Weighted Lattice Paths Enumeration by Gaussian Polynomials

Ivica Martinjak

Faculty of Science, University of Zagreb
Bijenička cesta 32, HR-10000 Zagreb, Croatia

Ivana Zubac

Faculty of Mechanical Engineering and Computing, University of Mostar
Matice hrvatske bb, BA-88000 Mostar, Bosnia and Herzegovina

Abstract

The Gaussian polynomial in variable q is defined as the q-analog of the binomial coefficient. In addition to remarkable implications of these polynomials to abstract algebra, matrix theory and quantum computing, there is also a combinatorial interpretation through weighted lattice paths. This interpretation is equivalent to weighted board tilings, which can be used to establish Gaussian polynomial identities. In particular, we prove duals of such identities and evaluate related sums.

Keywords: Gaussian binomial coefficient, q-binomial coefficient, Gaussian polynomial identities, q-analog, lattice paths, domino tiling, double counting

MSC: 05A30, 11B65

DOI: https://doi.org/10.5592/CO/CCD.2016.04

1.1 Introduction

We let n and k denote non-negative integers. The *Gaussian polynomial* or *q-binomial coefficient* is defined to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

Thus, we replace each factor m in the formula for binomial coefficient, by $q^m - 1$ to get Gaussian coefficient. It is easily seen that Gaussian binomial coefficient reduces to binomial coefficient when $q = 1$. More precisely, by the L'Hôpital's Rule, we have

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

More on this subject one can find in the book by P. Cameron [3]. Recall a classical implication of Gaussian polynomial. The number of elements in a finite field is a prime power and up to isomorphism there is a unique field of any prime power order (E. Galois). We let GF(q) denote the field with q elements. Then the number of k-dimensional subspaces of an n-dimensional vector space over GF(q) is $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$. A proof of this one can find in a nice overview by H. Cohn [4]. It is also worth mentioning that Gaussian polynomial are related to *quantum calculus* and more on this one can find in a book by V. Kac and P. Cheung [7].

Having the q-analog of the factorial defined as

$$[n]_q! = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1})$$

we also have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Recently, V. Guo and D. Yang [5] found Gaussian polynomial identities

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+1 \\ n-2k \end{bmatrix}_q q^{\binom{n-2k}{2}} = \begin{bmatrix} m+n \\ n \end{bmatrix}_q,$$

$$\sum_{k=0}^{\lfloor n/4 \rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^4} \begin{bmatrix} m+1 \\ n-4k \end{bmatrix}_q q^{\binom{n-4k}{2}} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+n-2k \\ n-2k \end{bmatrix}_q$$

which are q-analogues of binomial coefficient identities of Y. Sun,

$$\begin{aligned}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{m+k}{k}\binom{m+1}{n-2k} &= \binom{m+n}{n},\\ \sum_{k=0}^{\lfloor n/4\rfloor}\binom{m+k}{k}\binom{m+1}{n-4k} &= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{m+k}{k}\binom{m+n-2k}{m},\end{aligned}$$

respectively. (For an overview of binomial coefficients we refer the reader to classical reference by R. Graham, D. Knuth and O. Patashnik [6].) Being motivated by these results, in this work we aim at finding further Gaussian polynomial identities.

1.2 q-binomial coefficients and weighted lattice paths

As the basic recurrences for Gaussian polynomials we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{1}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{2}$$

where $0 < k < n$ and

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1.$$

This follows immediately from the definition. To prove (1) we have

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q &= \left(\frac{q^n-1}{q^k-1}-1\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q\\ &= q^k\left(\frac{q^{n-k}-1}{q^k-1}\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q\\ &= q^k\begin{bmatrix} n \\ k-1 \end{bmatrix}_q.\end{aligned}$$

In the same fashion one can prove the property of *symmetry* for Gaussian polynomials,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \tag{3}$$

as well as an analogy of the property of *absorption* for binomial coefficients,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{1-q^n}{1-q^{n-k}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q . \tag{4}$$

A well known combinatorial interpretation of Gaussian polynomial $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is that it enumerates the number of *integer partitions* that fit into the *square lattice* region of size $k \times (n-k)$ (see the book [2] by J. Azose, A. Benjamin and K. Garrett). More precisely, the Gaussian polynomial can be defined as the *generating function* for this type of partitions. For example, consider the polynomial

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = q^4 + q^3 + 2q^2 + q + 1.$$

This means that there are six distinct partitions whose *Young diagrams* fit into a 2×2 lattice region and that these partitions are

$$(2, 2), (2, 1), (2), (1, 1), (1), (0)$$

(there are 2 partitions of the number 2 while the other partitions are of numbers 4, 3, and 1. Now the symmetry property (3) follows immediately by conjugation of the Young diagrams. Similar bijective proofs can also be done for recurrences (1) and (2) (see the book [1] by G. Andrews and K. Ericcson).

Obviously, instead by partitions, one can interpret Gaussian polynomial by lattice paths from $(0, 0)$ to $(k, n-k)$, where an increment by 1 right has the weight of 1 while an increment by 1 up has the weight of q^s where s is the number of previous horizontal increments. Furthermore, we establish a bijection between these paths and a *board tilings* by squares and dominoes as follows:

i) we code an increment by 1 right in a path by a domino, and

ii) we code an increment by 1 up by a square.

We illustrate this 1 to 1 correspondence in Figure 1. The reasoning above proves the next lemma.

Lemma 1 *The Gaussian polynomial* $\left[{n \atop k} \right]_q$ *represents the number of* $(n+k)$*-board tilings by* $n-k$ *squares and* k *dominoes where the weight of every domino is equal to 1 while the weight of a square is equal to* q^s*, with* s *being the number of preceding dominoes.*

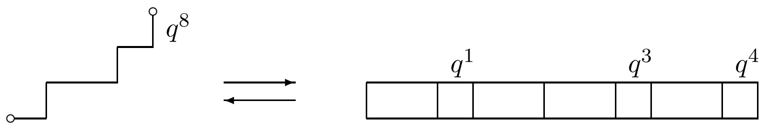

Figure 1: There is a 1 to 1 correspondence between weighted lattice paths with k horizontal increments and weighted board tilings with k dominoes.

Note that the parameter n in Lemma 1 represents the number of parts in a tiling (the number of increments of length 1 in a lattice path). Figure 2 shows 6-board tilings that correspond to partitions in the example above. According to Lemma 1, these tilings are represented by the Gaussian polynomial $\left[{4 \atop 2} \right]_q$. In what follows we use Lemma 1 to prove some Gaussian polynomial identities.

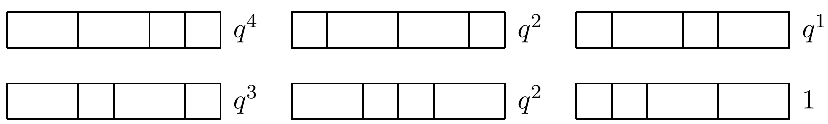

Figure 2: All six tilings of a board of length 6 having two dominoes, which are enumerated by the polynomial $q^4 + q^3 + 2q^2 + q + 1$.

1.3 Two pairs of polynomial identities

We shall now present obtained identities for the Gaussian polynomials, in the following statements. To get these, we enumerate elements of the set of all n-board weighted tilings in respect to certain criterions.

Theorem 1 *For natural numbers* n *and* k*, where* $n \geq k$*, we have*

$$\sum_{i=0}^{k} q^i \begin{bmatrix} n-k-1+i \\ i \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Proof. We consider weighted $(n+k)$-board tilings with k dominoes and $n-k$ squares, in respect to the last square in a tiling. The set S of all such tilings we separate into k disjoint subsets $S_1, S_2, \ldots, S_k$ such that in set S_1 there are tilings having the last square on the position $n+k-1$. In S_2 there are tilings having the last square on position $n+k-3$ (meaning that there are $k-1$ dominoes left from the last square and a sole domino is placed right from the last square). Furthermore, in S_3 the last square is on position $n+k-5$, etc.

Now, the tilings in S_1 are represented by

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^k.$$

Namely, by Lemma 1, tilings of the length $n+k-1$ are represented by $\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ and the weight of the last square is q^k (since there are k dominoes placed on the left of the last square). Furthermore, by an analogue argument, tilings in S_2 are represented by

$$\begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q q^{k-1},$$

etc. Finally, tilings in the set S_k are represented by

$$\begin{bmatrix} n-k-1 \\ 0 \end{bmatrix}_q,$$

since there are no dominoes on the left of the last square. Having in mind that the sets S_i are disjoint, the sum

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q q^{k-1} + \cdots + \begin{bmatrix} n-k-1 \\ 0 \end{bmatrix}_q$$

represents the set S and this fact completes the proof.

■

For the purpose of proving the following theorem, again we consider weighted $(n+k)$-board tilings with k dominoes and $n-k$ squares, but now with respect to the last domino in a tiling.

Theorem 2 *For natural numbers n and k, where $n \geq k$, we have*

$$\sum_{i=0}^{n-k} q^{k(n-k-i)} \begin{bmatrix} k-1+i \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Proof. The set S of weighted tilings of length $n+k$ we decompose into partition of $n-k+1$ disjoint subsets $S_0, S_1, S_2, \ldots, S_{n-k}$, such that in the subset S_i there are exactly $n-k-i$ squares on the right hand side of the last domino in a tiling.

The set S_0 is represented by

$$\begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q q^{k(n-k)}.$$

This holds true since on the left hand side from the last domino in S_0 there are $k-1$ dominoes and from the fact that there are $n-k$ squares on the right hand side of the last domino - in every of these tilings. Tilings in the set S_1 have $n-r-1$ dominoes on the right hand side of the last domino which means that they are enumerated by

$$\begin{bmatrix} k \\ k-1 \end{bmatrix}_q q^{k(n-k-1)},$$

etc. Finally, tilings in the set S_{n-k} are represented by the polynomial

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Thus, the sum of these terms,

$$\begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q q^{k(n-k)} + \begin{bmatrix} k \\ k-1 \end{bmatrix}_q q^{k(n-k-1)} + \cdots + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

is equal to the polynomial $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$. ■

Figure 3 shows tilings of length 7 with two dominoes and three squares separated into subsets S_0, S_1, S_2, S_3, with respect to the last domino. One can easily establish that $|S_0| = 1, |S_1| = 2, |S_2| = 3, |S_3| = 4$ and that polynomial $\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]_q$ enumerates these tilings.

In order to prove Theorem 3 we consider a class of tilings with an odd number of dominoes. More precisely, we take into account weighted $(n+2r+1)$-board tilings with $2r+1$ dominoes. The set S of all such tilings we separate into $n-2r$ subsets $S_1, S_2, \ldots, S_{n-2r}$, by means of the median domino. Tilings in set S_i have the median domino covering cells $2r+i$ and $2r+i+1$. Thus, in S_1 there are tilings having the median domino covering cells $2r+1$ and $2r+2$. This means that on the left hand side of the median domino there are r dominoes covering cells 1 through $2r$. Now we separate a tiling into parts t_1 - from cells 1 through $2r$ and t_2 - from cells $2r+3$ through $n+2r+1$.

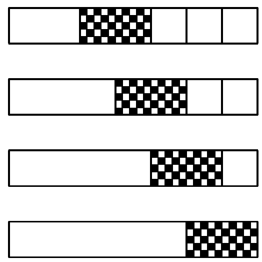

Figure 3: Tilings of length 7 with two dominoes and three squares separated into four subsets, with respect to the last domino in a tiling.

Clearly, the tilings in the set t_1 are enumerated by $\begin{bmatrix} r \\ r \end{bmatrix}_q$. The tilings in t_2 are enumerated by

$$\begin{bmatrix} n-r-1 \\ r \end{bmatrix}_q q^{(r+1)(n-2r-1)},$$

where the factor $q^{(r+1)(n-2r-1)}$ is added because there are $r+1$ dominoes left from a tiling in t_2 and there are $n-2r-1$ squares in a such tiling. By the same arguments one can conclude that the tilings in S_2 are enumerated by the polynomial

$$\begin{bmatrix} r+1 \\ r \end{bmatrix}_q \begin{bmatrix} n-r \\ r \end{bmatrix}_q q^{(r+1)(n-2r-2)},$$

tilings in S_3 by

$$\begin{bmatrix} r+2 \\ r \end{bmatrix}_q \begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q q^{(r+1)(n-2r-3)},$$

..., and tilings in S_{n-2r} by

$$\begin{bmatrix} n-r-1 \\ r \end{bmatrix}_q \begin{bmatrix} r \\ r \end{bmatrix}_q.$$

Finally, these terms sum up to

$$\begin{bmatrix} r \\ r \end{bmatrix}_q \begin{bmatrix} n-r-1 \\ r \end{bmatrix}_q q^{(r+1)(n-2r-1)} + \cdots + \begin{bmatrix} n-r-1 \\ r \end{bmatrix}_q \begin{bmatrix} r \\ r \end{bmatrix}_q$$

which is the polynomial $\begin{bmatrix} n \\ 2r+1 \end{bmatrix}_q$. This reasoning completes the proof of Theorem 3.

Theorem 3 *For natural numbers n and r, where $n \geq 2r+1$, we have*

$$\sum_{i=1}^{n-2r} q^{(r+1)(n-2r-i)} \begin{bmatrix} r-1+i \\ r \end{bmatrix}_q \begin{bmatrix} n-r-i \\ r \end{bmatrix}_q = \begin{bmatrix} n \\ 2r+1 \end{bmatrix}_q .$$

As an illustration of Theorem 3 we can take that there are 20 tilings of the length 9, tiled by three dominoes and three squares. These can be separated by means of the median domino into sets S_1, S_2, S_3 and S_4, where $|S_1| = 4, |S_2| = 6, |S_3| = 6, |S_4| = 4$ (Figure 4). These cardinalities correspond to the terms in the equality stated in Theorem 3 for $n = 6$ and $r = 1$. The Gaussian polynomial $\left[{6 \atop 3}\right]_q$ representing these tilings reads as

$$q^9 + q^8 + 2q^7 + 3q^6 + 3q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1.$$

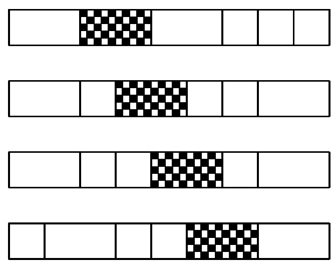

Figure 4: Representatives of sets $S_1, \ldots, S_4$, respectively, of tilings of length 9 with three dominoes, with the central domino marked.

Having in mind the symmetry of pairs of factors under the sum in Theorem 3, we have a simple consequence to binomial coefficients. By the substitution $m = n - 2r$ we immediately obtain the following corollary.

Corollary 1 *For natural numbers n and r, where n is even we have*

$$2 \sum_{i=1}^{n/2} \binom{r-1+i}{r} \binom{n+r-i}{r} = \binom{n+2r}{2r+1}.$$

In order to prove our next result, we consider the class of $(n+r)$-board tilings having r dominoes where the constraint on the length of a tiling is that $n-r$ must be odd.

Thus, the number of squares in a tiling is odd which provides us a double counting of tilings. By Lemma 1, the coefficient $\begin{bmatrix} n \\ r \end{bmatrix}_q$ enumerates these tilings. On the other hand, the sum

$$\begin{bmatrix} \frac{n-r-1}{2} \\ 0 \end{bmatrix}_q \begin{bmatrix} \frac{n-r-1}{2}+r \\ k \end{bmatrix}_q + \begin{bmatrix} \frac{n-r-1}{2}+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} \frac{n-r-1}{2}+r-1 \\ k-1 \end{bmatrix}_q q^{(n-r+1)/2} + \cdots + \begin{bmatrix} \frac{n-r-1}{2}+r \\ k \end{bmatrix}_q \begin{bmatrix} \frac{n-r-1}{2} \\ 0 \end{bmatrix}_q q^{r(n-r+1)/2}$$

consisting of $r+1$ terms also enumerates them. This follows by analogue arguments as in the previous proof, with the difference that here we enumerate tilings with respect to the median square (which always exists since their number is odd). Now, by the substitution $m := n-r$, we have the following statement.

Theorem 4 *For natural numbers n and r, where n is odd, we have*

$$\sum_{i=0}^{r} q^{i(n+1)/2} \begin{bmatrix} \frac{n-1}{2}+i \\ i \end{bmatrix}_q \begin{bmatrix} \frac{n-1}{2}+r-i \\ r-i \end{bmatrix}_q = \begin{bmatrix} n+r \\ r \end{bmatrix}_q .$$

In the same fashion as in case of Theorem 3 we consider, as an illustration of Theorem 4, tilings of the length 13. The Gaussian polynomial $\begin{bmatrix} 9 \\ 4 \end{bmatrix}_q$ representing these tilings reads as

$$q^{20} + q^{19} + 2q^{18} + 3q^{17} + 5q^{16} + 6q^{15} + 8q^{14} + 9q^{13} + 11q^{12} + 11q^{11} + 12q^{10} + 11q^{9} + 11q^{8} + 9q^{7} + 8q^{6} + 6q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + q + 1.$$

Again we have a consequence to binomial coefficients and we state it in Corollary 2.

Corollary 2 *For odd natural numbers n and r we have*

$$2\sum_{i=0}^{(r+1)/2} \binom{\frac{n-1}{2}+i}{i} \binom{\frac{n-1}{2}+r-i}{r-i} = \binom{n+r}{r}.$$

1.4 Concluding remarks

In this paper we use a combinatorial interpretation of q-binomial coefficients through the weighted lattice paths to establish identities for these polynomials. In particular, we establish two duals of Gaussian polynomial identities. A pair of identities is stated in Theorems 1 and 8. In Theorems 3 and 4 we evaluate the sum with terms of two factors.

We believe that ideas and results presented here could be used to get further Gaussian polynomial identities. Here we enumerate weighted tilings with respect to position of the last square, the last domino, the median domino and the median square. We are fairly convinced that one can found other similar criteria and employ them to obtain further identities. More refined constraints on tilings could possibly lead to even more interesting relations for Gaussian polynomials.

Acknowledgement

Our paper is a result of research in the frame of the project funded by the Federal Ministry of Education and Science.

References

[1] G. Andrews, K. Ericsson, *Integer Partitions*, Cambridge University Press, Cambridge UK, 2004.

[2] J. Azose, A. T. Benjamin, K. C. Garrett, *A Tiling Interpretation of q-Binomial Coefficients*, CiteSeer, 2007.

[3] P. Cameron, *Notes on Counting: An Introduction to Enumerative Combinatorics*, Cambridge University Press, 2017.

[4] H. Cohn, *Projective geometry over F1 and the Gaussian binomial coefficients*, Amer. Math. Monthly, 111 (2004), 487-495.

[5] V. J. W. Guo, D. Yang, *A q-analogue of some binomial coefficient identitites of Y. Sun*, Electron. J. Combin., 18 (2011) #P78, 1-6.

[6] R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.

[7] V. Kac, P. Cheung, *Quantum Calculus*, Springer, 2002.