On some partition identities related to affine Lie algebra representations

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Abstract

In this paper, meant for combinatorialists, we give a proof of equipotence of two families of partitions ('Primc conjecture') which appeared in the early stage of representation theory of affine Lie algebra $A_1^{(1)}$ (or $Sl(2,\mathbb{C})^\wedge$). The result was obtained already in 1983 and it lead to the very first character formulas for the level 3 standard modules (this is mentioned in Ref[1]) on pages 2 and 77 respectively). We hope that the intricacy of handling the interlocking recursion relations may be of some interest to combinatorialists not working in Lie algebra theory.

Keywords: partition function, partition identity, Lie algebra

MSC: 05A17, 11P84

1.1 Introduction

For $\nu = 0, 1, 2 \pmod 3$ we introduce the following families of partitions:

$$\begin{aligned} Q_s^\nu(n) := \{(n_0, n_1, ..., n_s) \in \mathbb{N}^{s+1} | n = n_0 + n_1 + ... + n_s, \\ n_i \equiv -i - \nu \pmod 3, \\ n_i \geq n_{i+1} - 2, \ \ n_i > n_{i+2}\} \end{aligned} \tag{9}$$

$$Q^\nu(n) := \bigcup_{s \geq 0} Q_s^\nu(n)$$

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and denote by

$$q_s^\nu(n) := \mathrm{Card} Q_s^\nu(n) \tag{10}$$
$$q^\nu(n) := \mathrm{Card} Q^\nu(n) \tag{11}$$

the numbers of partitions in these families of partitions.

The partitions from (9) we shall relate to the partitions appearing in the Gordon's generalization of the Rogers-Ramanujan identities:

$$\begin{aligned} P_{k,i}(n,m) := \{(n_1,...,n_m) \in \mathbb{N}^m | n = n_1 + ... + n_m, \\ n_j \geq n_{j+1}, n_j \geq n_{j+k-1} + 2, \\ \mathrm{Card}\{j | n_j = 1 < i\}\} \end{aligned} \tag{12}$$

$$p_{k,i}(n,m) := \mathrm{Card} P_{k,i}(n,m) \tag{13}$$

Proposition 1 *The family $Q_s^\nu(n)$ is nonempty if $s \pmod 3$ is given by the following table,*

$n \pmod 3 \setminus \nu$	0	1	2
0	0 *or* 2	1 *or* 2	2
1	–	–	0 *or* 1
2	1	0	–

(14)

Proof. *For any given n, ν if $\alpha = (n_0,...,n_s) \in Q_s^\nu(n) \neq \emptyset$, then*

$$n \equiv \sum_{i=0}^{s}(-i-\nu) \pmod 3 = -\frac{s(s+1)}{2} - (s+1)\nu \pmod 3. \tag{15}$$

■

The recursions for the number theoretic functions $q_s^\nu(n)$ (defined by (23)) are given by the following proposition.

Proposition 2 *For $n \in \mathbb{N}$, $s \in \mathbb{Z}_+$, $\nu \in \mathbb{Z}_3$ we have:*

$$q_s^\nu(n) = \begin{cases} q_s^{\nu+2}(n-2s+2) + q_{s-2}^{\nu+2}(n-2s-2), & \text{if } s+\nu \equiv 0 \pmod 3 \\ q_s^{\nu+1}(n-s-1), & \text{if } s+\nu \equiv 1 \pmod 3 \\ q_s^{\nu+1}(n-s-1) + q_{s-1}^{\nu+2}(n-s-1), & \text{if } s+\nu \equiv 2 \pmod 3 \end{cases} \tag{16}$$

with initial conditions

$$q_s^\nu(1) = \delta_{s,2} \cdot \delta_{s,0} \tag{17}$$

together with the convention $q_s^\nu(0) = 1$ *if* $s = \nu = 0$ *or* $s = -1, \nu = 1$; *0 otherwise. We put*

$$q_s^\nu(n) = 0 \tag{18}$$

if ($s \leq 0$ *or* $n \leq 0$) *and* $n \neq s$, *so that the right-hand side of (16) is well defined. With the recursions above all functions* $q_s^\nu(n)$ *are uniquely defined.*

Proof. *We distinguish three cases depending on* $(s+\nu)$ (mod 3). $s + \nu \equiv 0 (mod 3)$. *If* $s = 0$, *then* $\alpha = (n_0), n_0 \equiv -\nu (mod 3), n_0 > 0$; *so*

$$q_0^\nu(n) = \begin{cases} q_0^2(n-2), & \text{for } \nu = 0 \\ q_0^2(n-1), & \text{for } \nu = 1 \\ q_0^0(n-1), & \text{for } \nu = 2 \end{cases}.$$

If $s > 0$ *and* $\alpha = (n_0, ..., n_s) \in Q_s^\nu(n)$, *then* $n_{s-1} \equiv -(s-1) - \nu \equiv 1 (mod 3)$, $n_s = -s - \nu \equiv 0 (mod 3)$, *hence* $n_{s-1} \geq 1$, $n_s \geq 3$. *Then in the subcase a1)* $n_{s-1} \geq 2$ *we have* $n_{s-1} \geq 4$ *because* $n_{s-1} \equiv 1 (mod 3)$. *By letting* $\hat{\alpha} := (n_0 - 2, ..., n_s - 2)$ *we get all partitions from* $Q_s^{\nu+2}(n - 2s - 2)$.

In the subcase a 2) $n_{s-1} = 1$, *from (9) it follows* $n_{s-1} \geq n_s - 2$, *hence* $n_s = 3$. *So* $Q_1^2(4) = \{(1,3)\}$ *and* $q_1^2(4) = 1$, *and for* $s \geq 2$ *by letting*

$$\hat{\alpha} = (n_0 - 2, ..., n_{s-2} - 2)$$

we obtain all partitions from $Q_{s-2}^{\nu+2}(n - 2s - 2$, *because* $n_{s-2} > n_s (= 3)$, $n_{s-3} \geq n_{s-2} - 2 (\geq 1)$.

By combining the two bijections from a1) and a2) we obtain the first relation in (16).

b) $s + \nu \equiv 1$ (mod 3). *For* $\alpha = (n_0, ..., n_s) \in Q_s^\nu(n)$, $n_s \equiv -s - \nu \equiv 2$ (mod 3), $n_{s-1} \equiv -s + 1 - \nu \equiv 0$ (mod 3).

So $n_s \geq 2$ *and* $n_{s-1} \geq 3$ *(if* $s > 0$*). Then the map* $\alpha \mapsto \bar{\alpha} = (n_0 - 1, ..., n_s - 1)$ *is a bijection onto* $Q_s^{\nu+1}(n - s - 1)$.

c) $s + \nu \equiv 2$ (mod 3). *For* $\alpha = (n_0, ..., n_s) \in Q_s^\nu(n)$ *we have* $n_s \equiv 1$ (mod 3) *and* $n_{s-1} \equiv 2$ (mod 3) *(if* $s > 0$*). Like in a) we have two subspaces:*

c1) $n_s = 1$. *Then* $\alpha \mapsto \bar{\alpha} = (n_0 - 1, ..., n_{s-1} - 1)$ *is a bijection onto* $Q_{s-1}^{\nu+1}(n - s - 1)$.

c2) $n_s \geq 4$. *Then* $n_{s-1} \geq n_s - 2 (\geq 2)$, *so we get a bijection* $\alpha \mapsto \bar{\alpha} = (n_0 - 1, ..., n_s - 1)$ *onto* $Q_s^{\nu+1}(n-s-1)$. *In the case* $s = 0$, *we have* $\nu \equiv 2 \pmod 3$ *we need to check that* $q_0^2(n) = q_0^0(n-1)$. *For* $n > 1$ *it is obvious, and for* $n = 1$, $q_0^{(2)}(1) = 1 = q_0^0(0)$ *by (18).*

The fact that (16) uniquely determines the functions q_s^ν *one can prove by induction on* $n + s$. ■

Corollary 3 *For* $n \in \mathbb{N}, s \in \mathbb{Z}_+, \nu \in \mathbb{Z}_3$ *(with the conventions from the Proposition 2) we have*

$$\begin{aligned} q_{s+2}^\nu(n) + q_s^\nu(n) &= q_{s+2}^{\nu+1}(n + 2s + 6), \quad \textit{if } s + \nu \equiv 0 \pmod 3 \\ q_s^\nu(n) &= q_s^{\nu+2}(n + s + 1), \quad \textit{if } s + \nu \equiv 2 \pmod 3 \\ q_{s+1}^\nu(n) + q_s^\nu(n) &= q_{s+1}^{\nu+2}(n + s + 2), \quad \textit{if } s + \nu \equiv 2 \pmod 3 \end{aligned} \tag{19}$$

Corollary 4 *For the number theoretic functions* $q^\nu(n)$, $(\nu = 0, 1, 2)$ *defined in (11) we have the following formulas:*

$$
\begin{array}{lrl}
i) & q^0(3n) = & \sum_{s\geq 1} q^1_{3s-1}(3n+6s) \\
ii) & q^0(3n) = & \sum_{s\geq 0} q^2_{3s}(3n+3s+1) = \sum_{s\geq 0} q^0_{3s}(3n+6s+2) \\
iii) & q^1(3n) = & \sum_{s\geq 1} q^2_{3s-1}(3n+6s+4) \\
iv) & q^1(3n) = & \sum_{s\geq 1} q^0_{3s-1}(3n+3s) \\
v) & q^2(3n+1) = & \sum_{s\geq 1} q^1_{3s-2}(3n+3s) = \sum_{s\geq 1} q^0_{3s-2}(3n+6s-1) \\
vi) & q^2(3n+1) = & \sum_{s\geq 0} q^0_{3s}(3n+6s+3) \\
vii) & q^2(3n) = & \sum_{s\geq 0} q^2_{3s+2}(3n) = \sum_{s\geq 1} q^0_{3s-1}(3n-3s) \\
viii) & q^0(3n+2) = & \sum_{s\geq 0} q^0_{3s+1}(3n+2) = \sum_{s\geq 0} q^1_{3s+1}(3n-3s) \\
ix) & q^1(3n+2) = & \sum_{s\geq 0} q^1_{3s}(3n+2) = \sum_{s\geq 0} q^2_{3s}(3n-3s+1) \\
x) & q^0(3n+1) = & q^1(3n+1) = q^2(3n+2) = 0.
\end{array}
\tag{20}
$$

Proof. *Follows easily by Corollary 3 with the help of Proposition 1.* ■

1.2 Relation to Gordon partitions

In this subsection we shall relate partition families $Q^\nu_s(n)$ from (9) firstly to standard partitions with additional difference conditions and secondly to Gordon partitions. For $n \in \mathbb{N}$, $s \in \mathbb{Z}_+$, $\nu \in \mathbb{Z}_3$ we introduce the following families of partitions:

$$
\begin{aligned}
\bar{Q}^\nu_s(n) &:= \{(n_0, n_1, ..., n_s) \in \mathbb{N}^{s+1} | n_j \equiv -s-\nu (mod 3), \\
&\quad n = n_0 + ... + n_s, n_j \geq n_{j+1}, n_j - n_{j+2} \geq 5, n_{s-1} > 2\} \\
\bar{Q}^\nu(n) &:= \bigcup_{s\geq 0} \bar{Q}^\nu_s(n)
\end{aligned}
\tag{21}
$$

and denote by

$$\bar{q}^{\nu}_s(n) := \mathrm{Card}\bar{Q}^{\nu}_s(n) \quad (22)$$
$$\bar{q}^{\nu}(n) := \mathrm{Card}\bar{Q}^{\nu}(n). \quad (23)$$

Proposition 3 *For any $n \in \mathbb{N}$ and $0 \le s \le n$ we have*

$$q^{\nu}_s(n) = \bar{q}^{\nu}_s(n + s(s+1)) \quad (24)$$

Proof. *For $\alpha = (n_0, ..., n_s) \in Q^{\nu}_s(n)$ let $\alpha' = (n_0^{'}, ..., s_s^{'})$ be defined by*

$$n_i^{'} = n_i + 2(s-i).$$

We claim that $\alpha \mapsto \alpha^{'}$ is a bijection of $Q^{\nu}_s(n)$ with $\bar{Q}^{\nu}_s(n)$. From $\sum_{i=0}^{s} 2(s-i) = s(s+1)$ and $n_i^{'} = n_i + 2(s-i) \ge n_{i+1} - 2 + 2(s-i) = n_{i+1}^{'}$ it follows that $\alpha^{'}$ is a partition of $n + s(s+1)$. Since $n_i^{'} \equiv -i - \nu + 2(s-i) \equiv -s - \nu \pmod 3)$,

$$n_i^{'} = n_i + 2(s-i) > n_{i+2} + 2(s-i) = n_{i+2}^{'} + 4$$

and

$$n_{s-1}^{'} = n_{s-1} + 2 > 2.$$

then the partition $\alpha^{'}$ belongs to $\bar{Q}^{\nu}_s(n + s(s+1))$.

Conversely for any partition $\beta = (m_0, m_1, ..., m_s) \in \bar{Q}^{\nu}_s(n + s(s+1))$ let $\alpha = (n_0, ..., n_s)$, where $n_i := m_i - 2(s-i)$. Then $n_i \in \mathbb{N}$, because for $i = s - 2j$

$$n_{s-2j} = m_{s-2j} - 4j > m_{s-2j+2} - 4(j-1) > ... > m_s > 0$$

and for $i = s - 2j - 1$

$$n_{s-2j-1} = m_{s-2j-1} - 4j - 2 > ... > m_{s-1} - 2 > 0$$

where we have used the defining inequalities in $\bar{Q}^{\nu}_s(n)$ twice. Thus we have proved that $\alpha \mapsto \alpha^{'}$ is a bijection, hence the identity (24) follows. ■

Now we shall relate the partition families $\bar{Q}^{\nu}_s(n)$ to the Gordon's partitions families defined in (12).

Proposition 4 *In terms of the numbers $p_{k,i}(n,m)$ of Gordon partitions defined in (12) we have the following formulas:*

$$\bar{q}_s^{\nu}(n) = \begin{cases} p_{3,3}(\frac{n}{3}, s+1), & \text{if } s+\nu \equiv 0 \pmod 3, n \equiv 0 \pmod 3 \\ p_{3,2}(\frac{n+s+1}{3}, s+1), & \text{if } s+\nu \equiv 1 \pmod 3, n \equiv 2s+2 \pmod 3 \\ p_{3,2}(\frac{n+2s+2}{3}, s+1), & \text{if } s+\nu \equiv 2 \pmod 3, n \equiv s+1 \pmod 3 \end{cases}$$

Proof. *1.) Let $s+\nu \equiv 0 \pmod 3$. Then each $\alpha = (n_0, n_1, ..., n_s) \in \bar{Q}_s^{\nu}(n)$ satisfies $n_i \equiv 0 \pmod 3$, $n_i \geq n_{i+1}$, $n_i - n_{i+2} \geq 5$, $n_{s-1} > 2$. By defining $\alpha' = (m_0, ..., m_s)$, with $m_i = n_i/3$ we see that $m_i \geq m_{i+1}$, $m_i \in \mathbb{N}$, $m_i - m_{i+2} \geq 2$, $m_{s-1} > 1$. This implies that $\alpha' \in P_{3,3}(\frac{n}{3}, s+1)$ and clearly $\alpha \mapsto \alpha'$ is a bijection.*

2.) $s+\nu \equiv 1 \pmod 3$. Then $n_i \equiv 2 \pmod 3$, $n_i \geq n_{i+1}$, $n_i - n_{i+2} \geq 5$, $n_{s-1} > 2 (\implies n_{s-1} \geq 5)$. By defining $\alpha' = (m_0, ..., m_s)$ where $m_i = (n_i+1)/3$ we see that $m_i \geq m_{i+1}$, $m_i - m_{i+2} \geq 2$, $m_{s-1} > 1$ so α' belongs to $P_{3,2}(\frac{n+s+1}{3}, s+1)$ and again $\alpha \mapsto \alpha'$ is a bijection.

3.) $s+\nu \equiv 2 \pmod 3$. Then $n_i \equiv 1 \pmod 3$, $n_{s-1} > 2 (\implies n_{s-1} \geq 4)$. Let $\alpha' = (m_0, ..., m_s)$ where $m_i = (n_i+2)/3$. Then $m_i \in \mathbb{N}$, $m_i - m_{i+2} \geq 2$, $m_{s-1} > 1$. So $\alpha' \in P_{3,2}(\frac{n+2s+2}{3}, s+1)$ and $\alpha \mapsto \alpha'$ is a bijection. The proof is finished. ■

By combining the Proposition 3 and Proposition 4 we obtain

Corollary 5 *For any $n \in \mathbb{N}$ and $0 \leq s \leq n$, $\nu \in \{0,1,2\}$ we have*

$$q_s^{\nu}(n) = \begin{cases} p_{3,3}(\frac{n+s(s+1)}{3}, s+1), & \text{if } s+\nu \equiv 0 \pmod 3 \\ p_{3,2}(\frac{n+(s+1)^2}{3}, s+1), & \text{if } s+\nu \equiv 1 \pmod 3 \\ p_{3,2}(\frac{n+(s+1)(s+2)}{3}, s+1), & \text{if } s+\nu \equiv 2 \pmod 3 \end{cases}$$

where $p_{3,i}(n,m)$ is the number of partitions $(n_1, ..., n_m)$ of $n (= n_1 + ... + n_m)$ such that $n_j - n_{j+2} \geq 2$, $Card\{j | n_j = 1\} < i$. (cf. (12))

Remark 5 *We have*

i) $p_{3,3}(n,m) \neq 0 \implies m^2 \leq 2n$

ii) $p_{3,2}(n,m) \neq 0 \implies m(m+1) \leq 2n$

For i), if $(n_1,...,n_m) \in P_{3,3}(n,m)$ *then*

$$n = n_1 + ... + n_m \geq \begin{cases} (m + (m-1) + (m-1) + ... + 3 + 3 + 1 + 1 = \frac{m^2+1}{2}, & \textit{if } m \textit{ odd} \\ ((m-1) + (m-1) + ... + 3 + 3 + 1 + 1 = \frac{m^2}{2}, & \textit{if } m \textit{ even} \end{cases}$$

For ii), if $(n_1,...,n_m) \in P_{3,2}(n,m)$ *then*

$$n = n_1 + ... + n_m \geq m + (m-1) + ... + 2 + 1 = m(m+1)/2.$$

Remark 6 *We have*

$$q_s^\nu(n) \neq 0 \implies \begin{cases} (s+2)^2 < 2n, & \textit{if } s+\nu \equiv 0 \pmod 3 \\ (s+1)(s+4) \leq 2n, & \textit{if } s+\nu \equiv 1 \pmod 3 \\ (s+1)(s+2) \leq 2n, & \textit{if } s+\nu \equiv 2 \pmod 3. \end{cases}$$

1.3 Equipotence of the families $Q^\nu(n), \nu = 0, 1$ of partitions

Theorem 7 *For* $n \in 3\mathbb{N}$, $\nu = 0, 1$, *the set of partitions*

$$\begin{aligned} Q^\nu(n) := \{(n_0, n_1, ..., n_s) \in \mathbb{N}^{s+1} \quad | \quad & n_i \equiv -i - \nu \pmod 3, s \geq 0, \\ & n = n_0 + ... + n_s, \\ & n_i > n_{i+2}, \\ & n_i \geq n_{i+1} - 2\} \end{aligned}$$

are equinumerous, i.e. Card $Q^0(n) =$ *Card* $Q^1(n)$.

Proof. We have to prove that $\mathrm{Card}Q^0(n) = \mathrm{Card}Q^1(n)$ $(n \in 3\mathbb{N})$ or, by using the notation (11) that

$$q^0(n) = q^1(n), \;\; n \in 3\mathbb{N} \tag{25}$$

By relations (20) i), iv) of Corollary 4 this is equivalent to

$$\sum_{s \equiv 2 \pmod 3} q_s^1(n + 2s + 2) = \sum_{s \equiv 2 \pmod 3} q_s^0(n + s + 1), \;\; n \in 3\mathbb{N}, \;\; s \geq 0 \tag{26}$$

and, by Corollary 5. this is equivalent to

$$\sum_{s\equiv 0 \pmod 3} p_{3,3}\left(\frac{n+s(s+1)}{3}, s\right) = \sum_{s\equiv \pmod 3} p_{3,2}(\frac{n+s(s+2)}{3}, s), \tag{27}$$

with $n \in 3\mathbb{N},\ \ s > 0$.

Now we employ the Theorem 1 of reference [2] in the special case $k = 3, i = 2, 3, d = 0$. In our notations (13) and by replacing q for q^3 special case reads as follows:

$$\sum_{n\geq 0}\sum_{m\geq 0} p_{3,i}(n,m)q^{3n}z^m = \sum_{n_1\geq 0}\sum_{n_2\geq 0} \frac{z^{N_1+N_2}}{(q^3)_{n_1}(q^3)_{n_2}} \begin{cases} q^{3(N_1^2+N_2^2)}, & \text{if } i=3 \\ q^{3(N_1^2+N_2^2+N_2)}, & \text{if } i=2 \end{cases} \tag{28}$$

where $N_1 = n_1 + n_2$, $N_2 = n_2$ and $(q)_k = (1-q)(1-q^2)...(1-q^k)$. From (28) it follows easily

$$\begin{aligned} p_{3,3}\left(\frac{n+s(s+1)}{3}, s\right) &= [q^{n+s(s+1)}] \sum_{n_1,n_2\geq 0, N_1+N_2=s} \frac{q^{3(N_1^2+N_2^2)}}{(q^3)_{n_1}(q^3)_{n_2}} \\ &= [q^n] \sum_{n_1,n_2\geq 0, N_1+N_2=s} \frac{q^{3N_1^2+3N_2^2-s(s+1)}}{(q^3)_{n_1}(q^3)_{n_2}} \\ &= [q^n] \sum_{n_1,n_2\geq 0, N_1+N_2=s} \frac{q^{2(N_1^2-N_1N_2+N_2^2)-N_1-N_2}}{(q^3)_{n_1}(q^3)_{n_2}}. \end{aligned} \tag{29}$$

Similarly we get

$$p_{3,2}\left(\frac{n+s(s+2)}{3}, s\right) = [q^n] \sum_{n_1,n_2\geq 0, N_1+N_2=s} \frac{q^{2(N_1^2-N_1N_2+N_2^2)+N_2-2N_1}}{(q^3)_{n_1}(q^3)_{n_2}}. \tag{30}$$

From (25) $\Longleftrightarrow$ (26) $\Longleftrightarrow$ (27) and (29) , (30) to prove (25) it is enough to prove the equality of generating functions

$$1+\sum_{n\geq 1, n\equiv 0 \pmod 3} q^0(n)x^n = \sum_{n_1,n_2\geq 0, N_1+N_2\equiv 0 \pmod 3} \frac{x^{2(N_1^2-N_1N_2+N_2^2)-N_1-N_2}}{(x^3)_{n_1}(x^3)_{n_2}} \quad (31)$$

and

$$1+\sum_{n\geq 1, n\equiv 0 \pmod 3} q^1(n)x^n = \sum_{n_1,n_2\geq 0, N_1+N_2\equiv 0 \pmod 3} \frac{x^{2(N_1^2-N_1N_2+N_2^2)+N_2-2N_1}}{(x^3)_{n_1}(x^3)_{n_2}} \quad (32)$$

Observe now that for $(n_1, n_2 \in \mathbb{N}^2)$ we have

$$n_1 + 2n_2 \equiv 0 \pmod 3 \Longleftrightarrow n_2 + 2n_1 \equiv 0 \pmod 3.$$

By change of variables $(n_1, n_2) \to (n_2, n_1)$ in (32), the quantity $N_2 - 2N_1 = n_2 - 2(n_1 + n_2) = -2n_1 - n_2$ gets transformed to $-n_1 - 2n_2 = -N_1 - N_2$ and the R.H.S of (32) becomes the R.H.S of (31) so the R.H.S of (31) and (32) are equal. This proves the Theorem 7

■

Theorem 8 *For the functions $q^{(0)}(n)$, $q^{(1)}(n)$, $q(n)$ we have the following generating functions:*

$$i)\ 1+\sum_{n\equiv 0 \pmod 3} q^{(0)}(n)x^n = \sum_{n_1,n_2\geq 0, N_1+N_2\equiv 0 \pmod 3} \frac{x^{2(N_1^2-N_1N_2+N_2^2)-N_1-N_2}}{(x^3)_{n_1}(x^3)_{n_2}} \quad (33)$$

$$ii)\ \sum_{n\equiv 0 \pmod 3} q^{(1)}(n)x^n = \sum_{n_1,n_2\geq 0, N_1+N_2\equiv 2 \pmod 3} \frac{x^{2(N_1^2-N_1N_2+N_2^2)-N_1-N_2}}{(x^3)_{n_1}(x^3)_{n_2}} \quad (34)$$

$$iii)\ \sum_{n\equiv 1 \pmod 3} q^{(2)}(n)x^n = \sum_{n_1,n_2\geq 0, N_1+N_2\equiv 1 \pmod 3} \frac{x^{2(N_1^2-N_1N_2+N_2^2)-N_1-N_2}}{(x^3)_{n_1}(x^3)_{n_2}} \quad (35)$$

Proof. The relation *i*) is obtained in (31). In order to prove relation *ii*) we use (20), *iii*) to get

$$q^{(1)}(n) = \sum_{s\in 3\mathbb{Z}_+ +1} q_s^{(2)}(n+2s+2).$$

Furthermore, according to Corollary 5 we have

$$\begin{aligned} q^{(1)}(n) &= \sum_{s\in 3\mathbb{Z}_+ +1} p_{3,3}\left(\frac{n+(s+1)(s+2)}{3}, s+1\right) \\ &= \sum_{t\in 3\mathbb{Z}_+ +2} [x^{n+t(t+1)}] \sum_{n_1,n_2\geq 0, N_1+N_2\equiv 2 \pmod 3} \frac{x^{(3N_1^2+3N_2^2)}}{(x^3)_{n_1}(x^3)_{n_2}} \\ &= [x^n] \sum_{n_1,n_2\geq 0, N_1+N_2\equiv 2 \pmod 3} \frac{x^{2(N_1^2-N_1N_2+N_2^2)-N_1-N_2}}{(x^3)_{n_1}(x^3)_{n_2}}. \end{aligned}$$

Now, by substitution into generating function for $q^{(1)}$ we obtain relation *ii*).

For the purpose to to prove relation *iii*) firstly we use relation *vi*) in Corollary 5 to get:

$$\begin{aligned} q^{(2)}(n) &= \sum_{s\in 3\mathbb{Z}_+ +1} q_s^{(0)}(n+2s+2), \quad n\equiv 1 \pmod 3 \\ &= \sum_{s\in 3\mathbb{Z}_+} p_{3,3}\left(\frac{n+(s+1)(s+2)}{3}, s+1\right) \\ &= \sum_{t\in 3\mathbb{Z}_+ +1} [x^{n+t(t+1)}] \sum_{n_1,n_2\geq 0, N_1+N_2\equiv t \pmod 3} \frac{x^{(3N_1^2+3N_2^2)}}{(x^3)_{n_1}(x^3)_{n_2}} \\ &= [x^n] \sum_{n_1,n_2\geq 0, N_1+N_2\equiv 1 \pmod 3} \frac{x^{2(N_1^2-N_1N_2+N_2^2)-N_1-N_2}}{(x^3)_{n_1}(x^3)_{n_2}}. \end{aligned}$$

Once having this relation, the statement *iii*) follows directly. ■

Corollary 6 *We let* χ *denote the generating function which is the sum of generating function of i), ii) and iii) in Theorem 8. Then, we have*

$$\chi = \sum_{n_1,n_2\geq 0} \frac{x^{2(N_1^2-N_1N_2+N_2^2)-N_1-N_2}}{(x^3)_{n_1}(x^3)_{n_2}}. \tag{36}$$

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