Lindström – Gessel – Viennot theorem as a common point of linear algebra and combinatorics

Edin Liđan
University of Bihać
Bosnia and Herzegovina
edin.lidan@unbi.ba

Abstract

Lindström – Gessel – Viennot theorem connects linear algebra and combinatorics with graph theory. We will present proof of LGV theorem and its application on Cauchy – Binnet and generalized Cauchy – Binnet formula as well as calculation of binomial determinants and some other specific determinants.

Keywords: LGV theorem, calculating determinants, weighted directed acyclic graph, path, Cauchy – Binnet formula

MSC: 05A15, 05A05

1 Introduction

Linear algebra and combinatorics are one of the oldest mathematical disciplines which even today significantly influenced further development of other disciplines and computer science. Although if we think about modern mathematics as a collection of many overlapping disciplines whose subjects may look far distant from each other, mathematics was always strongly integrated science with unexpected, mysterious and beautiful links among diverse subjects. Here we present one such deep result which connects determinants and graphs.

On the website of KAIST Math Problem of the Week (Weekly Math Challenges in KAIST) in December 2016 the following problem was posted.

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Problem 1 (Koon and Yun Bum)**.** *Let $S_n = (a_{ij})_{ij}$ be an $n \times n$ matrix such that*

$$a_{ij} = \binom{2(i+j-1)}{i+j-1}.$$

Find $\det S_n$*.*

The solution of this problem was given by Koon and Yun Bum in 2017. Using the properties of linear algebra and binomial coefficients they reduced matrix an upper triangular matrix which determinant is $S_n = 2^n$. We sketch his solution:
Proof: Let L_n be the lower triangular matrix with entries given by

$$L_n := \begin{cases} \binom{2i-1}{i+j-1} & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

and let $U_n := L_n^T$. Note that

$$(L_n U_n)_{ij} = \sum_{k=1}^{2i-1} \binom{2i-1}{k}\binom{2j-1}{k+j-i} \text{for } i \geq j.$$

Observe the following identity

$$\binom{2(i+j-1)}{i+j-1} = \sum_{k=0}^{2i-1} \binom{2i-1}{k}\binom{2j-1}{k+j-i}.$$

As

$$2(L_n U_n)_{ij} = \sum_{k=0}^{2i-1} \binom{2i-1}{k}\binom{2j-1}{k+j-i},$$

so $2(L_n U_n)_{ij} = a_{ij}$, where $S_n = (a_{ij})$. Hence, $S_n = (2L_n)U_n$,

$$\det S_n = (\det 2L_n) \det U_n = 2^n \det L_n \det U_n = 2^n.$$

□

In this paper we will view this matrix as the matrix of path systems of some graph. We will say something about its determinant based on the Lindström – Gessel – Viennot theorem.

Theorem 1.1 (LGV theorem)**.** *Let G be a directed acyclic graph, with a weight function $\omega : E \to \mathbb{R}$, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be two (not necessarily disjoint) sets of vertices. Let M be the path matrix from $\mathcal{A}$ to $\mathcal{B}$, and let VD be the set of all vertex disjoint path systems of $\mathcal{A}$ to $\mathcal{B}$. Then*

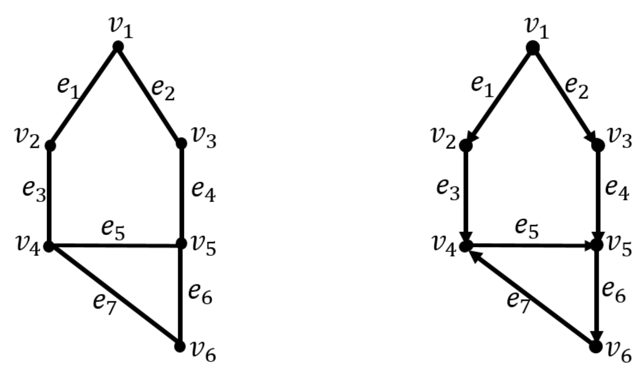
$$\det M = \sum_{\mathcal{P} \in VD} sign(\mathcal{P})\omega(\mathcal{P}).$$

In Section 2, we will define basic terms and properties related to directed weighted acyclic graphs. In Section 3, the proof of LGV theorem and its application in Cauchy-Binnet's and generalized Cauchy - Binnet's formula will be provided. Thus, we will show how to apply LGV theorem to a binomial determinant. In the last section the solution of Problem 1 will be provided using LGV theorem for $n = 2$.

2 Weighted graphs and directed paths in graphs

In this section we review basics of graphs and explain path systems in a graph in order to state Lindström – Gessel – Viennot theorem.

Definition 2.1. *Graph G is pair of sets $G = (V, E)$, where V is a set of vertices, and E is a set of edges, formed by pairs of vertices.*



(a) Graph G (b) Directed graph G

Figure 1: Example of graphs

For example in Figure 1(a) we have a graph $G = (V, E)$ with the finite set of vertices $V = \{v_1, \dots, v_6\}$, and finite set of edges $E = \{e_1, e_2, \dots, e_7\}$. If we give a directions to edges (Figure 1(b)) then we we call a graph directed.

Definition 2.2. *A walk is a sequence v_0, e_1, v_1, ..., v_k of graph vertices v_i and graph edges e_i such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. The length of a walk is its number of edges.*

Definition 2.3. *A path in G is a walk with with all distinct vertices in sequence.*

Definition 2.4. *A trail is a walk v_0, e_1, v_1, ..., v_k with no repeated edge. The length of a trail is its number of edges.*

A u, v trail is a trail with first vertex u and last vertex v, where u and v are known as the endpoints. A walk of trail is closed if the first vertex is equal to last vertex and is the only vertex that is repeated.

Definition 2.5. *A cycle of a graph G is a subset of the edge set of G that forms a path such that the first node of the path corresponds to the last.*

Definition 2.6. *A directed acyclic graph is a graph with directed edges containing no cycles.*

Throughout the paper we consider only simple graphs (no loops and no multiple edges). Let us suppose that for each edge e of graph G it is associated a real number $w(e)$ called its *weight*. Then graph G together with these weights is called a *weighted graph*. If their edges are directed, the graph is called *directed acyclic graph G*. For us, paths of directed weighted acyclic graph G will be the most interesting.

Definition 2.7. *A path system $\mathcal{P}$ is given by a permutation $\sigma \in S_n$ and n paths $P_1 : A_1 \to B_{\sigma(1)}$, $P_2 : A_2 \to B_{\sigma(2)}$, ..., $P_n : A_n \to B_{\sigma(n)}$. Weight of a path system $\mathcal{P}$ is given by*

$$\omega(\mathcal{P}) = \prod_{i=1}^{n} \omega(P_i)$$

where $w(P)$ is the weight of path P and sign($\mathcal{P}$) = sign(σ). Weight of a path P is defined by the product of the edges in the path

$$\omega(P) = \prod_{e \in P} \omega(e).$$

For a trivial path P (from a vertex v to itself), we define $\omega(P) = 1$.

For two vertices A and B of G we define weight from A to B, with

$$\omega(A, B) = \sum_{P:A\to B} \omega(P). \tag{1}$$

Example 2.1. *We now illustrate the weights of paths from v_1 to v_6 in the following weighted graph G:*

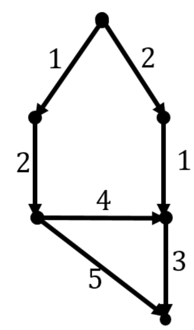


Figure 2: Directed weighted acyclic graph G

There are three such paths $P_1 : e_2e_4e_6$, path $P_2 : e_1e_3e_5e_6$ and path $P_3 : e_1e_3e_7$. Determine now the weight for all possible paths which we noticed and then we obtain that their weights are

$$\begin{aligned} \omega(P_1) &= \omega(e_2)\omega(e_4)\omega(e_6) = 2 \cdot 1 \cdot 3 = 6, \\ \omega(P_2) &= \omega(e_1)\omega(e_3)\omega(e_5)\omega(e_6) = 1 \cdot 2 \cdot 4 \cdot 3 = 24, \\ \omega(P_3) &= \omega(e_1)\omega(e_3)\omega(e_7) = 1 \cdot 2 \cdot 5 = 10. \end{aligned}$$

Now we deduce that

$$\omega(v_1, v_6) = \omega(P_1) + \omega(P_2) + \omega(P_3) = 6 + 24 + 10 = 40.$$

□

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\} \subset V$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_n\} \subset V$ be a two subsets of V having the same cardinality n.

Definition 2.8. *Vertex - disjoint path system $\mathcal{P} : \mathcal{A} \to \mathcal{B}$ is a collection of all paths where in every collection there are no two paths P_i, $P_j \in \mathcal{P}$ with a common vertex.*

Definition 2.9. *The matrix of paths* $M = [m_{i,j}]_{i,j=1}^{n}$ *from* $\mathcal{A}$ *to* $\mathcal{B}$ *is defined by*

$$m_{i,j} = \sum_{P:A_i \to B_j} \omega(P) = \omega(A_i, B_j).$$

3 Lindström – Gessel – Viennot theorem

Lindström - Gessel - Viennot Lemma (LGV theorem) or the nonintersecting paths theorem gives some characterization of determinant of the matrix of paths in terms of special path systems. The proof of this theorem was presented by Bernt Lindström (1973) in the context of matroid theory [4], but all beauty of this theorem was revealed by Ira Gessel and Gerard Viennot in their paper *Binomial Determinants, Paths, and Hook Length Formulae* ([2]). In this paper it is described how to apply the theorem to the combinatorics problems. However, we must mention that similar idea appeared earlier in the work of Karlin and McGregor (1959) in a probabilistic framework ("Slater determinant") in quantum mechanics ([3]). Now we will give the proof of LGV theorem.

Proof of LGV theorem: Determinant of $n \times n$ matrix is defined as

$$\det(M) = \sum_{\sigma \in S_n} \left(\operatorname{sign}(\sigma) \prod_{i=1}^{n} m_{i\sigma(i)} \right).$$

Consider $\sigma \in S_n$, where σ is a permutation of set $\{1, 2, \ldots, n\}$

$$\operatorname{sign}(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)} = \operatorname{sign}(\sigma) m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{n\sigma(n)},$$

where $m_{i\sigma(i)}$ is the sum of weights of collection path system from A_i to $B_{i\sigma(i)}$. Now apply the definition of weight from some vertex to some other vertex within the graph to get that

$$\begin{aligned} & \operatorname{sign}(\sigma) m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{n\sigma(n)} \\ = \; & \operatorname{sign}(\sigma) \left(\sum_{P_1:A_1 \to B_{\sigma(1)}} \omega(P_1) \right) \cdots \left(\sum_{P_n:A_n \to B_{\sigma(n)}} \omega(P_n) \right) \\ = \; & \sum_{\mathcal{P}:_{P_1:A_1 \to B_{\sigma(1)}, \ldots, P_n:A_n \to B_{\sigma(n)}}} \operatorname{sign}(\mathcal{P}) \omega\left(\mathcal{P}\right). \end{aligned}$$

If we make the sum over all σ, we get

$$\det M = \sum_{\mathcal{P}} (\mathrm{sign}(\mathcal{P}_\sigma))\, \omega(\mathcal{P}_\sigma),$$

where $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ is collection of all path systems which run from $\mathcal{A}$ to $\mathcal{B}$ and

$$\mathcal{P}_\sigma = \{\mathcal{P} : \mathcal{P} \text{ system of paths } \mathcal{A} \text{ to } \mathcal{B} \text{ given with } \sigma\}.$$

From the sum over all path systems $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{B}$ we obtain

$$\det M = \sum_{\mathcal{P}} \mathrm{sign}(\mathcal{P})\omega(\mathcal{P}). \tag{2}$$

Let ND be collection of all path systems which have at least two common vertices. Then we can show the right side of equality (2) as

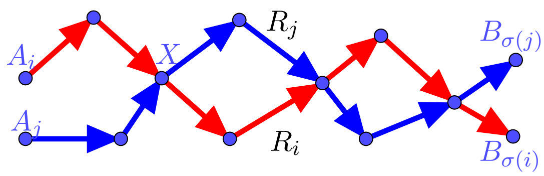
$$\sum_{\mathcal{P}} \mathrm{sign}(\mathcal{P})\omega(\mathcal{P}) = \sum_{\mathcal{P}\in VD} \mathrm{sign}(\mathcal{P})\omega(\mathcal{P}) + \sum_{\mathcal{P}\in ND} \mathrm{sign}(\mathcal{P})\omega(\mathcal{P}).$$

The goal is to show that we have

$$\sum_{\mathcal{P}\in ND} (\mathrm{sign}\mathcal{P})\omega(\mathcal{P}) = 0.$$

For a path system $\mathcal{R} = (R_1, R_2, \ldots, R_n) \in ND$, define

- i to be the smallest index such that R_i intersected with some R_j,
- X to be the first vertex at which R_i intersects some other path $\mathcal{R}$,
- j to be the smallest index of all the paths in $\mathcal{R}$ that intersects R_i u X (equivalently the smallest index of all paths such that $X \in P_i \cap P_j$, $(j > i)$,
- L_{iX} to be part of path R_i from A_i to X, and R_{iX} part of path R_i from X to $B_{\sigma(i)}$, so it is $\omega(R_i) = \omega(L_{iX}) \cdot \omega(R_{iX})$,
- L_{jX} to be part of path R_j from A_j to X, and R_{jX} part of the path R_j from X to $B_{\sigma(j)}$, so it is $\omega(R_j) = \omega(L_{jX}) \cdot \omega(R_{jX})$.

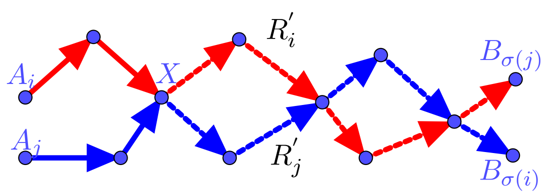


Now, we define an involution φ on ND by setting

$$\varphi : ND \to ND \quad \varphi(\mathcal{R}) = \mathcal{T} = (T_1, T_2, T_3, \ldots, T_n),$$

where $T_k = R_k$ when $k \neq i, j$, and T_i and T_j are defined as

- T_i is the path from A_i using the edges L_{iX} to X, after that we use the edges from R_{jX} to $B_{\sigma(j)}$, so that $\omega(T_i) = \omega(L_{iX}) \cdot \omega(R_{jX})$,
- T_j is the path from A_j using the edges L_{jX} to X, after that we use the edges from R_{iX} to $B_{\sigma(i)}$, so that $\omega(T_j) = \omega(L_{jX}) \cdot \omega(R_{iX})$.



$\mathcal{T} = (R_1^{'}, R_2^{'}, \ldots, R_n^{'})$ $\mathcal{T}$ have $\sigma^{'} \in S_n$ where is $\sigma^{'} = \sigma \circ (i, j)$

$$\begin{aligned} \text{sign } \sigma^{'} &= \text{sign } \sigma \ \text{sign}(i,j) \\ \text{sign } \sigma^{'} &= -\text{sign } \sigma \end{aligned}$$

From which we obtained

$$\begin{aligned} \text{sign } \mathcal{R} &= -\text{sign } \mathcal{T} \\ -\text{sign } \mathcal{R} &= \text{sign } \mathcal{T} \end{aligned}$$

Both path systems $\mathcal{T}$ and $\mathcal{R}$ are contained from the same set of the edges, so

$$\begin{aligned}\omega(\mathcal{T}) &= \prod_{i=1}^{n}\omega(T_i) = \omega(T_1)\omega(T_2)\cdots\omega(T_n)\\ &= \left(\prod_{k\in\{1,2,\ldots,n\}\setminus\{i,j\}}\omega(T_k)\right)\omega(T_i)\omega(T_j)\\ &= \left(\prod_{k\in\{1,2,\ldots,n\}\setminus\{i,j\}}\omega(R_k)\right)\omega(T_i)\omega(T_j).\end{aligned}$$

Based on above obtained paths T_i and T_j it follows that

$$\begin{aligned}\omega(T_i)\omega(T_j) &= (\omega(L_{iX})\cdot\omega(R_{jX}))(\omega(L_{jX})\cdot\omega(R_{iX}))\\ &= (\omega(L_{iX})\cdot\omega(R_{iX}))\cdot(\omega(L_{jX})\cdot\omega(R_{jX}))\\ &= \omega(R_i)\omega(R_j).\end{aligned}$$

Thus, $\omega(\mathcal{T}) = \omega(\mathcal{R})$. From definition φ is an involution so it follows that $\varphi = \varphi^{-1}$, i.e. φ is bijection. Thus we found $1-1$ correspondence for matching pairs of system paths $(\mathcal{R}, \varphi(\mathcal{R}))$ in ND where every pair of system paths is

$$\begin{aligned}\omega(\mathcal{R}) &= \omega(\varphi(\mathcal{R}))\\ \operatorname{sign}(\mathcal{R}) &= -\operatorname{sign}(\varphi(\mathcal{R})).\end{aligned}$$

It follows that

$$\sum_{\mathcal{P}\in ND}\operatorname{sign}(\mathcal{P})\omega(\mathcal{P}) = 0.$$

The Theorem is proved.

□

Apart from the connection with linear algebra, it offers a nice connection between graph theory and combinatorics, which will be illustrated on the theorem and examples.

Proposition 3.1. *For $m, n \in \mathbb{N}$*

$$\det M = \begin{bmatrix} \binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{n-1}\\ \binom{m+1}{0} & \binom{m+1}{1} & \cdots & \binom{m+1}{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{m+n-1}{0} & \binom{m+n-1}{1} & \cdots & \binom{m+n-1}{n-1}\end{bmatrix} = 1.$$

Proof: This Proposition we will prove using the LGV theorem. The idea consists of creating the directed weighted acyclic graph which weights of paths is equal to the appropriate binomial coeficient in determinant. On the other words

$$\begin{bmatrix} \binom{m}{0} & \binom{m}{1} & \dots & \binom{m}{n-1} \\ \binom{m+1}{0} & \binom{m+1}{1} & \dots & \binom{m+1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m+n-1}{0} & \binom{m+n-1}{1} & \dots & \binom{m+n-1}{n-1} \end{bmatrix} = \begin{bmatrix} \omega(A_1,B_1) & \omega(A_1,B_2) & \dots & \omega(A_1,B_n) \\ \omega(A_2,B_1) & \omega(A_2,B_2) & \dots & \omega(A_2,B_n) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(A_n,B_1) & \omega(A_n,B_2) & \dots & \omega(A_n,B_n) \end{bmatrix}.$$

That graph we can construct in the following way (Figure 3). The edges of matrix M are directed on the right (horizontally) and upwards (vertically).

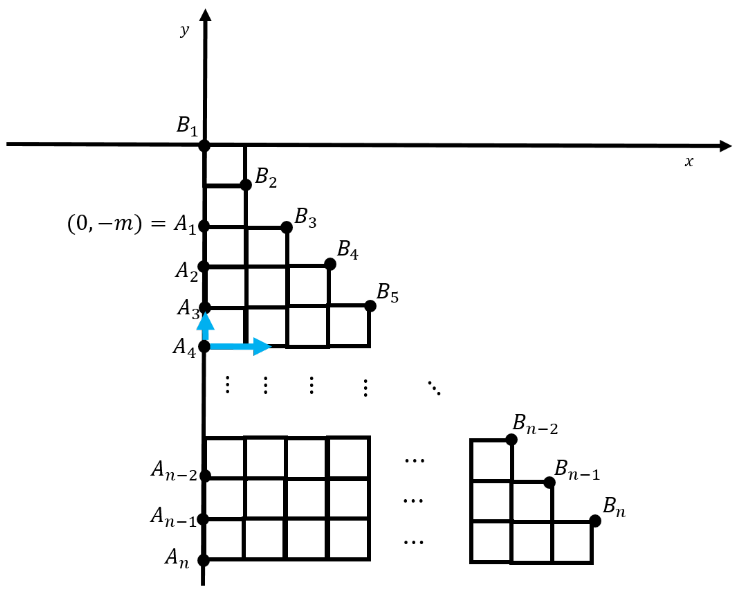


Figure 3: Directed weighted acyclic graph of determinant M

Consider now all disjoint path system in graph constructed graph. If we observe the path from A_1 to B_1, we notice that we have only one such path. In case we start from A_2 upwards, paths would be intersected. Thus, one option is to go right so that the path from A_2 to B_2 is determined by only one possible path. The analogy applies to all other paths in the graph, which means that only one disjoint paths system exists.

$$\det M = \#\text{number of disjoint path systems} = 1.$$

□

Theorem 3.1 (Chauchy - Binnet's formula)**.** *For every two $n \times n$ square matrices M_1 and M_2*

$$\det(M_1 M_2) = \det(M_1)\det(M_2).$$

Proof: Let us take the following sets of vertices

$$\begin{aligned} \mathcal{A} &= \{A_1, A_2, \ldots, A_n\}, \\ \mathcal{B} &= \{B_1, B_2, \ldots, B_n\}, \\ \mathcal{C} &= \{C_1, C_2, \ldots, C_n\}. \end{aligned}$$

Now we construct directed graph with vertices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, where the edges are directed from $\mathcal{A}$ to $\mathcal{B}$ and from $\mathcal{B}$ to $\mathcal{C}$ with weights $\omega(A_i, B_j) = m_1[i,j]$ and $\omega(B_j, C_k) = m_2[j,k]$. If $M = M_1 M_2$, then

$$m[i,j] = \sum_{k=1}^{k} m_1[i,j] m_2[j,k].$$

Consider random system of paths $\mathcal{P}$ in which verices are disjoint from $\mathcal{A}$ to $\mathcal{C}$. $\mathcal{P}$ must go through $\mathcal{B}$. Every system of paths from $\mathcal{A}$ to $\mathcal{C}$ is divided in two parts $\mathcal{Q}$ and $\mathcal{R}$, where $\mathcal{Q}$ is the system of disjoint paths from $\mathcal{A}$ to $\mathcal{B}$, and $\mathcal{R}$ is the system of disjoint paths from $\mathcal{B}$ to $\mathcal{C}$. $\mathcal{W}$ is the set of all disjoint paths systems from $\mathcal{A}$ to $\mathcal{B}$, and $\mathcal{Z}$ is set of all disjoint path systems from $\mathcal{B}$ to $\mathcal{C}$. Now consider

$$\begin{aligned} \det(M_1)\det(M_2) &= \sum_{\mathcal{Q}\in\mathcal{W}} \operatorname{sign}(\mathcal{Q})\omega(\mathcal{Q}) \sum_{\mathcal{R}\in\mathcal{Z}} \operatorname{sign}(\mathcal{R})\omega(\mathcal{R}) \\ &= \sum_{\mathcal{P}\in\mathcal{W}\times\mathcal{Z}} \operatorname{sign}(\mathcal{R})\operatorname{sign}(\mathcal{Q})\omega(\mathcal{R})\omega(\mathcal{Q}), \end{aligned}$$

where $\mathcal{W} \times \mathcal{Z}$ is set of ordered pairs $(\mathcal{Q}, \mathcal{R})$ suitable disjoint paths $P_i : A_i \to B_{\sigma(\mathcal{Q}(i))} \to C_{\sigma(\mathcal{R}(\sigma(\mathcal{Q}(i))))}$, and $\sigma_{\mathcal{Q}} \circ \sigma_{\mathcal{R}} = \sigma$. Then weight of random disjoint path system $\mathcal{P}$ is

$$\omega(\mathcal{P}) = \omega(\mathcal{Q})(\mathcal{R}), \tag{3}$$

and

$$\operatorname{sign}(\sigma) = \operatorname{sign}(\sigma_{\mathcal{Q}} \circ \sigma_{\mathcal{R}}) = \operatorname{sign}(\sigma_{\mathcal{Q}})\operatorname{sign}(\sigma_{\mathcal{R}}), \tag{4}$$

from which it follows

$$\operatorname{sign}\mathcal{P} = \operatorname{sign}(\mathcal{Q})\operatorname{sign}(\mathcal{R}) \tag{5}$$

If we now apply LGV theorem, we get

$$\begin{aligned} \det(M_1)\det(M_2) &= \sum_{\mathcal{P} \in \mathcal{W} \times \mathcal{Z}} \operatorname{sign}(\mathcal{R})\operatorname{sign}(\mathcal{Q})\omega(\mathcal{R})\omega(\mathcal{Q}) \\ &= \sum_{\mathcal{P}} \operatorname{sign}(\mathcal{P})\omega(\mathcal{P}) = \det(M_1 M_2). \end{aligned}$$

□

Theorem 3.2 (Generalized Cauchy - Binet's formula)**.** *Let M_1 be a $n \times r$ matrix and let M_2 be a $r \times n$ matrix where $n \leq r$. Then we have*

$$\det(M_1 M_2) = \sum_{X \subset \{1,2,\ldots,r\}, |X| = n} \det(M_1[X]) \det(M_2[X]),$$

where $M_1[X]$ is square submatrix formed by columns matrix limited to columns indexed as X and $M_2[X]$ is matrix limited on rows indexed as X.

Proof: Construct directed graph $G = \{\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}, \mathrm{E}\}$, where is

$$\begin{aligned} \mathcal{A} &= \{A_1, A_2, \ldots, A_n\}, \\ \mathcal{B} &= \{B_1, B_2, \ldots, B_n\}, \\ \mathcal{C} &= \{C_1, C_2, \ldots, C_n\}, \\ E &= \{(A_i, B_j) : i \in \{\overline{1,n}\}, j \in \{\overline{1,r}\}\} \cup \{(B_j, C_k) : j \in \{\overline{1,r}\}, k \in \{\overline{1,n}\}\}. \end{aligned}$$

Define weights of edges with

$$\begin{aligned} \omega(A_i, B_j) &= m_1[i,j], \\ \omega(B_i, C_j) &= m_2[j,k]. \end{aligned}$$

If $M = M_1M_2$, then

$$m[i,j] = \sum_{k=1}^{r} m_1[i,k]m_2[j,k].$$

Fix some arbitrary $X \subset \{1, 2, \ldots, r\}$. Let $\mathcal{P}_{\mathcal{A}\mathcal{X}}$ be set of all disjoint paths from $\mathcal{A}$ to $B[\mathcal{X}]$, where is $B[\mathcal{X}]$ subset of B limited with indexes of X, and $\mathcal{P}_{\mathcal{X}\mathcal{B}}$ the set of all disjoint path systems from $B[\mathcal{X}]$ to $\mathcal{C}$. Consider now

$$\begin{aligned}
\det(M_1) \cdot \det(M_2) &= \sum_{Q \in \mathcal{P}_{AX}} \operatorname{sign}(\mathcal{Q})\omega(\mathcal{Q}) \sum_{R \in \mathcal{P}_{XB}} \operatorname{sign}(\mathcal{R})\omega(\mathcal{R}) \\
&= \sum_{\mathcal{P} \in \mathcal{P}_{AX} \times \mathcal{P}_{XB}} \operatorname{sign}(\mathcal{P})\omega(\mathcal{P}),
\end{aligned}$$

where $\mathcal{P}_{AX} \times \mathcal{P}_{XB}$ contains set of all disjoint path systems from $\mathcal{A}$ to $\mathcal{C}$ which go through all vertices $\mathcal{B}[X]$. Now, in principle, we get

$$\sum_{X \subset \{1,2,\ldots,r\}, |X|=n} \det(M_1[X]) \det(M_2[X]).$$

This sum also gives us a sum over all system paths from $\mathcal{A}$ to $\mathcal{C}$. Furthermore, we have

$$\begin{aligned}
&\sum_{X \subset \{1,\ldots,r\}, |X|=n} \det(M_1[X]) \det(M_2[X]) \\
= &\sum_{X \subset \{1,\ldots,r\}, |X|=n} \left(\sum_{\mathcal{P} \in \mathcal{P}_{AX} \times \mathcal{P}_{XB}} \operatorname{sign}(\mathcal{P})\omega(\mathcal{P}) \right) \\
= &\sum_{\mathcal{P}} \operatorname{sign}(\mathcal{P})\omega(\mathcal{P}) = \det(M_1 \cdot M_2).
\end{aligned}$$

□

4 LGV and Problem 1

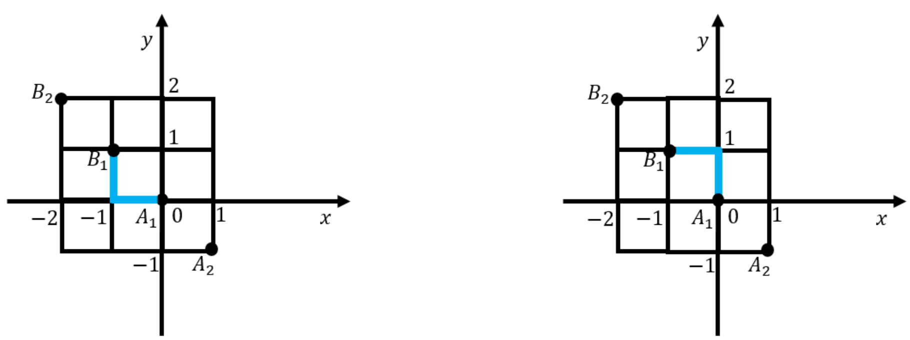
We calculate matrix using LGV theorem in the case $n = 2$.

$$\begin{bmatrix} \binom{2}{1} & \binom{4}{2} \\ \\ \binom{4}{2} & \binom{6}{3} \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ \\ 6 & 20 \end{bmatrix}$$

The Idea is same as like in the Proposition 3.1. We want to create the directed weighted acyclic graph in which weights of paths are equal to the value in our matrix, i.e.

$$\begin{bmatrix} \omega(A_1, B_1) & \omega(A_1, B_2) \\ \\ \omega(A_2, B_1) & \omega(A_2, B_2) \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ \\ 6 & 20 \end{bmatrix}$$

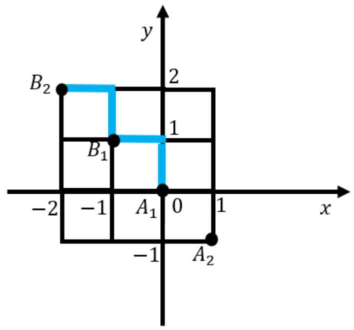
So consider our case of matrix S_2 and create a directed weighted graph of matrix S_2. First, we will consider the possible paths from A_1 to B_2 (Figure 4).



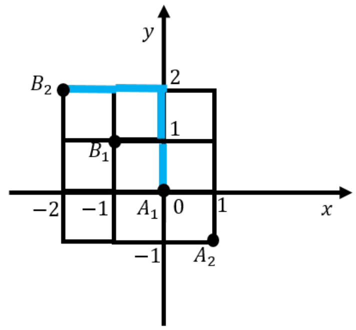
(a) Path P_1 (b) Path P_2

Figure 4: Paths from A_1 to B_1

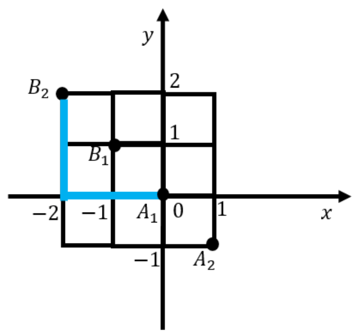
Check now the weight of path from A_1 to B_1. Path P_1 and P_2 in our graph have a weight one, using (2) we get that the weight of path form A_1 to B_1 is 2. Now, consider all possible paths from A_1 to B_2 (Figure 5).



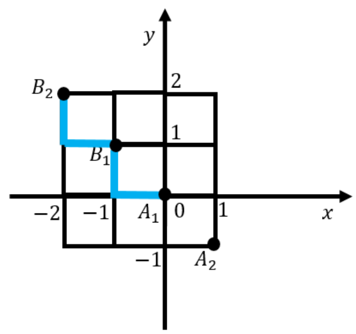
(a) Path P_3



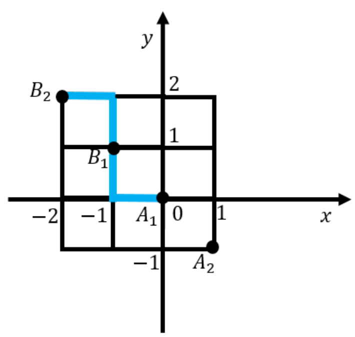
(b) Path P_4



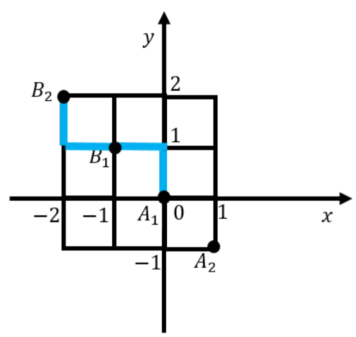
(c) Path P_5



(d) Path P_6



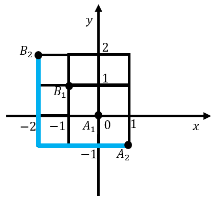
(e) Path P_4



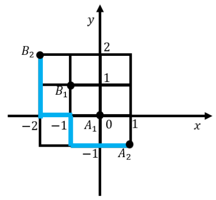
(f) Path P_8

Figure 5: Paths from A_1 to B_2

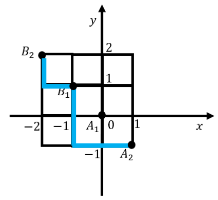
Every paths P_i, $i = 1, \ldots, 6$ have a weight one. Using the (2) we obtain that the weight of path from A_1 to B_2 is equal to 6. Analogously we will obtained the weight of paths from A_2 to B_1. Now, we will consider the possible paths from A_2 to B_2 (Figure 6).



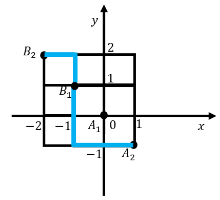
(a) Path P_{15}



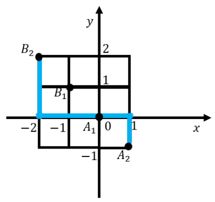
(b) Path P_{16}



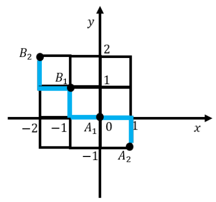
(c) Path P_{17}



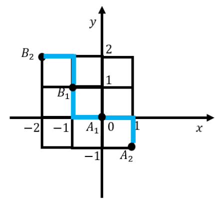
(d) Path P_{18}



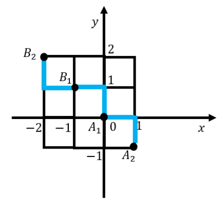
(e) Path P_{19}



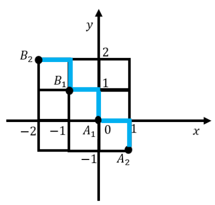
(f) Path P_{20}



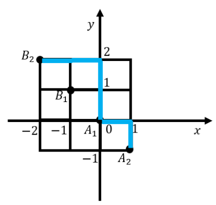
(g) Path P_{21}



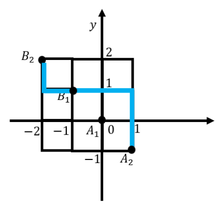
(h) Path P_{22}



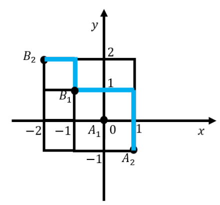
(i) Path P_{23}



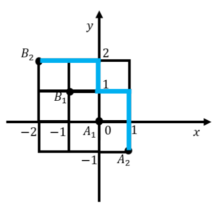
(j) Path P_{24}



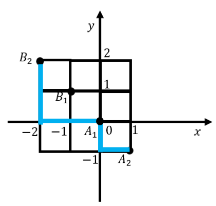
(k) Path P_{25}



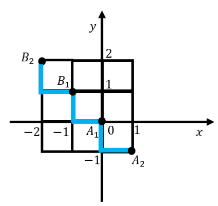
(l) Path P_{26}



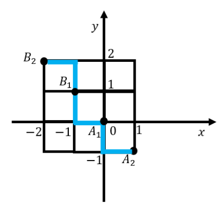
(m) Path P_{27}



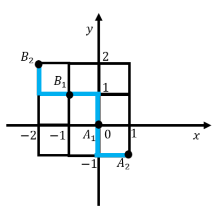
(n) Path P_{28}



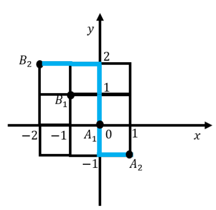
(o) Path P_{29}



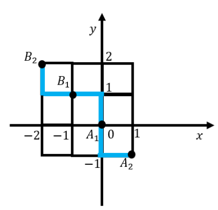
(p) Path P_{30}



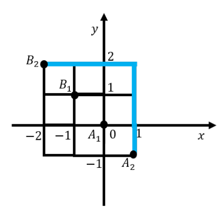
(q) Path P_{31}



(r) Path P_{32}



(s) Path P_{33}



(t) Path P_{34}

Figure 6: Paths from A_2 to B_2

Using (2) we obtained $\omega(A_2, B_2) = 20$. If we now consider disjoint path systems in the graph and their signs we conclude that $\det S_2 = 4$ (Figure 7).

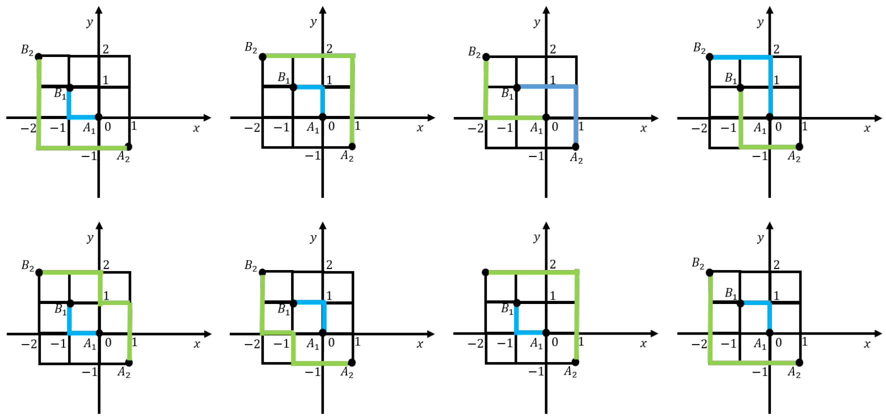


Figure 7: Disjoint system of paths

In the case of S_n, we use the same idea and create the graph of matrix S_n in the following way (Figure 8). It's known that in rectangular dimensions of $m \times n$ we have a $\frac{(m+n)!}{m!n!}$ different nonintersecting paths.

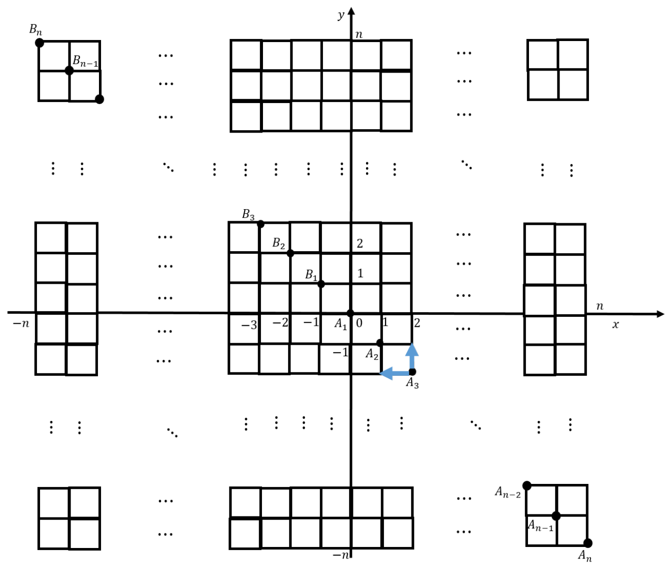


Figure 8: Directed weighted acyclic graph of matrix s_n

From Koon and Yun Bum's solution we know that the difference between the number of non-intersecting paths with positive sign and the number of non-intersecting paths with negative sign is 2^n. However, in general the number of all non-intersecting path systems is large, even for $n = 3$ so deducing the result directly from the LGV theorem requires this result, which seems non-trivial.

Problem 2. *Find a combinatorial argument that the difference between the number of non-intersecting paths with positive sign and the number of non-intersecting paths with negative sign from* $\{A_1, \ldots, A_n\}$ *and* $\{B_1, \ldots, B_n\}$ *in the graph on the Figure 8 is* 2^n*.*

5 Conclusion

We illustrated some possibilities of applying LGV theorem. This theorem can be implemented many problems in combinatorics and some other mathematicians area, for example we can LGV theorem apply on: Dyck paths, Motzkin numbers, Hankel determinants, Catalan numbers, rhombus tilings and many others problems.

Acknowledgment

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[5] https://mathsci.kaist.ac.kr/pow/2016/12/