Relating Brunn-Minkowski and Rogers-Shephard inequalities with the Minkowski asymmetry measure

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Abstract
In this paper we explain how the Minkowski asymmetry measure sharpens several classic results. Especially, we were able to tighten the Brunn-Minkowski and the Rogers-Shephard inequalities in terms of the Minkowski asymmetry measure using some stability results on those inequalities.

Keywords: Brunn-Minkowski inequality, Rogers-Shephard inequality, Minkowski asymmetry measure

MSC: 52A10, 52A38

1 Introduction
Let \( \text{vol}(K) \) be the \( n \)-dimensional volume (or Lebesgue measure) of \( K \). We will write \( \text{vol}_n(K) \) whenever it is necessary to specify the dimension. Let \( \mathcal{K}^n \) be the set of all convex compact sets in \( \mathbb{R}^n \), \( \mathbb{B}_2^n \) be the Euclidean ball of radius 1, \( K + L \) be the Minkowski sum of convex bodies \( K \) and \( C \), i.e. \( K + L = \{ a + b | a \in K, b \in L \} \).

The Brunn-Minkowski inequality establishes that for any convex compact sets \( K, L \) holds
\[
\text{vol}(K + L)^{\frac{1}{n}} \geq \text{vol}(K)^{\frac{1}{n}} + \text{vol}(L)^{\frac{1}{n}}.
\]
Equality holds if and only if one of the three following cases are true (see [FiMaPr]):

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(i) in case for sets $K$ and $L$ with positive volume if and only if $K$ is a homothet of $L$,

(ii) in case when one of them has volume 0, namely $\text{vol}(K) = 0$ and $\text{vol}(L) > 0$, then if and only if $K$ is a singleton,

(iii) in case when both of them have volume 0, then if and only if $K$ and $L$ are contained in parallel hyperplanes.

The Brunn-Minkowski inequality was proved in the 19th century by Brunn for compact convex sets in low dimensions ($n \leq 3$) (see [Bru]), and Minkowski for general compact convex sets in $\mathbb{R}^n$ (see [Mi]). With the time the Brunn-Minkowski inequality became the starting point of the Brunn-Minkowski theory and a powerful tool for problems involving metric quantities such as volume, surface area and mean width.

The Rogers-Shephard inequality (see [RoSh, Thm. 1]), which can be regarded as a reverse inequality to the Brunn-Minkowski inequality, yields

$$\text{vol}(K + L) \leq \binom{2n}{n} \frac{\text{vol}(K)\text{vol}(L)}{\text{vol}(K \cap (-L))}.$$  

Moreover, equality holds if and only if $K = -L$ is an $n$-dimensional simplex.

The Minkowski asymmetry $s(K)$ of a convex compact sets $K$ is the smallest rescalation of $K$ which contains a translation of $-K$. It is known that $1 \leq s(K) \leq n$ with $s(K) = 1$ if and only if $K$ is centrally symmetric, while $s(K) = n$ if and only if $K$ is an $n$-dimensional simplex (see [BrKo, Cor. 2.7]).

Computing the volume is always a computationally hard task. However, computing the Minkowski asymmetry measure may be a computationally easy task (see [BrKo]). We wonder whether we could improve the Brunn-Minkowski and the Rogers-Shephard inequalities for prescribed $s(K) = s \in [1, n]$.

In order to start finding values of $c(s)$ and $C(s)$, we need to recall the proofs of the Brunn-Minkowski and the Rogers-Shephard inequalities. Secondly, we will show also the characterizations of their equality cases. Finally, we will show the ideas in order to obtain some stability results, in sense of near-equalities, of the Brunn-Minkowski and the Rogers-Shephard inequalities, which will help in providing bounds for $c(s)$ and $C(s)$ from some particular values of $s \in [0, 1]$.

The results presented in this paper are based on the master thesis (see [Di]).
1.1 Minkowski asymmetry measure

For an \( n \)-dimensional convex and compact set \( K \), its Minkowski asymmetry (measure) \( s(K) \) (cf. Minkowski 1911) is the smallest \( \rho > 0 \) such that \( -K \subset x + \rho K \) for some \( x \in \mathbb{R}^n \), i.e.

\[
s(K) := \inf \{ \lambda > 0 | K \subset x + \lambda(-K) \text{ for some } x \in \mathbb{R}^n \}.
\]

Note that for Minkowski asymmetry \( s(K) \) holds that \( K \subset x + s(K)(-K) \) for some \( x \in \mathbb{R}^n \). Figure 1 shows two examples of computing the Minkowski asymmetry measure.

\[\begin{array}{c}
\includegraphics{figure1.png}
\end{array}\]

Figure 1: \( K \subset x + s(K)(-K) \) for some \( x \in \mathbb{R}^n \) and \( \Delta \subset y + 2(-\Delta) \) for some \( y \in \mathbb{R}^n \)

1.2 The Brunn-Minkowski inequality

Figure 2 shows the Minkowski sum of two 0-centered triangles.

\[\begin{array}{c}
\includegraphics{figure2.png}
\end{array}\]

Figure 2: Minkowski sum of the regular triangles \( \Delta \) and \( \nabla \)

We are now ready to state the Brunn-Minkowski inequality for compact sets, which gives a lower bound for \( \text{vol}(K + L) \) in terms of \( \text{vol}(K) \) and \( \text{vol}(L) \).
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Theorem 1 (Brunn 1887, Minkowski 1896). For any compact sets $K, L$ holds

$$\frac{\text{vol}(K + L)^{\frac{1}{n}}}{\text{vol}(K)^{\frac{1}{n}}} \geq \frac{\text{vol}(L)^{\frac{1}{n}}}{\text{vol}(L)^{\frac{1}{n}}}.$$  

(1)

The Brunn-Minkowski inequality (1) implies a well-known fact from Analysis, namely the Isoperimetric inequality.

Theorem 2 (Isoperimetric inequality). Let $K$ be a convex set in $\mathbb{R}^n$ and $P(K)$ its surface area. Then

$$P(K) \geq n\text{vol}(\mathbb{B}_2^n)^{\frac{1}{n}}\text{vol}(K)^{\frac{n-1}{n}}.$$  

The Isoperimetric inequality implies that among sets with a fixed surface area, Euclidean balls maximize the volume.

Next let us state the Rogers-Shephard inequality, which gives us an upper bound for $\text{vol}(K + L)$ in terms of $\text{vol}(K), \text{vol}(L)$ and $\text{vol}(K \setminus (-L))$. Note that it is in general impossible to find an upper bound for $\text{vol}(K + L)$ just in terms of $\text{vol}(K)$ and $\text{vol}(L)$.

Theorem 3 (Rogers and Shephard, 1958). Let $K, L$ be convex bodies. Then

$$\text{vol}(K + L)\text{vol}(K \setminus (-L)) \leq \binom{2n}{n}\text{vol}(K)\text{vol}(L).$$  

(2)

There exist also another version of the Rogers-Shephard inequality, namely with projections and intersections.

Theorem 4 (Rogers and Shephard, 1958). Let $K$ be a compact body, $H$ an $i$-dim subspace of $\mathbb{R}^n$. Then

$$\text{vol}_i(P_H K) \max_{x \in \mathbb{R}^n} \text{vol}_{n-i}(K \cap (x + H^\perp)) \leq \binom{n}{i}\text{vol}_n(K).$$  

(3)

An application of the Rogers-Shephard inequality (3) can be the following. One could approximate the volume $\text{vol}(K)$ of a compact set $K \in \mathbb{R}^n$ (for instance a brain tumor) by measuring the volume of the projection $\text{vol}_i(P_H K)$ and the volume of the biggest section $\max_{x \in \mathbb{R}^n} \text{vol}_{n-i}(K \cap (x + H^\perp))$.

In order to see that (3) implies (2), we introduce the set $M := \{(z_1, z_2)^T \in \mathbb{R}^{2n} | z_2 \in K, z_1 \in z_2 + L\}$. Using the definition of the set $M$ it can be easily seen that $P_H M = K + L$, $M \cap H^\perp = K \cap (-L)$ and $\text{vol}_{2n}(M) = \text{vol}_{n}(K)\text{vol}_{n}(L)$. And therefore two versions of the Rogers-Shephard inequality (2) and (3) are equivalent.
Choosing $L = -K$ and combining the Rogers-Shephard and the Brunn-Minkowski inequalities leads to

$$2^n \leq \frac{\text{vol}(K - K)}{\text{vol}(K)} \leq \left(\frac{2n}{n}\right).$$

We aim to find constants $c(s), C(s)$ with $2^n \leq c(s) \leq C(s) \leq \left(\frac{2n}{n}\right)$, such that for every convex body $K$ such that $s(K) = s \in [1, n]$ holds

$$c(s) \leq \frac{\text{vol}(K - K)}{\text{vol}(K)} \leq C(s).$$

In order to do it we study the equality cases of the Brunn-Minkowski inequality (1) and the Rogers-Shephard inequalities (2) and (3), their stability results and then involve the Minkowski asymmetry measure in that inequalities.

### 1.3 Equality case of the Brunn-Minkowski inequality

The equality case of the Brunn-Minkowski inequality (1) is the following.

**Theorem 5** (Klain, 2011, Kneser, Süss, 1932). Let $K$ and $L$ be convex compact sets with positive volumes. Then

$$\text{vol}(K + L)^{\frac{1}{n}} = \text{vol}(K)^{\frac{1}{n}} + \text{vol}(L)^{\frac{1}{n}}$$

holds if and only if $K$ and $L$ are homothets.

Moreover, if exactly one of the sets is lower dimensional, namely $\text{vol}(K) = 0$, and $\text{vol}(L) > 0$, then equality holds if and only if $K$ is a singleton; and if $\text{vol}(K) = \text{vol}(L)$, then equality holds if and only if $K$ and $L$ are contained in parallel hyperplanes.

### 1.4 Stability of the Brunn-Minkowski inequality

We consider the stability as a sharpening of the inequality at the near-quality case. Let $K \triangle L$ be the symmetric difference of $K$ and $L$, i.e. $K \triangle L = (K \setminus L) \cup (L \setminus K)$. For the Brunn-Minkowski inequality (1) it holds that

**Theorem 6** (Figalli, 2009). Let $K$ and $L$ be convex compact sets in $\mathbb{R}^n$. Then

$$\text{vol}(K + L)^{\frac{1}{n}} \geq \left(\text{vol}(K)^{\frac{1}{n}} + \text{vol}(L)^{\frac{1}{n}}\right) \left(1 + \frac{A(K,L)^2}{c(n)\sigma(K,L)^{\frac{2}{n}}} \right)$$
with $A(K, L) := \inf_{x \in \mathbb{R}^n} \left\{ \frac{\text{vol}(K \triangle (x + \lambda L))}{\text{vol}(K)} \right\}$, \( \lambda = \left( \frac{\text{vol}(K)}{\text{vol}(L)} \right)^{\frac{1}{n}} \), \( \sigma(K, L) := \max \left\{ \frac{\text{vol}(K)}{\text{vol}(L)}, \frac{\text{vol}(L)}{\text{vol}(K)} \right\} \)
and \( c(n) = 14n^24^{n-1} \).

One can see that due to the stability result in case of near-equality of (1) we have that \( K \) and \( L \) are almost homothets. Therefore \( \text{vol}(K \triangle (x + \lambda L)) \approx 0 \) and therefore \( A(K, L) \approx 0 \).

### 1.5 Equality case of the Rogers-Shephard inequality

The equality case of the Rogers-Shephard inequality (3) can be stated as following.

**Theorem 7** (Rogers and Shepard, 1958). Let \( K \) be a convex compact sets in \( \mathbb{R}^n \), \( H \) an \( i \)-dim subspace of \( \mathbb{R}^n \) and \( x \in \mathbb{R}^n \). Then
\[
\text{vol}_i(P_{H}K) \max_{x \in \mathbb{R}^n} \text{vol}_{n-i}(K \cap (x + H^\perp)) = \binom{n}{i} \text{vol}_i(K)
\]
if and only if
\[
K \cap (x + H^\perp) =_{h} K \cap H^\perp \quad \forall x \in \mathbb{R}^n
\]
and
\[
\text{vol}_{n-i}(K \cap (x + H^\perp)) = (1 - ||x||_K)^{n-i} \text{vol}_{n-i}(K \cap H^\perp).
\]

Next we present a more recent result that is the equality case of the Rogers-Shephard inequality (2).

**Theorem 8** (Alonso-Gutierrez, Jimenez, Villa, 2013). Let \( K \) and \( L \) be convex compact sets in \( \mathbb{R}^n \). Then
\[
\text{vol}_n(K + L)\text{vol}_n(K \cap (-L)) = \binom{2n}{n} \text{vol}_n(K)\text{vol}_n(L)
\]
if and only if \( K \) and \( L \) are simplices such that \( K = -L \).

### 1.6 Stability of the Rogers-Shephard inequality

Let \( d_{BM}(K, L) := \min\{\rho \geq 1|K \subset A(L) \subset x + \rho K \text{ for some } x \in \mathbb{R}^n\} \) with \( A(L) \) being any affine transformation of \( L \) be the Banach-Mazur distance of \( K \) with respect to \( L \) and \( T \) be an \( n \)-simplex.
Theorem 9 (Boroczky, 2005). Let $K$ be a convex compact sets in $\mathbb{R}^n$. If

$$\text{vol}(K - K) = (1 - \varepsilon) \left(\frac{2n}{n}\right) \text{vol}(K),$$

then

$$1 + \frac{1}{n} \varepsilon \leq d_{BM}(K, T) \leq 1 + n^{50n^2} \varepsilon.$$

1.7 Results

Let us recall that the Brunn-Minkowski and Rogers-Shephard inequalities for convex compact sets $K$ and $-K$ state that

$$2^n \leq \frac{\text{vol}(K - K)}{\text{vol}(K)} \leq \left(\frac{2n}{n}\right).$$

Moreover, $\frac{\text{vol}(K - K)}{\text{vol}(K)} = 2^n$ if and only if $K$ is symmetric and $\frac{\text{vol}(K - K)}{\text{vol}(K)} = \left(\frac{2n}{n}\right)$ if and only if $K$ is an $n$-dimensional simplex.

But at the same time we have that

$$1 \leq s(K) \leq n.$$

Moreover, $s(K) = 1$ if and only if $K$ is symmetric and $s(K) = n$ if and only if $K$ is an $n$-dimensional simplex.

Combining those facts enable us to involve the asymmetry measure into these inequalities. We state some improvements on the Brunn-Minkowski and the Rogers-Shephard inequalities by means of the Minkowski asymmetry measure (see [Di]).

Theorem 10. Let $K$ be a convex compact set in $\mathbb{R}^n$ and $s = s(K)$. Then

$$c(s) \geq 2^n \left(1 + \frac{1}{n} \frac{(s - 1)^n \text{vol}_{n-1}(\mathbb{B}_2^{n-1})}{2^{n-1} n 2^n \text{vol}_{n}(\mathbb{B}_2^n)}\right)^n$$

and

$$C(s) \leq (1 + s)^n.$$  \hspace{1cm} (4) \hspace{1cm} (5)

Moreover, if $n - \frac{1}{4n} < s < n$, then

$$c(s) \geq \left(\frac{2n}{n}\right) (1 - 4n^2(n - s)) \quad \text{and} \quad C(s) \leq \left(\frac{2n}{n}\right) \left(1 - \frac{n - s}{n^{1+50n^2}}\right).$$ \hspace{1cm} (6)
It is worth mentioning that (4) and (5) (resp. (6)) are specially good when $s(K) \approx 1$ (resp. $s(K) \approx n$).

**Theorem 11.** The diagram $f(K^n)$, where $f : K^n \to [1, n] \times \left[ 2^n, \binom{2n}{n} \right]$ is given by $f(K) := \left( s(K), \frac{\text{vol}(K-K)}{\text{vol}(K)} \right)$, is simply connected, contains $(1, 2^n)$ and $(n, \binom{2n}{n})$.

We now investigate $f(K^2)$.

**Remark 1.** Let $T = \text{conv}(\{(0, 1)^T, (\pm \sqrt{3}/2, -1/2)^T\})$, let $s \in [1, 2]$, and let $K_s := T \cap (-sT)$ and $C_s := \text{conv}(T \cup (-sT))$.

Then $s(K_s) = s(C_s) = s$, $\text{vol}(K_s) = \frac{2s-(s-1)^2}{4}$, $\text{vol}(K_s - K_s) = \frac{(s+1)^2}{2}$, $\text{vol}(C_s) = \frac{3\sqrt{3}}{2}s$, and $\text{vol}(C_s - C_s) = 3\sqrt{3}s(1 + s)$.

We finally provide upper and lower bounds for the constants $c(n), C(n)$ in the planar case derived from the Theorem 10 and Remark 1.

**Corollary 1.** Let $K \in K^2$ and let $s = s(K)$. Then

$$4 \left( 1 + \frac{(s-1)^4}{2^{11}\pi^2} \right)^2 \leq c(s) \leq 2 \frac{(s+1)^2}{2s - (s-1)^2}$$

and

$$2(s+1) \leq C(s) \leq (1 + s)^2.$$

Moreover, if $s > \frac{8}{7}$, then

$$6(16s - 31) \leq c(s) \quad \text{and} \quad C(s) \leq 6 \left( 1 - \frac{2 - s}{2201} \right).$$
Figure 3: The diagram in Theorem 11 and the bounds obtained in
Theorem 11 (dashed blue and red lines, respectively) and Corollary
1 (dashed black lines) in the case of $n = 2$; the light grey area is
obtained in Theorems 1 and 3.
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