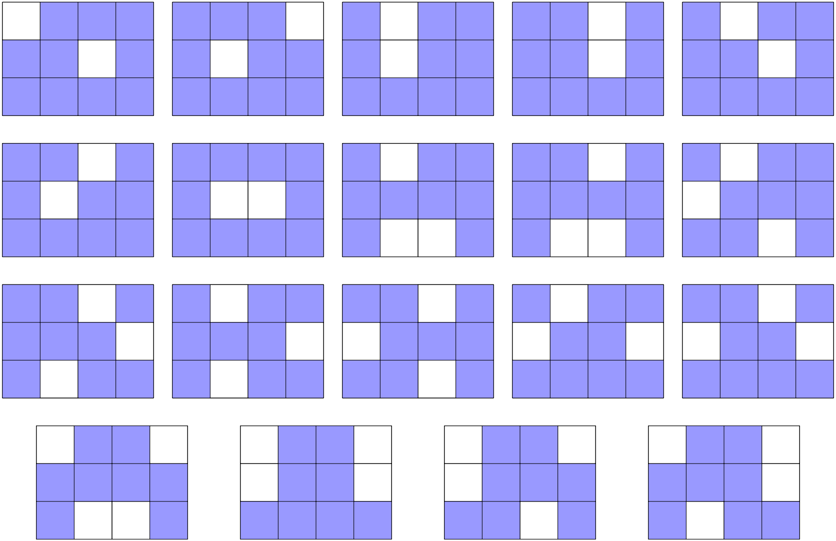


Figure: Maximal configurations on 3×4 grid.

Simplified model — Riviera model

Problem

How many maximal configurations exist on a tract of land with dimensions $m \times n$*?*

Simplified model — Riviera model

In the continuation we consider the case $m = 2$ where the lower row is completely occupied and focus only on the top row. Thus, we are considering 1-D configurations where "the sun comes only from east **or** west".

Remark

If we would insist that each occupied lot obtains light from the east **and** the west, the model would be equivalent to the Flory model.

Taken from Lucas Gerin. *The Page-Rényi parking process.* EJC, 2015, 22 (4)

Flory model and transfer matrix method

Example of a maximal configuration: OHOOHOHOH

$$\begin{tikzcd}
\mathrm{OO} \arrow[r, bend left] & \mathrm{OH} \arrow[r, bend left] & \mathrm{HO} \arrow[l, bend left] \arrow[ll, bend left]
\end{tikzcd}$$

The associated adjacency matrix:

$$\begin{array}{c c}
 & \begin{array}{ccc} \mathrm{OO} & \mathrm{OH} & \mathrm{HO} \end{array} \\
\begin{array}{c} \mathrm{OO} \\ \mathrm{OH} \\ \mathrm{HO} \end{array} & \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]
\end{array}$$

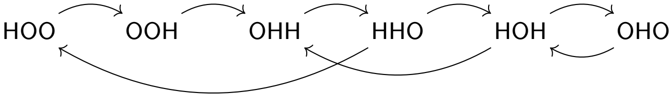
Maximal configurations are related to walks on the directed graph.

$$\text{Number of maximal configurations of length } n \ \approx \lambda^n$$

where $\lambda = 1.32471796$ (Plastic number) is Perron-Frobenius eigenvalue of the adjacency matrix.

Transfer matrix method applied to the Riviera model

Maximal configurations on a 1-D tract of land of length n where the light must obtained from east or west.
Example of a maximal configuration: HHOOHHOHOHHOH



The associated adjacency matrix:

$$
\begin{array}{c c}
 & \begin{array}{cccccc} \text{HOO} & \text{OOH} & \text{OHH} & \text{HHO} & \text{HOH} & \text{OHO} \end{array} \\
\begin{array}{c} \text{HOO} \\ \text{OOH} \\ \text{OHH} \\ \text{HHO} \\ \text{HOH} \\ \text{OHO} \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

Perron-Frobenius eigenvalue $\lambda = 1.40126837$.

Generating function for the Riviera model

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad a = b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

maximal configurations of length $n = a^T \cdot A^{n-3} \cdot b, \quad n \geqslant 3$

$$f(y) = 1 + y + y^2 + \sum_{n=3}^{\infty} a^T \cdot A^{n-3} \cdot b \cdot y^n$$

Surprise

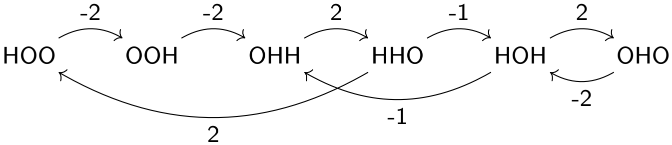
The function $-\dfrac{y^5 - y^3 - y - 1}{y^6 - y^4 - y^3 - y^2 + 1}$ is the generating function of the sequence counting the maximal configurations of length n.
That is precisely the sequence `https://oeis.org/A080013` with an offset of 4.

A080013 – 1, 1, 3, 3, 4, 6, 9, 12, 16, 24, 33, 46, 64, . . .

Theorem

The number of maximal configurations of length n *in the Riviera model is equal to the number of permutations* π *of length* $n+4$ *which satisfy the constraint*

$$\pi(i) - i \in \{-2, -1, 2\}.$$



A080013 – 1, 1, 3, 3, 4, 6, 9, 12, 16, 24, 33, 46, 64, ...

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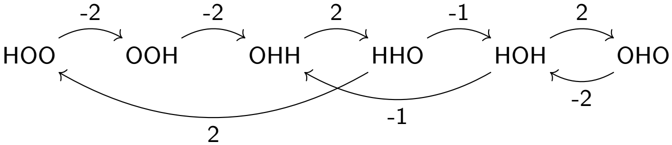


HOHHO $\sim$ OHHOHOHHOOHH

$$\text{OHH} \xrightarrow{2} \text{HHO} \xrightarrow{-1} \text{HOH} \xrightarrow{2} \text{OHO} \xrightarrow{-2} \text{HOH} \xrightarrow{-1} \text{OHH} \xrightarrow{2} \text{HHO} \xrightarrow{2} \text{HOO} \xrightarrow{-2} \text{OOH} \xrightarrow{-2} \text{OHH}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 5 & 2 & 4 & 8 & 9 & 6 & 7 \end{pmatrix}$$

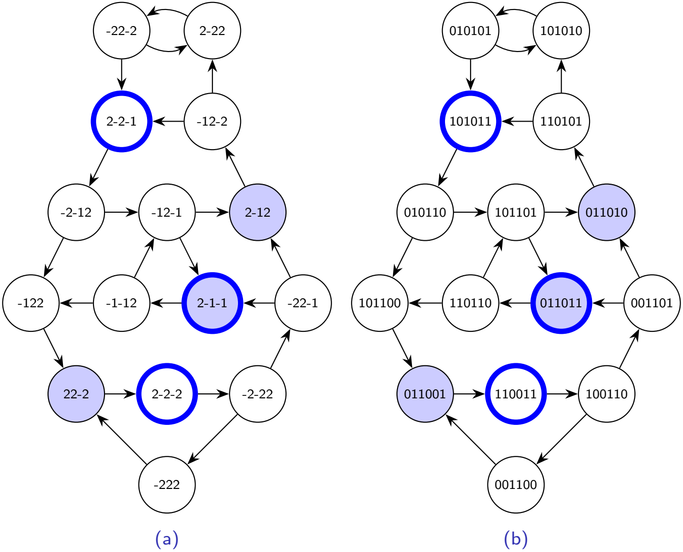


Figure: The above graphs encode strongly restricted permutations and maximal configurations of Riviera model. The starting nodes are shaded and thicker outlines indicate the ending nodes.

Complexity

For $\rho \in [0,1]$ (building density) denote with $J_n(\rho)$ the number of maximal configurations of length n with exactly $\lfloor \rho n \rfloor$ occupied lots.
The function

$$f(\rho) = \lim_{n\to\infty} \frac{\ln J_n(\rho)}{n}$$

is called **the complexity** of the sequence $J_n(\rho)$.

One has:

$$J_n(\rho) \sim e^{nf(\rho)}.$$

Problem

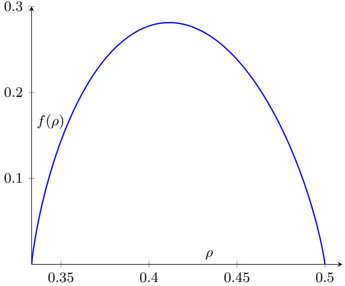
To determine the complexity $f(\rho)$ of Riviera model.

What is known so far:

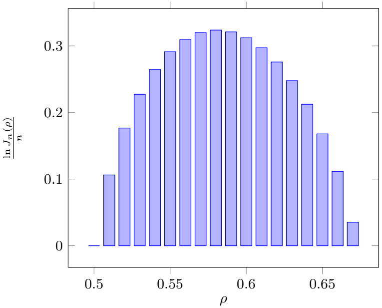
- $\max_\rho f(\rho) = \ln \lambda = \ln 1.401268 = 0.337377.$
- $\operatorname{supp} f = \left[\frac{1}{2}, \frac{2}{3}\right].$

Complexity

The complexity function of Flory model:
$f(\rho) = \rho \ln \rho - (1 - 2\rho) \ln(1 - 2\rho) - (3\rho - 1) \ln(3\rho - 1)$



(a) Complexity of Flory model, $f(\rho)$



(b) $\frac{\ln J_n(\rho)}{n}$, $n = 100$. (Riviera)

Jamming limit

Probability model (Random sequential adsorption)

- Sequentially choose a lot, in a random order.
- If the lot permits occupation, then occupy it.
- If it does not, then it remains unoccupied.
- The process stops when we arrive at maximal configuration.

$X^s_{m,n}-$ the occupancy of a maximal configuration obtained from the above procedure.

Problem

To determine

$$\theta := \lim_{m,n\to\infty} \frac{X^s_{m,n}}{mn}$$

in some sense (in expectation, in distribution, probability).

The above limit is called **jamming limit (saturation coverage)**.

Simulations: $\theta \approx 0.636$

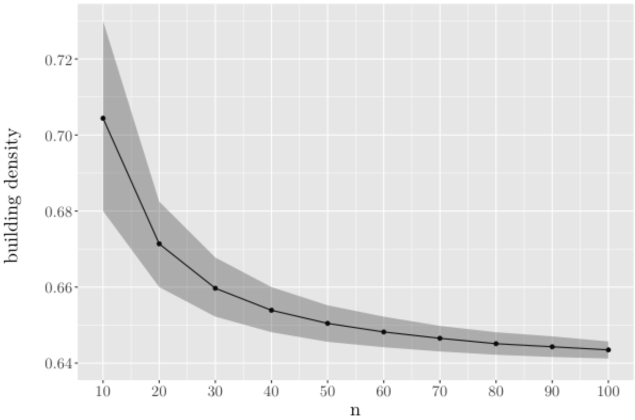


Figure: Behavior of $\mathbb{E}\left[X^s_{m,n}/(mn)\right]$ when we let both of the parameters, m and n, to infinity at the same rate.

- The jamming limit of Riviera model: 0.60039.
- The jamming limit of Flory model is $\frac{1-e^{-2}}{2} \approx 0.432332$.

Probability distribution of the occupancy?

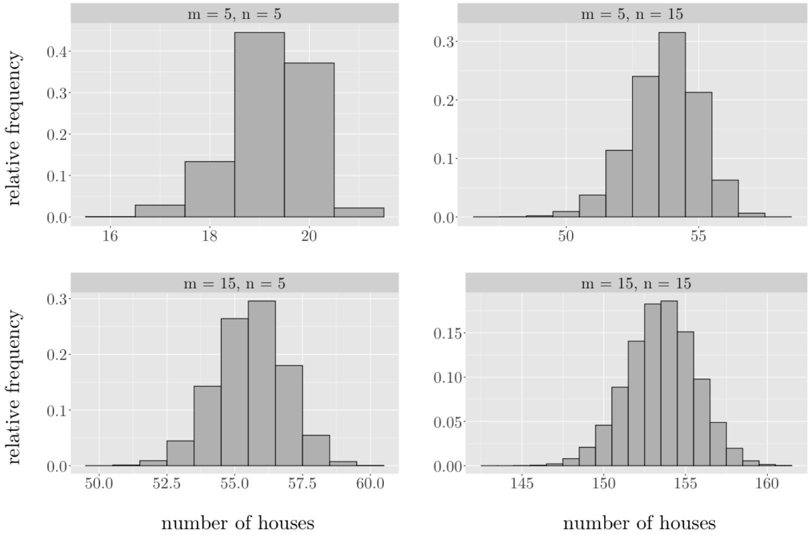


Figure: Histograms that approximate the distribution of the random variable $X^s_{m,n}$, i.e. the occupancy in a maximal configuration, in the case of sequential building.

More problems:

- Do all that, but for multi-storey models...

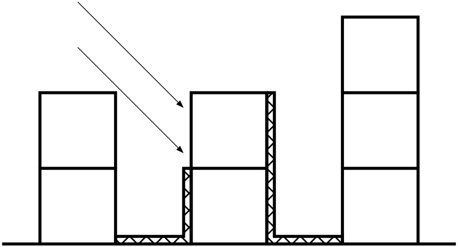


Figure: Both stories of the middle house are blocked from the sunlight from the east, while only the first story is blocked from the west.

The end

Thank you for your attention!