The Prouhet–Tarry–Escot Problem

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Abstract

An known elementary proof of existence of infinitely many solutions (and a procedure of their construction) to the Prouhet-Tarry-Escott problem is given together with its connection to the Prouhet–Thue–Morse sequence. Another very short and possibly not yet published proof is presented utilizing Taylor expansion of a suitably chosen function represented in two different ways.

The Prouhet-Tarry-Escott problem is a classical problem in Diophantine equations that consists of finding two different lists (i.e. multisets) of integers $[a_1, a_2, \dots a_n]$ and $[b_1, b_2, \dots b_n]$ (repetitions are allowed) such that

$$\begin{array}{ccccccccccc} a_1 & + & \cdots & + & a_n & = & b_1 & + & \cdots & + & b_n \\ a_1^2 & + & \cdots & + & a_n^2 & = & b_1^2 & + & \cdots & + & b_n^2 \\ \vdots & & & & & \vdots & & & & & \vdots \\ a_1^m & + & \cdots & + & a_n^m & = & b_1^m & + & \cdots & + & b_n^m \end{array}$$

where m and n are some natural numbers. It is known that n must be strictly greater than m (of special interest are so called ideal solutions where $m = n - 1$). The above system can be written as $[a_i] =_m [b_i]$. The earliest occurrence of a problem of this kind can be traced down to Euler and Goldbach 1750–1751 when they noted that

$$[a, b, c, a + b + c] =_2 [a + b, a + c, b + c].$$

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The problem was named after E. Prouhet, who studied it in the early 1850s, and G. Tarry and Escott, who studied it in the early 1910s.

We shall restrict the problem to disjoint sets (or multisets).

Definition 1 *Two finite sets (or multisets) $A, B \subset \mathbb{C}$ are said to have (or satisfy) n–property if $A, B \neq \emptyset$, $A \cap B = \emptyset$ and for any $k \in \{0, 1, 2, \ldots, n-1\}$ we have*

$$\sum A^k := \sum_{a \in A} a^k = \sum_{b \in B} b^k = \sum B^k.$$

Example 2 *The sets $A := \{0, 3, 5, 6\}$ and $B := \{1, 2, 4, 7\}$ have 3–property because*

$$\begin{aligned}
\sum A^0 &= 0^0 + 3^0 + 5^0 + 6^0 = 4 = 1^0 + 2^0 + 4^0 + 7^0 = \sum B^0 \\
\sum A^1 &= 0^1 + 3^1 + 5^1 + 6^1 = 14 = 1^1 + 2^1 + 4^1 + 7^1 = \sum B^1 \\
\sum A^2 &= 0^2 + 3^2 + 5^2 + 6^2 = 70 = 1^2 + 2^2 + 4^2 + 7^2 = \sum B^2.
\end{aligned}$$

Definition 3 *Let $S \subseteq \mathbb{C}$ and $c \in \mathbb{C}$. Then we define*

$$S + c := \{s + c \mid s \in S\}.$$

Lemma 4 *Let $A, B \subset \mathbb{C}$ satisfy n–property. Then for any $c \in \mathbb{C}$ the sets $A + c$ and $B + c$ also have n–property.*

Proof.

$$\begin{aligned}
\sum (A+c)^k &= \sum_{a \in A} (a+c)^k = \sum_{a \in A} \sum_{j=0}^{k} \binom{k}{j} a^j c^{k-j} = \sum_{j=0}^{k} \binom{k}{j} c^{k-j} \sum_{a \in A} a^j = \\
&= \sum_{j=0}^{k} \binom{k}{j} c^{k-j} \sum_{b \in B} b^j = \sum (B+c)^k \qquad \blacksquare
\end{aligned}$$

Example 5 *The sets $C = A - 4 = A + (-4) = \{-4, -1, 1, 2\}$ and $D = B - 4 = \{-3, -2, 0, 3\}$, where A and B are the same as in Example 2, also have 3–property:*

$$\begin{aligned}
\sum C^0 &= 4 = \sum D^0 \\
\sum C^1 &= -2 = \sum D^1 \\
\sum C^2 &= 22 = \sum D^2.
\end{aligned}$$

Lemma 6 *Let $A, B, C, D \subset \mathbb{C}$ and $(A \cup B) \cap (C \cup D) = \emptyset$. Then if two pairs of sets A, B and C, D both have n–property then the sets $A \cup C$ and $B \cup D$ also have n–property.*

Proof. Obvious. ■

Proposition 7 *If sets $A, B \subset \mathbb{C}$ have n–property then for any $c \in \mathbb{C}$ such that $(A \cup B) \cap ((A+c) \cup (B+c)) = \emptyset$ sets $A_1 = (B+c) \cup A$ and $B_1 = (A+c) \cup B$ have $(n+1)$–property.*

Proof. Lemmas 4 and 6 imply that the sets A_1 and B_1 have n–property. It is easy to check that $A_1 \cap B_1 = \emptyset$. Now we have to prove that equality $\sum A_1^n = \sum B_1^n$ holds:

$$\begin{aligned}
\sum A_1^n &= \sum ((B+c) \cup A)^n = \sum (B+c)^n + \sum A^n \\
&= \sum B^n + \sum_{j=0}^{n-1} \binom{n}{j} c^{n-j} \sum_{b \in B} b^j + \sum A^n \\
&= \sum B^n + \sum_{j=0}^{n-1} \binom{n}{j} c^{n-j} \sum_{a \in A} a^j + \sum A^n \\
&= \sum B^n + \sum (A+c)^n \\
&= \sum B_1^n.
\end{aligned}$$
■

Example 8 *Let A and B are as in Example 2. If we set $c = 8$ then $A_1 = \{0, 3, 5, 6, 9, 10, 12, 15\}$ and $B_1 = \{1, 2, 4, 7, 8, 11, 13, 14\}$ have 4–property:*

$$\begin{aligned}
\sum A_1^0 &= 8 = \sum B_1^0 \\
\sum A_1^1 &= 60 = \sum B_1^1 \\
\sum A_1^2 &= 620 = \sum B_1^2 \\
\sum A_1^3 &= 7200 = \sum B_1^3
\end{aligned}$$

Example 9 *Starting with $A_0 := \{0\}$ and $B_0 := \{i\}$ (note that any pair of singletons has 1–property) and taking $c_1 := 1$ and then $c_2 := \sqrt{2}$ we obtain (applying the procedure* *from Proposition 7 twice)* $A_2 = \{0, 1+i, i+\sqrt{2}, 1+\sqrt{2}\}$ *and* $B_2 = \{1, i, \sqrt{2}, 1+\sqrt{2}+i\}$ *that have* 3*–property:*

$$\begin{array}{rcl} \sum A_2^0 = & 4 & = \sum B_2^0 \\ \sum A_2^1 = & 2(1+\sqrt{2}+i) & = \sum B_2^1 \\ \sum A_2^2 = & 4+2\sqrt{2}+(2+2\sqrt{2})i & = \sum B_2^2 \end{array}$$

Example 10 *The sets* $A_0 := \{0\}$ *and* $B_0 := \{1\}$ *have* 1*–property. Suppose that sets* A_{n-1} *and* B_{n-1} *have* n*–property. Then by Proposition 7 for* $c_n := 1+\max(A_{n-1}\cup B_{n-1})$ *the sets* $A_n := A_{n-1}\cup(B_{n-1}+c_n)$ *and* $B_n := B_{n-1}\cup(A_{n-1}+c_n)$ *have* $(n+1)$*–property. By the definition of* c_n *it follows that* $\max(A_{n-1}\cup B_{n-1}) < c_n \le \min((A_{n-1}+c_n)\cup(B_n+c_n))$ *which proves that* $(A_{n-1}\cup B_{n-1})\cap((A_{n-1}+c)\cup(B_{n-1}+c)) = \emptyset$*.*

Lemma 11 *Let* $\alpha_2(m)$ *be the sum of digits of binary representation of number* m*. The sets* A_n *and* B_n *defined in Example 10 can be characterized in the following way:*

$$A_n = \{m \in \{0,1,2,\ldots,2^{n+1}-1\} \mid \alpha_2(m) \in 2\mathbb{N}\},$$
$$B_n = \{m \in \{0,1,2,\ldots,2^{n+1}-1\} \mid \alpha_2(m) \in 2\mathbb{N}+1\},$$

i.e., A_n *is the set of all natural numbers less than* 2^{n+1} *having even number of digits* 1 *in their binary representations. Likewise,* B_n *consists of all natural numbers less than* 2^{n+1} *having odd number of digits* 1 *in their binary representations. Note that* $c_n = 2^{n+1}$*.*

Proof. By induction on n we prove the statement of the Lemma 11 and that $c_n = 2^{n+1}$. Induction base is clear. Suppose now that

$$A_{n-1} = \{m \in \{0,1,2,\ldots,2^{n}-1\} \mid \alpha_2(m) \in 2\mathbb{N}\},$$
$$B_{n-1} = \{m \in \{0,1,2,\ldots,2^{n}-1\} \mid \alpha_2(m) \in 2\mathbb{N}+1\}.$$

and $c_n = 2^n$. By definition we have

$$A_n = A_{n-1}\cup(B_{n-1}+c_n),$$
$$B_n = B_{n-1}\cup(A_{n-1}+c_n).$$

Also, $\max(A_n\cup B_n) = c_n + \max(A_{n-1}\cup B_{n-1}) = 2c_n - 1 = 2^{n+2}-1$, so $c_{n+1} = 2^{n+2}$. Now, $(B_{n-1}+c_n) = \{m \in \{0,1,2,\ldots,2^n-1\} \mid \alpha_2(m) \in 2\mathbb{N}+1\} + 2^{n+1}$, so $A_n = A_{n-1}\cup(B_{n-1}+c_n) = \{m \in \{m \in \{0,1,2,\ldots,2^{n+1}-1\} \mid \alpha_2(m) \in 2\mathbb{N}\}$. (Adding $c_n = 2^n$ to numbers of B_{n-1} is equivalent to prefixing their binary representation with 1 together with some zeroes if necessary.) The proof for B_n is analogous.■

Remark 12 *The sequence $t_n := \alpha_2(n) \bmod 2$ is known as the Prouhet–Thue–Morse sequence (see [1] or [2], example 12.). Sets A_n and B_n can be characterized as follows:*

$$A_n = \{m \in \{0, 1, 2, \dots, 2^{n+1} - 1\} \mid t_m = 0\}, \ B_n = \{m \in \{0, 1, 2, \dots, 2^{n+1} - 1\} \mid t_m = 1\}.$$

In other wording A_n (resp. B_n) consists of all nonnegative integers less than 2^{n+1} which are expressible as a sum of even (resp. odd) number of powers of 2.

Now we can elucidate this remarkable property by generalizing A_n (resp. B_n) to be sum of even (resp. odd) numbers of indeterminates $x_0, \dots, x_{n-1}$. (Then Lemma 11 is just a very special case $x_k = 2^k$.)
The existence of different finite sets (of numbers) A and B having the n–property for any n can be proved quickly by using Taylor expansion of the following function $F := \prod_{j=0}^{n-1}(1 - e^{tx_j})$. We have

$$\begin{aligned}
F &= \prod_{j=0}^{n-1}(1 - e^{tx_j}) = \\
&= 1 - (e^{tx_0} + \dots + e^{tx_{n-1}}) + \left(e^{t(x_0+x_1)} + \dots + e^{t(x_{n-2}+x_{n-1})}\right) - \dots = \\
&= \sum_{r=0}^{n}(-1)^r \sum_{0 \le j_1 < \dots < j_r < n} e^{t(x_{j_1} + \dots + x_{j_r})} = \\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{n} \frac{(-1)^r}{k!} t^k \sum_{0 \le j_1 < \dots < j_r < n} (x_{j_1} + \dots + x_{j_r})^k .
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
F &= \prod_{j=0}^{n-1}\left(-tx_j - \frac{t^2x_j^2}{2!} - \frac{t^3x_j^3}{3!} - \cdots\right) = \\
&= (-1)^n x_0 x_1 \cdots x_{n-1} t^n \prod_{j=0}^{n-1}\left(1 + \frac{tx_j}{2!} + \frac{t^2x_j^2}{3!} + \cdots\right) .
\end{aligned}$$

Now for $k < n$ we get

$$0 = k![t^k]F = \sum_{r=0}^{n}(-1)^r \sum_{0 \le j_1 < \dots < j_r < n} (x_{j_1} + \dots + x_{j_r})^k,$$

i.e.

$$\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \sum_{0 \le j_1 < \cdots < j_r < n} (x_{j_1} + \cdots + x_{j_r})^k = \sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \sum_{0 \le j_1 < \cdots < j_r < n} (x_{j_1} + \cdots + x_{j_r})^k \ .$$

It follows immediately that the sets

$$A = \{x_{j_1} + \cdots + x_{j_r} \mid 0 \le r \le n, r \in 2\mathbb{N},\ 0 \le j_1 < \cdots < j_r < n\}$$

and

$$B = \{x_{j_1} + \cdots + x_{j_r} \mid 0 < r \le n, r \in 2\mathbb{N}+1,\ 0 \le j_1 < \cdots < j_r < n\}$$

have n–property.

Example 13 *Starting with* $x_1 = 1, x_2 = -2, x_3 = 3, x_4 = -4, x_5 = 5$ *we arrive (after removing numbers belonging to the both sets) at the sets* $A = \{-6, -1, 0, 1, 7, 8\}$ *and* $B = \{-5, -4, 2, 3, 4, 9\}$ *that have* 5*–property. (Zero in set* A *is obtained as an empty sum.)*

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