Relating Brunn-Minkowski and Rogers-Shephard inequalities with the Minkowski asymmetry measure

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Abstract

In this paper we explain how the Minkowski asymmetry measure sharpens several classic results. Especially, we were able to tighten the Brunn-Minkowski and the Rogers-Shephard inequalities in terms of the Minkowski asymmetry measure using some stability results on those inequalities.

Keywords: Brunn-Minkowski inequality, Rogers-Shephard inequality, Minkowski asymmetry measure

MSC: 52A10, 52A38

1 Introduction

Let $\mathrm{vol}(K)$ be the n-dimensional volume (or Lebesgue measure) of K. We will write $\mathrm{vol_n}(K)$ whenever it is necessary to specify the dimension. Let $\mathcal{K}^n$ be the set of all convex compact sets in $\mathbb{R}^n$, $\mathbb{B}_2^n$ be the Euclidean ball of radius 1, $K+L$ be the Minkowski sum of convex bodies K and C, i.e. $K+L=\{a+b|a \in K,\, b \in L\}$.
The Brunn-Minkowski inequality establishes that for any convex compact sets K, L holds

$$\mathrm{vol}(K+L)^{\frac{1}{n}} \geq \mathrm{vol}(K)^{\frac{1}{n}} + \mathrm{vol}(L)^{\frac{1}{n}}.$$

Equality holds if and only if one of the three following cases are true (see [FiMaPr]):

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(i) in case for sets K and L with positive volume if and only if K is a homothet of L,

(ii) in case when one of them has volume 0, namely $\mathrm{vol}(K) = 0$ and $\mathrm{vol}(L) > 0$, then if and only if K is a singleton,

(iii) in case when both of them have volume 0, then if and only if K and L are contained in parallel hyperplanes.

The Brunn-Minkowski inequality was proved in the 19th century by Brunn for compact convex sets in low dimensions ($n \leq 3$) (see [Bru]), and Minkowski for general compact convex sets in $\mathbb{R}^n$ (see [Mi]). With the time the Brunn-Minkowski inequality became the starting point of the Brunn-Minkowski theory and a powerful tool for problems involving metric quantities such as volume, surface area and mean width.

The Rogers-Shephard inequality (see [RoSh, Thm. 1]), which can be regarded as a reverse inequality to the Brunn-Minkowski inequality, yields

$$\mathrm{vol}(K+L) \leq \binom{2n}{n} \frac{\mathrm{vol}(K)\mathrm{vol}(L)}{\mathrm{vol}(K \cap (-L))}.$$

Moreover, equality holds if and only if $K = -L$ is an n-dimensional simplex.

The Minkowski asymmetry $s(K)$ of a convex compact sets K is the smallest rescalation of K which contains a translation of $-K$. It is known that $1 \leq s(K) \leq n$ with $s(K) = 1$ if and only if K is centrally symmetric, while $s(K) = n$ if and only if K is an n-dimensional simplex (see [BrKo, Cor. 2.7]).

Computing the volume is always a computationally hard task. However, computing the Minkowski asymmetry measure may be a computationally easy task (see [BrKo]). We wonder whether we could improve the Brunn-Minkowski and the Rogers-Shephard inequalities for prescribed $s(K) = s \in [1, n]$.

In order to start finding values of $c(s)$ and $C(s)$, we need to recall the proofs of the Brunn-Minkowski and the Rogers-Shepard inequalities. Secondly, we will show also the characterizations of their equality cases. Finally, we will show the ideas in order to obtain some stability results, in sense of near-equalities, of the Brunn-Minkowski and the Rogers-Shepard inequalities, which will help in providing bounds for $c(s)$ and $C(s)$ from some particular values of $s \in [0, 1]$.

The results presented in this paper are based on the master thesis (see [Di]).

1.1 Minkowski asymmetry measure

For an n-dimensional convex and compact set K, its Minkowski asymmetry (measure) $s(K)$ (cf. Minkowski 1911) is the smallest $\rho > 0$ such that $-K \subset x + \rho K$ for some $x \in \mathbb{R}^n$, i.e.

$$s(K) := \inf\{\lambda > 0 | K \subset x + \lambda(-K) \quad \text{for some} \quad x \in \mathbb{R}^n\}.$$

Note that for Minkowski asymmetry $s(K)$ holds that $K \subset x + s(K)(-K)$ for some $x \in \mathbb{R}^n$. Figure 1 shows two examples of computing the Minkowski asymmetry measure.

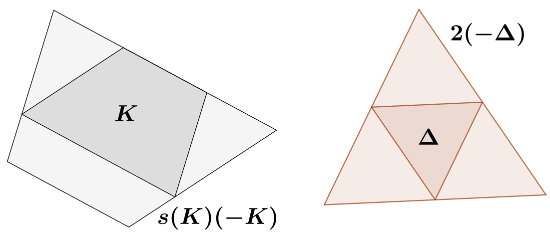

Figure 1: $K \subset x + s(K)(-K)$ for some $x \in \mathbb{R}^n$ and $\triangle \subset y + 2(-\triangle)$ for some $y \in \mathbb{R}^n$

1.2 The Brunn-Minkowski inequality

Figure 2 shows the Minkowski sum of two 0-centered triangles.

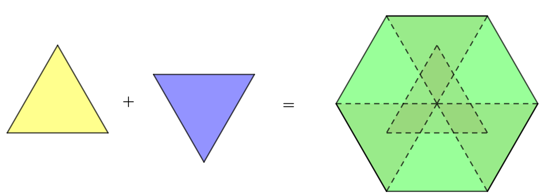

Figure 2: Minkowski sum of the regular triangles $\triangle$ and $\triangledown$

We are now ready to state the Brunn-Minkowski inequality for compact sets, which gives a lower bound for $\mathrm{vol}(K + L)$ in terms of $\mathrm{vol}(K)$ and $\mathrm{vol}(L)$.

Theorem 1 (Brunn 1887, Minkowski 1896)**.** *For any compact sets K, L holds*

$$\mathrm{vol}(K+L)^{\frac{1}{n}} \geq \mathrm{vol}(K)^{\frac{1}{n}} + \mathrm{vol}(L)^{\frac{1}{n}}. \tag{1}$$

The Brunn-Minkowski inequality (1) implies a well-known fact from Analysis, namely the Isoperimetric inequality.

Theorem 2 (Isoperimetric inequality)**.** *Let K be a convex set in $\mathbb{R}^n$ and $P(K)$ its surface area. Then*

$$P(K) \geq n\mathrm{vol}(\mathbb{B}_2^n)^{\frac{1}{n}}\mathrm{vol}(K)^{\frac{n-1}{n}}.$$

The Isoperimetric inequality implies that among sets with a fixed surface area, Euclidean balls maximize the volume.

Next let us state the Rogers-Shephard inequality, which gives us an upper bound for $\mathrm{vol}(K+L)$ in terms of $\mathrm{vol}(K)$, $\mathrm{vol}(L)$ and $\mathrm{vol}(K\cap(-L))$. Note that it is in general impossible to find an upper bound for $\mathrm{vol}(K+L)$ just in terms of $\mathrm{vol}(K)$ and $\mathrm{vol}(L)$.

Theorem 3 (Rogers and Shephard, 1958)**.** *Let K, L be convex bodies. Then*

$$\mathrm{vol}(K+L)\mathrm{vol}(K\cap(-L)) \leq \binom{2n}{n}\mathrm{vol}(K)\mathrm{vol}(L). \tag{2}$$

There exist also another version of the Rogers-Shephard inequality, namely with projections and intersections.

Theorem 4 (Rogers and Shephard, 1958)**.** *Let K be a compact body, H an i-dim subspace of $\mathbb{R}^n$. Then*

$$\mathrm{vol}_i(P_H K)\max_{x\in\mathbb{R}^n}\mathrm{vol}_{n-i}(K\cap(x+H^{\perp})) \leq \binom{n}{i}\mathrm{vol}_n(K). \tag{3}$$

An application of the Rogers-Shepard inequality (3) can be the following. One could approximate the volume $\mathrm{vol}(K)$ of a compact set $K\in\mathbb{R}^n$ (for instance a brain tumor) by measuring the volume of the projection $\mathrm{vol}_i(P_H K)$ and the volume of the biggest section $\max_{x\in\mathbb{R}^n}\mathrm{vol}_{n-i}(K\cap(x+H^{\perp}))$.

In order to see that (3) implies (2), we introduce the set $M := \{(z_1,z_2)^T \in \mathbb{R}^{2n} | z_2\in K, z_1\in z_2+L\}$. Using the definition of the set M it can be easily seen that $P_H M = K+L$, $M\cap H^{\perp} = K\cap(-L)$ and $\mathrm{vol}_{2n}(M) = \mathrm{vol}_n(K)\mathrm{vol}_n(L)$. And therefore two versions of the Rogers-Shepard inequality (2) and (3) are equivalent.

Choosing $L = -K$ and combining the Rogers-Shephard and the Brunn-Minkowski inequalities leads to

$$2^n \leq \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \leq \binom{2n}{n}.$$

We aim to find constants $c(s), C(s)$ with $2^n \leq c(s) \leq C(s) \leq \binom{2n}{n}$, such that for every convex body K such that $s(K) = s \in [1, n]$ holds

$$c(s) \leq \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \leq C(s)\,.$$

In order to do it we study the equality cases of the Brunn-Minkowski inequality (1) and the Rogers-Shephard inequalities (2) and (3), their stability results and then involve the Minkowski asymmetry measure in that inequalities.

1.3 Equality case of the Brunn-Minkowski inequality

The equality case of the Brunn-Minkowski inequality (1) is the following.

Theorem 5 (Klain, 2011, Kneser, Süss, 1932)**.** *Let K and L be convex compact sets with positive volumes. Then*

$$\mathrm{vol}(K+L)^{\frac{1}{n}} = \mathrm{vol}(K)^{\frac{1}{n}} + \mathrm{vol}(L)^{\frac{1}{n}}$$

holds if and only if K and L are homothets.

Moreover, if exactly one of the sets is lower dimensional, namely $\mathrm{vol}(K) = 0$, and $\mathrm{vol}(L) > 0$, then equality holds if and only if K is a singleton; and if $\mathrm{vol}(K) = \mathrm{vol}(L)$, then equality holds if and only if K and L are contained in parallel hyperplanes.

1.4 Stability of the Brunn-Minkowski inequality

We consider the stability as a sharpening of the inequality at the near-quality case. Let $K \triangle L$ be the symmetric difference of K and L, i.e. $K \triangle L = (K \setminus L) \cup (L \setminus K)$. For the Brunn-Minkowski inequality (1) it holds that

Theorem 6 (Figalli, 2009)**.** *Let K and L be convex compact sets in $\mathbb{R}^n$. Then*

$$\mathrm{vol}(K+L)^{\frac{1}{n}} \geq \left(\mathrm{vol}(K)^{\frac{1}{n}} + \mathrm{vol}(L)^{\frac{1}{n}}\right)\left(1 + \frac{A(K,L)^2}{c(n)\sigma(K,L)^{\frac{1}{n}}}\right)$$

with $A(K,L) := \inf_{x\in\mathbb{R}^n}\left\{\frac{\mathrm{vol}(K\triangle(x+\lambda L))}{\mathrm{vol}(K)}\right\}$, $\lambda = \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(L)}\right)^{\frac{1}{n}}$, $\sigma(K,L) := max\left\{\frac{\mathrm{vol}(K)}{\mathrm{vol}(L)}, \frac{\mathrm{vol}(L)}{\mathrm{vol}(K)}\right\}$ *and* $c(n) = 14n^2 4^{n-1}$.

One can see that due to the stability result in case of near-equality of (1) we have that K and L are almost homothets. Therefore $\mathrm{vol}(K \triangle (x+\lambda L)) \approx 0$ and therefore $A(K,L) \approx 0$.

1.5 Equality case of the Rogers-Shephard inequality

The equality case of the Rogers-Shephard inequality (3) can be stated as following.

Theorem 7 (Rogers and Shepard, 1958)**.** *Let K be a convex compact sets in $\mathbb{R}^n$, H an i-dim subspace of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then*

$$\mathrm{vol}_i(P_H K)\max_{x\in\mathbb{R}^n}\mathrm{vol}_{n-i}(K\cap(x+H^\perp)) = \binom{n}{i}\mathrm{vol}_n(K)$$

if and only if

$$K\cap(x+H^\perp) =_h K\cap H^\perp \quad \forall\, x\in\mathbb{R}^n$$

and

$$\mathrm{vol}_{n-i}(K\cap(x+H^\perp)) = (1-||x||_K)^{n-i}\mathrm{vol}_{n-i}(K\cap H^\perp).$$

Next we present a more recent result that is the equality case of the Rogers-Shephard inequality (2).

Theorem 8 (Alonso-Gutierrez, Jimenez, Villa, 2013)**.** *Let K and L be convex compact sets in $\mathbb{R}^n$. Then*

$$\mathrm{vol}_n(K+L)\mathrm{vol}_n(K\cap(-L)) = \binom{2n}{n}\mathrm{vol}_n(K)\mathrm{vol}_n(L)$$

if and only if K and L are simplices such that $K=-L$.

1.6 Stability of the Rogers-Shephard inequality

Let $d_{BM}(K,L) := \min\{\rho \geq 1 | K \subset A(L) \subset x+\rho K \text{ for some } x\in\mathbb{R}^n\}$ with $A(L)$ being any affine transformation of L be the Banach-Mazur distance of K with respect to L and T be an n-simplex.

Theorem 9 (Boroczsky, 2005). *Let K be a convex compact sets in $\mathbb{R}^n$. If*

$$\mathrm{vol}(K-K) = (1-\varepsilon)\binom{2n}{n}\mathrm{vol}(K),$$

then

$$1 + \frac{1}{n}\varepsilon \le d_{BM}(K,T) \le 1 + n^{50n^2}\varepsilon.$$

1.7 Results

Let us recall that the Brunn-Minkowski and Rogers-Shephard inequalities for convex compact sets K and $-K$ state that

$$2^n \le \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \le \binom{2n}{n}.$$

Moreover, $\frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} = 2^n$ if and only if K is symmetric and $\frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} = \binom{2n}{n}$ if and only if K is an n-dimensional simplex.

But at the same time we have that

$$1 \le s(K) \le n.$$

Moreover, $s(K) = 1$ if and only if K is symmetric and $s(K) = n$ if and only if K is an n-dimensional simplex.

Combining those facts enable us to involve the asymmetry measure into these inequalities. We state some improvements on the Brunn-Minkowski and the Rogers-Shephard inequalities by means of the Minkowski asymmetry measure (see [Di]).

Theorem 10. *Let K be a convex compact set in $\mathbb{R}^n$ and $s = s(K)$. Then*

$$c(s) \ge 2^n \left(1 + \frac{1}{n\,4^{n-1}} \left(\frac{(s-1)^n \mathrm{vol}_{n-1}(\mathbb{B}_2^{n-1})}{2^{n-1} n^{2n} \mathrm{vol}_n(\mathbb{B}_2^n)}\right)^2\right)^n \tag{4}$$

and

$$C(s) \le (1+s)^n. \tag{5}$$

Moreover, if $n - \frac{1}{4n} < s < n$, then

$$c(s) \ge \binom{2n}{n}(1 - 4n^2(n-s)) \quad \textit{and} \quad C(s) \le \binom{2n}{n}\left(1 - \frac{n-s}{n^{1+50n^2}}\right). \tag{6}$$

It is worth mentioning that (4) and (5) (resp. (6)) are specially good when $s(K) \approx 1$ (resp. $s(K) \approx n$).

Theorem 11. *The diagram* $f(\mathcal{K}^n)$*, where* $f : \mathcal{K}^n \to [1,n] \times \left[2^n, \binom{2n}{n}\right]$ *is given by* $f(K) := \left(s(K), \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)}\right)$*, is simply connected, contains* $(1, 2^n)$ *and* $(n, \binom{2n}{n})$*.*

We now investigate $f(\mathcal{K}^2)$.

Remark 1. *Let* $T = \mathrm{conv}(\{(0,1)^T, (\pm\sqrt{3}/2, -1/2)^T\})$*, let* $s \in [1,2]$*, and let*

$$K_s := T \cap (-sT) \quad \textit{and} \quad C_s := \mathrm{conv}(T \cup (-sT)).$$

Then $s(K_s) = s(C_s) = s$*,* $\mathrm{vol}(K_s) = \frac{2s-(s-1)^2}{4}$*,* $\mathrm{vol}(K_s - K_s) = \frac{(s+1)^2}{2}$*,* $\mathrm{vol}(C_s) = \frac{3\sqrt{3}}{2}s$*, and* $\mathrm{vol}(C_s - C_s) = 3\sqrt{3}s(1+s)$*.*

We finally provide upper and lower bounds for the constants $c(n), C(n)$ in the planar case derived from the Theorem 10 and Remark 1.

Corollary 1. *Let* $K \in \mathcal{K}^2$ *and let* $s = s(K)$*. Then*

$$4\left(1 + \frac{(s-1)^4}{2^{11}\pi^2}\right)^2 \le c(s) \le 2\frac{(s+1)^2}{2s-(s-1)^2}$$

and

$$2(s+1) \le C(s) \le (1+s)^2.$$

Moreover, if $s > \frac{8}{7}$*, then*

$$6(16s-31) \le c(s) \quad \textit{and} \quad C(s) \le 6\left(1 - \frac{2-s}{2^{201}}\right).$$

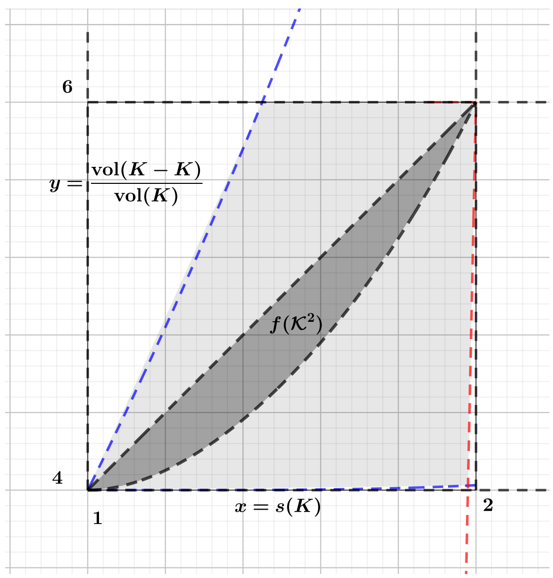

Figure 3: The diagram in Theorem 11 and the bounds obtained in Theorem 11 (dashed blue and red lines, respectively) and Corollary 1 (dashed black lines) in the case of $n = 2$; the light grey area is obtained in Theorems 1 and 3.

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