Tomislav Došlić, Snježana Majstorović (Eds.)

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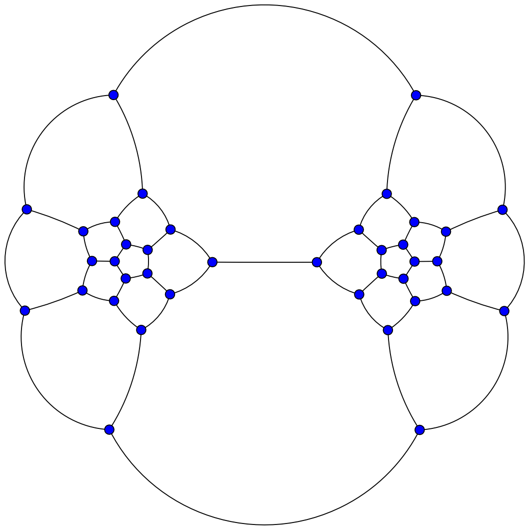

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This volume is dedicated to

Professor Dragutin Svrtan

on the occasion of his 70^{th} birthday

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Preface

Writing a preface is always a pleasure to me, this one even more than usual. Several are the reasons. The first one is that the CroCoDays meeting became traditional – three times certainly makes a tradition. The next one is that we managed to organize our conference in very difficult circumstances of a global pandemics. Many events and gatherings with traditions far richer than our were canceled or delayed, but not the **CroCoDays** conference. The next reason is that we succeeded in combining real life attendance with remote participation and that everything went smoothly. Next, several good quality contributions were submitted for our Proceedings, their little imperfections promptly expunged by our referees. And last, but not the least, reason for satisfaction is that the final result of our collective work is dedicated to our friend, colleague and teacher, **Professor Dragutin Svrtan**, on occasion of his seventieth birthday. It is a small sign of appreciation of his contribution to mathematics and to our community. Thank you, Dragec, and may you long keep helping us to learn and to enjoy mathematics!

I thank all participants, all members of our organizing team and all authors and referees for their efforts. Particular thanks go to our technical support who made the remote participation possible.

It is also a pleasure to thank our sponsors, the Faculty of Civil Engineering, the Croatian Academy of Sciences and Arts and the Croatian Science Foundation (IP-2016-06-1142), whose generous support made possible both the conference and these Proceedings.

I hope to meet you all, and many others, on the fourth Croatian Combinatorial Days in September 2022.

Zagreb, June 8, 2021 Tomislav Došlić

On the dimension of arcs in mixed labyrinth fractals

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Abstract

Mixed labyrinth fractals are dendrites in the unit square, which were recently studied with respect to the lengths of arcs in the fractals. In this article we first give a construction method for mixed labyrinth fractals with the property that all arcs in the fractal have box counting dimension 2. Subsequently, we show how a certain familiy of patterns can provide a mixed labyrinth fractal of any box counting dimension between 1 and 2, which also coincides with the box-counting dimension of the arc between any two distinct points of the fractal. Finally, we show how the results can be extended to a more general setting.

1 Introduction

Labyrinth fractals are dendrites in the unit square which were introduced in the last decade [2, 3]. They can also be viewed as special families of Sierpiński carpets. Labyrinth fractals were studied with respect to their topological and geometrical properties, with special focus on the arcs in these dendrites. The construction of labyrinth sets and labyrinth fractals is done by using labyrinth patterns: in the self-similar case one labyrinth pattern generates, inductively, a sequence of labyrinth sets, whose limit is a labyrinth fractal, and in the case of mixed labyrinth sets and mixed labyrinth fractals [4] more than one pattern (in general, even infinitely many) are used for their construction.

While for the self-similar labyrinth fractals [2, 3] there are results on the lengths and the dimension of the arcs in these dendrites, the papers on the mixed case [4, 5] deal mainly with the length of arcs, and how this depends on the properties of the patterns that generate a mixed labyrinth fractal. The main goal of the present article is to construct families of mixed labyrinth fractals with the property that the arcs between distinct points of the fractal have given box counting dimension, equal to the dimension of the whole fractal.

There exists quite a lot of interest in fractal dendrites lately. Self-similar dendrites were constructed with the help of polygonal systems and studied recently [16], but the

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emphasis is not on length of arcs, and, moreover, the methods are different, using tools like contractive iterated function systems and zippers, a different approach from ours, and restricted only to self-similar fractals. We also mention existing results on dimensions of homogenoeous Moran sets in a closed real interval of finite length [8], since the fractals studied in the present paper can be seen as a family of planar relatives of these.

In the last years, labyrinth fractals and results on them have already shown to have applications in physics. We mention, e.g., the fractal reconstruction of complicated images, signals and radar backgrounds [14], or the construction of prototypes of ultra-wide band radar antennas [15]. Very recently, fractal labyrinths combined with genetic algorithms were used in order to create big robust antenna arrays and nano-antennas in telecommunication [13]. Concerning further possible applications of mixed labyrinth fractals, we want to mention very recent results in materials engineering [11], which show that dendrite growth, a largely unsolved problem, plays an essential role when approaching high power and energy lithium-ion batteries. Furthermore, there is recent research on crystal growth [9] which suggests that, depending on the geometric structure of the studied material, labyrinth fractals can offer a suitable model for studying various phenomena and objects occurring in other fields of science. Finally, we mention that Koch curves, which are of interest in theoretical physics in the context of diffusion processes (e.g., [17]), are related to arcs in labyrinth fractals.

Let us now give a short outline of the article. Section 2 contains the needed preliminaries: we give the definition of labyrinth patterns, labyrinth sets and labyrinth fractals, and cite some of the main results regarding topological and geometric properties of labyrinth fractals proven in earlier research [2, 3, 4, 6], which are essential to our considerations throughout this paper. In Section 3 we introduce a special family of horizontally and vertically blocked labyrinth patterns called snake cross patterns. We analyse their shape, as well as the the objects generated by certain sequences of snake cross patterns. These are mixed labyrinth sets and, in the limit, mixed labyrinth fractals, dendrites with the property that the length of any non-trivial arc is infinite. Section 4 is dedicated to proving that the fractal dendrite obtained as the limit of the construction in the previous section has the property that the arc between any two distinct points in the fractal has box counting dimension two. Subsequently, we give in Section 5 a construction based on the use of snake cross patterns that provides mixed labyrinth fractals of any box-counting dimension $\delta \in [1, 2)$. In a short last section we take a glimpse at how these results could be used to infer analogons thereof in a more general setting. We conclude with a conjecture.

2 Preliminaries on self-similar and mixed labyrinth fractals

Labyrinth patterns are very convenient tools for the construction of labyrinth fractals. The notations and notations have already been introduced and used in work on self-similar labyrinth fractals [2, 3] and later on for the mixed ones [4]. Let $x, y, q \in [0,1]$ such that $Q = [x, x+q] \times [y, y+q] \subseteq [0,1] \times [0,1]$. For any point $(z_x, z_y) \in [0,1] \times [0,1]$ we define the map $P_Q(z_x, z_y) = (qz_x + x, qz_y + y)$. For any integer $m \geq 1$ let $S_{i,j,m} = \{(x,y) \mid \frac{i}{m} \leq x \leq \frac{i+1}{m}$ and $\frac{j}{m} \leq y \leq \frac{y+1}{m}\}$ and $\mathcal{S}_m = \{S_{i,j,m} \mid 0 \leq i \leq m-1$ and $0 \leq j \leq m-1\}$.

Any nonempty $\mathcal{A} \subseteq \mathcal{S}_m$ is called an *m-pattern* and m its *width*. Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of non-empty patterns and $\{m_k\}_{k=1}^{\infty}$ be the corresponding *width-sequence*, i.e., for all $k \geq 1$ we have $\mathcal{A}_k \subseteq \mathcal{S}_{m_k}$. We denote $m(n) = \prod_{k=1}^{n} m_k$, for all $n \geq 1$. Let $\mathcal{W}_1 = \mathcal{A}_1$. Ee call $\mathcal{W}_1$ the *set of white squares of level* 1, and the elements of $\mathcal{S}_{m_1} \setminus \mathcal{W}_1$ the *black squares of level* 1. For $n \geq 2$ the *set of white squares of level* n is defined as

$$\mathcal{W}_n = \bigcup_{W \in \mathcal{A}_n, W_{n-1} \in \mathcal{W}_{n-1}} \{P_{W_{n-1}}(W)\}. \tag{1}$$

Note that $\mathcal{W}_n \subset \mathcal{S}_{m(n)}$. We call the elements of $\mathcal{S}_{m(n)} \setminus \mathcal{W}_n$ the *black squares of level n*. For $n \geq 1$, we define $L_n = \bigcup_{W \in \mathcal{W}_n} W$. Thus, $\{L_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence of compact sets, and $L_\infty = \bigcap_{n=1}^{\infty} L_n$, is the *limit set defined by the sequence of patterns* $\{\mathcal{A}_k\}_{k=1}^{\infty}$.

Examples of patterns are shown in figures 1 and 2. These also illustrate the first two steps of the construction of a mixed labyrinth set.

We define, for $\mathcal{A} \subseteq \mathcal{S}_m$, *the graph of the pattern* $\mathcal{A}$, $\mathcal{G}(\mathcal{A}) \equiv (\mathcal{V}(\mathcal{G}(\mathcal{A})), \mathcal{E}(\mathcal{G}(\mathcal{A})))$: Its set of vertices $\mathcal{V}(\mathcal{G}(\mathcal{A}))$ consists of the white squares in $\mathcal{A}$, i.e., $\mathcal{V}(\mathcal{G}(\mathcal{A})) = \mathcal{A}$ and its edges $\mathcal{E}(\mathcal{G}(\mathcal{A}))$ are the unordered pairs of white squares, that have a common side. The *top row* in $\mathcal{A}$ is the set of all white squares in $\{S_{i,m-1,m} \mid 0 \leq i \leq m^n - 1\}$. The bottom row, left column, and right column in $\mathcal{A}$ are defined analogously. A *top exit* in $\mathcal{A}$ is a white square in the top row, such that there is a white square in the same column in the bottom row. A *bottom exit* in $\mathcal{A}$ is defined analogously. A *left exit* in $\mathcal{A}$ is a white square in the left column, such that there is a white square in the same row in the right column. A *right exit* in $\mathcal{A}$ is defined analogously. A top exit together with the corresponding bottom exit build a *vertical exit pair*, and a left exit together with the corresponding right exit build a *horizontal exit pair*. The above notions can also be defined in the special case $\mathcal{A} = \mathcal{W}_n$.

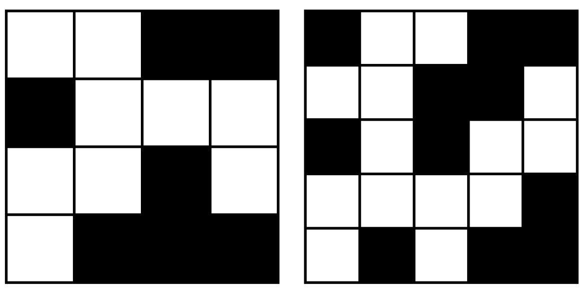

Figure 1: Two labyrinth patterns, $\mathcal{A}_1$ (a 4-pattern) and $\mathcal{A}_2$ (a 5-pattern)

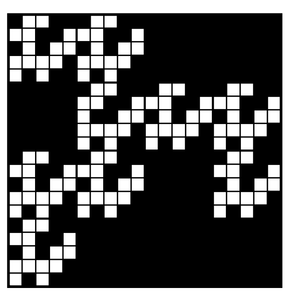

Figure 2: The set $\mathcal{W}_2$, constructed based on the above patterns $\mathcal{A}_1$ and $\mathcal{A}_2$, that can also be viewed as a 20-pattern

A non-empty m-pattern $\mathcal{A} \subseteq \mathcal{S}_m$, $m \geq 3$ is called a *$m \times m$-labyrinth pattern* (in short, *labyrinth pattern*) if $\mathcal{A}$ has the tree property, the exits property and the corner property.

Property 1 (The tree property)**.** *$\mathcal{G}(\mathcal{A})$ is a tree.*

Property 2 (The exits property)**.** *$\mathcal{A}$ has exactly one vertical exit pair, and exactly one horizontal exit pair.*

Property 3 (The corner property)**.** *If there is a white square in $\mathcal{A}$ at a corner of $\mathcal{A}$, then there is no white square in $\mathcal{A}$ at the diagonally opposite corner of $\mathcal{A}$.*

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of non-empty patterns, with $m_k \geq 3$, $n \geq 1$ and $\mathcal{W}_n$ the corresponding set of white squares of level n. We call $\mathcal{W}_n$ an $m(n) \times m(n)$-*mixed labyrinth set* (in short, *labyrinth set*) of level n, if $\mathcal{A} = \mathcal{W}_n$ has the tree property, the exits property and the corner property. It was shown [4, Lemma 1] that if all patterns in the sequence $\{\mathcal{A}_k\}_{k=1}^{\infty}$ are labyrinth patterns, then $\mathcal{W}_n$ is a labyrinth set, for any $n \geq 1$. The limit set L_∞ defined by a sequence $\{\mathcal{A}_k\}_{k=1}^{\infty}$ of labyrinth patterns is called *mixed labyrinth fractal.*

The fact that the labyrinth patterns have exits leads to the fact that any labyrinth fractal also has 4 exits, which are defined as follows. Let $T_n \in \mathcal{W}_n$ be the top exit of $\mathcal{W}_n$, for $n \geq 1$. The *top exit of* L_∞ is $\bigcap_{n=1}^{\infty} T_n$. The other exits of L_∞ are defined analogously. We note that the exits property yields that $(x,1),(x,0) \in L_\infty$ if and only if $(x,1)$ is the top exit of L_∞ and $(x,0)$ is the bottom exit of L_∞. For the left and the right exits the analogue statement holds.

For more details on mixed labyrinth sets and mixed labyrinth fractals and for results on topological properties of mixed labyrinth fractals we refer to the papers [4, 5].

In the special case when in the above construction $\mathcal{A}_k = \mathcal{A}$, for all $k \geq 1$, i.e., the sequence of labyrinth patterns is a constant sequence, we obtain a self-similar labyrinth fractal, like those studied initially [2, 3].

The following result was proven both in the case of self-similar labyrinth fractals [2, Theorem 1], and in the case of mixed labyrinth fractals [4, Theorem 1].

Theorem 1. *Every (self-similar or mixed) labyrinth fractal is a dendrite.*

We recall that a dendrite is a connected compact Hausdorff space that is locally connected and contains no simple closed curve. By Theorem 1 and a result in Kuratowski's book [12, Corollary 2, p. 301] it follows that between any pair of points $x \neq y$ in the dendrite L_∞ there is a unique arc in the fractal that connects them.

It was shown [2, 3, 4, 5] that certain patterns, called blocked labyrinth patterns, generate fractals with interesting properties concerning the lengths of arcs in the fractal. More precisely, a labyrinth pattern is called *horizintally blocked*, if in the row that contains the horizontal exit pair of the pattern there exists at least one black square. Analogously, a labyrinth pattern is *vertically blocked*, if there exists at least one black square in the column that contains the vertical exit pair of the pattern. Figures 1, 2 and 3 show examples of horizontally and vertically blocked labyrinth patterns.

In the self-similar case the following facts were proven [3, Theorem 3.18]:

Theorem 2. *Let L_∞ be the (self-similar) labyrinth fractal generated by a horizontally and vertically blocked $m \times m$-labyrinth pattern. Between any two points in L_∞ there is a unique arc a. The length of a is infinite. The set of all points, at which no tangent to a exists, is dense in a.*

It was proven, that in this case the box counting dimension of all arcs in the fractal is strictly greater than 1. Moreover, all these arcs have box counting and Hausdorff dimension strictly less than 2, which easily follows, e.g., from the fact that the dimension (both box counting and Hausdorff) of the whole fractal is strictly less than 2, see [2, 3].

In the mixed case, it was shown [5] that the fact that the patterns are blocked is not sufficient in order to obtain all arcs with infinite length.

Very recent research [6, Theorem 5] reveals the following sufficient but not necessary condition in order to obtain a mixed labyrinth fractal where all (non-trivial) arcs in the fractal have infinite length.

Theorem 3. *If the width-sequence $\{m_k\}_{k\geq 1}$ of the sequence $\{\mathcal{A}_k\}_{k\geq 1}$ of both horizontally and vertically blocked labyrinth patterns satisfies the condition*

$$\sum_{k\geq 1}^{\infty} \frac{1}{m_k} = \infty, \tag{2}$$

then the mixed labyrinth fractal L_∞ has the property that for any two distinct points $x, y \in L_\infty$ the length of the arc in L_∞ that connects x and y is infinite.

Below we recall a result [4, Lemma 4] that enables us to use paths in the graphs of labyrinth patterns in order to construct arcs in a labyrinth fractal L_∞, both in the self-similar, and in the mixed case.

Lemma 1. *(Arc Construction) Let $a, b \in L_\infty$, where $a \neq b$. For all $n \geq 1$, there are $W_n(a), W_n(b) \in V(\mathcal{G}(\mathcal{W}_n))$ such that*

- *(a)* $W_1(a) \supseteq W_2(a) \supseteq \ldots,$
- *(b)* $W_1(b) \supseteq W_2(b) \supseteq \ldots,$
- *(c)* $\{a\} = \bigcap_{n=1}^{\infty} W_n(a),$
- *(d)* $\{b\} = \bigcap_{n=1}^{\infty} W_n(b).$
- *(e) The set* $\bigcap_{n=1}^{\infty} \left(\bigcup_{W \in p_n(W_n(a), W_n(b))} W \right)$ *is an arc between a and b.*

Until now, only one result with respect to fractal dimension of mixed labyrinth fractals was proven [4, Proposition 6, p.121]:

Proposition 1. *If a is an arc between the top and the bottom exit in L_∞ then*

$$\liminf_{n\to\infty} \frac{\log(\boxbar(n))}{\sum_{k=1}^{n} \log(m_k)} = \underline{\dim}_B(a) \leq \overline{\dim}_B(a) = \limsup_{n\to\infty} \frac{\log(\boxbar(n))}{\sum_{k=1}^{n} \log(m_k)},$$

where $\boxbar(n)$ denotes the number of squares in the path in $\mathcal{G}(\mathcal{W}_n)$ that connects the top and the bottom exit of $\mathcal{W}_n$. For the other pairs of exits, the analogous statements hold.

In the considerations to follow we aim to construct mixed labyrinth fractals whose box counting dimension can be established. First, we introduce a special sequence of labyrinth patterns, called snake cross patterns, which enables us to construct mixed labyrinth fractals with the property that the box counting dimension of any non-trivial arc (i.e., arc that connects two distinct points of the fractal) equals 2. Subsequently, we give a method for constructing mixed labyrinth fractals of any given (rational) box counting dimension between 1 and 2. For more details on the box counting dimension and its properties we refer, e.g., to Falconer's book [7].

3 Snake cross patterns and mixed labyrinth sets, and the fractals generated by them

In recent research [5], a special family of labyrinth patterns, called "special cross patterns" were used in order to construct mixed labyrinth fractals with certain properties of their arcs. These patterns were chosen due to their symmetry properties, which make them convenient tools when studying lengths of arcs between exits of the fractal. Here we introduce a family of patterns called *snake cross patterns*, whose name is due to the fact that their shape and some of their properties remind us of the special cross patterns, but, since here the length of the four "arms" of the cross is meant to be as long as possible, they have in general more and much bigger windings than the special cross patterns. Figure 3 shows a special cross pattern, and in Figures 4,5 and 6 we see snake cross patterns.

In order to define the sequence $\{\mathcal{A}_k\}_{k\geq 1}$ of snake cross patterns, with corresponding width sequence $\{m_k\}_{k\geq 1}$, let us first set $m_k := 4k + 7$, for all $k \geq 1$. Each pattern has a central square C_k, namely the white square that lies in row $2k + 4$ and column $2k + 4$, and which is coloured in white. Moreover, the top and bottom exit of any snake cross pattern lie in column $2k + 4$, and the left and right exit lie in row $2k + 4$. In Figure 3 the central square of the pattern is coloured in light grey, in Figure 6 the central square of the

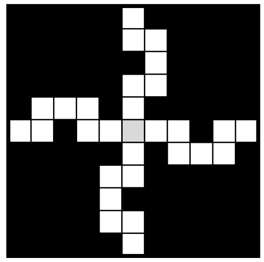

Figure 3: An example: a special cross pattern of width 11

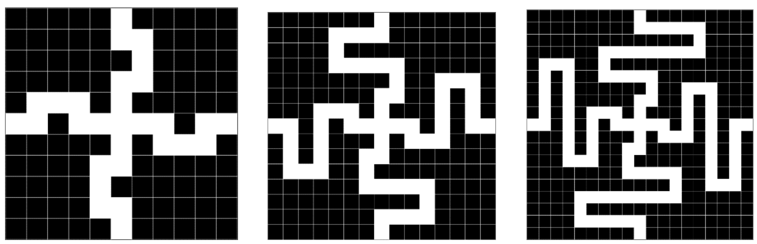

Figure 4: The snake cross patterns $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ of width 11, 15, and 19, respectively.

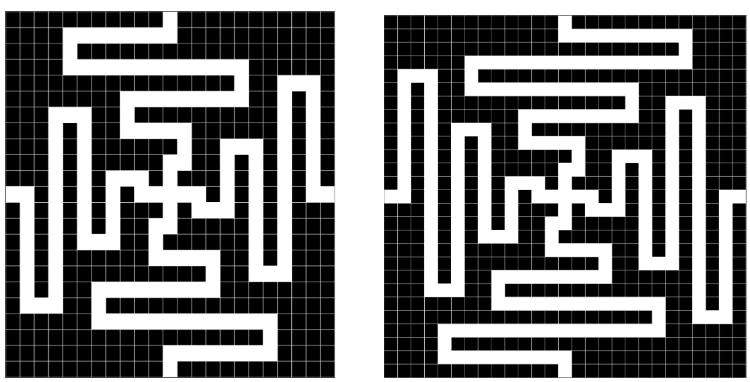

Figure 5: The snake cross patterns $\mathcal{A}_4, \mathcal{A}_5$ of width 23, and 27, respectively.

pattern is white (for better visibility on the coloured figure). Four "arms" emerge from the central square. In Figure 6 these are marked with different colours. The arms' shape, the windings on each of the four arms of the cross, resemble snakes in movement. The four arms are identical (up to rotation), and each of them can be obtained by the same procedure:

1. Start from the central square C_k.
2. The exit square on one of the sides of the unit square is selected as a distinguished vertex of $\mathcal{G}(\mathcal{A}_k)$, namely, the midpoint of one of its sides coincide with the midpoint of the mentioned side of the unit square.
3. "Move two steps" (two neighbouring squares) towards the selected exit.
4. Turn to the right and move through pairwise neigbouring squares until either a diagonal (see Figure 6) is reached or a neighbour of the selected exit is reached. In

the latter case stop the construction of the arm.

5. Turn to the left and "move two steps".
6. Turn to the left and move until either a diagonal is reached or a neighbour of the selected exit is reached. In the latter case stop the construction of the arm.
7. Turn to the right and "move two steps".
8. Repeat steps 4 to 7.

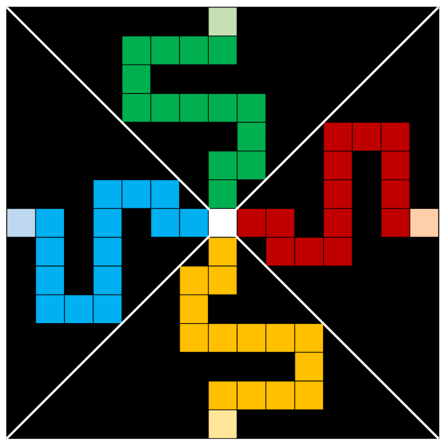

Figure 6: The four arms, the exits and the central square in the snake cross pattern $\mathcal{A}_2$

Remark 1. *The k-th snake cross pattern has k windings ("detours") on each of its four arms and $2k$ black squares in the row that contains its left and right exit, as well as in the column that contains its top and botom exit. We note, according to Figures 3 and 4, that the first snake cross pattern $\mathcal{A}_1$ is in particular a special cross pattern.*

It is easy to see that by replacing "left" with "right" and viceversa in the above construction algorithm of a snake cross pattern we obtain an other labyrinth pattern. Moreover, due to its shape, we can see it just as the "reflected" initial pattern. We call the initial one (defined above) a *right snake cross pattern* because the first turn was to the right, and its corresponding "reflected" pattern a *left snake cross pattern*. Throughout this article we assume, without loss of generality, that all snake cross patterns mentioned below are right snake cross patterns.

One can immediately see that the snake cross patterns are both horizontally and vertically blocked labyrinth patterns. One can use these patterns in order to define a mixed labyrinth fractal L_∞: let $\mathcal{A}_{k k\geq 1}$ be a sequence of snake cross patterns (as defined above) and L_∞ the corresponding labyrinth fractal. We call L_∞ a *mixed snake labyrinth fractal*.

Based on Theorem 1 and Theorem 3 one immediately obtains the proof of the following result.

Proposition 2. *Let L_∞ be a mixed snake labyrinth fractal. Then:*

1. *L_∞ is a dendrite.*
2. *For any distinct points $x, y \in L_\infty$, the arc in L_∞ that connects x and y has infinite length.*

4 Mixed labyrinth fractals with arcs of maximal box counting dimension

Now, our aim is to find the box counting dimension of arcs in a mixed snake labyrinth fractal. As a first step, we are interested in the arcs that connect exits of L_∞. Therefore, let us consider the lengths of the paths between exits in the labyrinth sets $\mathcal{W}_n$, for $n \geq 1$. For $n \geq 1$, let ◫(n) denote the length of the path in $\mathcal{G}(\mathcal{W}_n)$ that connects the top and bottom exit of the labyrinth set $\mathcal{W}_n$, and, analogously use the notations ⊟(n),◰(n),◲(n),◱(n),◳(n) for the lengths of the paths connecting the other six pairs of exits, respectively, as shown by the symbols. Moreover, for $k \geq 1$, let ◫$_k$,⊟$_k$,◰$_k$,◲$_k$,◱$_k$,◳$_k$ denote the lengths of the corresponding paths in $\mathcal{G}(\mathcal{A}_k)$. By the lengths of paths in the graphs we mean the number of graph vertices (i.e., the number of white squares) that lie on the path.

Lemma 2. *With the above notations,*

◫(n) = ⊟(n) = ◰(n) = ◲(n) = ◱(n) = ◳(n) $=: D(L_n)$, *for all* $n \geq 1$.

Proof. (*Sketch*) One can immediately see that ◫$_k$ = ⊟$_k$ = ◰$_k$ = ◲$_k$ = ◱$_k$ = ◳$_k$, for all $k \geq 1$, and thus in particular ◫(1) = ⊟(1) = ◰(1) = ◲(1) = ◱(1) = ◳(1). By induction, one can then immediately obtain the above result. □

We remark that one could also prove the above equalities by using the results on path matrices of mixed labyrinth fractals [4, Proposition 1], but here in our special case it is not necessary to use the path matrices.

Proposition 3. *Let L_∞ be the mixed labyrinth fractal generated by the sequence of snake cross patterns $\{\mathcal{A}_k\}_{k\geq 1}$ defined above. If a is an arc that connects two distinct exits of L_∞, then its box counting dimension is* $\dim_B(a) = 2$.

Proof. In order to compute the box counting dimension of the arc between two distinct exits of L_∞, we use an alternative definition of the box counting dimenstion [7, Definition 1.3], and we apply the property that one can use, instead of $\delta \to 0$, apropriate sequences $(\delta_k)_{k\geq 0}$ (see the cited reference). In our case, for $\delta_k = \frac{1}{m(k)}$, we have $\delta_k < \frac{1}{5}\delta_{k-1}$, and thus the box counting dimenstion of any arc between two exits in the fractal is given by

$$\dim_B(L_\infty) = \lim_{n\to\infty} \frac{\log D(L_n)}{\log m(n)}.$$

Therefore, let us now estimate $D(L_n)$. In order to do this, we count the number of squares that are along one of the four arms of the pattern $\mathcal{A}_k$ (we do not include the central square of the pattern), taking into account the following facts, according to the procedure described in Section 3:

(*i*) in the first step of the construction of the arm one square is added;

(*ii*) in step 3 two squares are added;

(*iii*) in the non-final steps 4 and 6 the number of added squares is

$$1 + 4 + 8 + \cdots + 4(k-1) = 1 + 4\frac{(k-1)k}{2} = 2k^2 - 2k + 1;$$

(*iv*) in the steps 3, 5 and 7 there are $2k$ added squares;

(*v*) in the final step 4 or 6 there are $2k - 1$ added squares.

The above counting yields a total number of

$$1 + 2 + (2k^2 - 2k + 1) + 2k + (2k - 1) = 2k^2 + 2k + 3$$

squares. Hence, the length of the path in $\mathcal{G}(\mathcal{A}_k)$ between two exits of the snake cross pattern $\mathcal{A}_k$ is $2(2k^2 + 2k + 3) + 1 = 4k^2 + 4k + 7$. Thus we obtain

$$D(L_n) = \prod_{k=1}^{n} \left(4k^2 + 4k + 7\right)$$

for the number of squares in the path in $\mathcal{G}(\mathcal{W}_n)$ that connects two distinct exits of the labyrinth set $\mathcal{W}_n$. Due to the arc construction given in Lemma 1 and the fact that the arc between two exits can be obtained as the limit of the nested sequence of sets defined, for each $n \geq 1$, as the union of the squares that lie in the path in $\mathcal{G}(\mathcal{W}_n)$ between the corresponding exits of the labyrinth set $\mathcal{W}_n$, and therefore is covered by the squares in this path, we obtain

$$\dim_B(a) = \lim_{n\to\infty} \frac{\log\left(\prod_{k=1}^{n}(4k^2 + 4k + 7)\right)}{\log\left(\prod_{k=1}^{n}(4k + 1)\right)} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n}\log(4k^2 + 4k + 7)}{\sum_{k=1}^{n}\log(4k + 1)}. \tag{3}$$

Note that for every $\varepsilon > 0$ and for sufficiently large k the inequality $4k^2+4k+7 > (4k+1)^{2-\varepsilon}$ holds, hence $\dim_B(a) \geq 2$. On the other hand, $4k^2 + 4k + 7 < (4k + 1)^2$, which then implies $\dim_B(a) = 2$. □

Proposition 3, the structure of mixed labyrinth fractals defined by snake cross patterns, and the properties of the box counting dimension immediately yield the following corollary.

Corollary 1. *Let L_∞ be a mixed labyrinth fractal generated by a sequence of snake cross patterns as above. Then* $\dim_B(L_\infty) = 2$.

Notation. Let $x \in L_\infty$ and $n \geq 1$. Along this paper the notation $W_n(x)$ has the following meaning (analogoulsy as introduced in the proof of [2, Lemma 6] for the case of self similar labyrinth fractals): Let $\mathcal{W}(x)$ be the set of all white squares in $\bigcup_{n\geq 1}\mathcal{W}_n$ that contains the point x. Let $W_1(x)$ be a white square in $\mathcal{W}_1$ that contains infinitely many (white) squares of $\mathcal{W}(x)$ as a subset. For $n \geq 2$ we define $W_n(x)$ as a white square in $\mathcal{W}_n$, such that $W_n(x) \subseteq W_{n-1}(x)$, and $W_n(x)$ contains infinitely many white squares of $W(x)$ as a subset.

Theorem 4. *Let L_∞ be a mixed labyrinth fractal generated by a sequence of snake cross patterns as above. Then, for any distinct points $x, y \in L_\infty$ the arc $a(x, y)$ that connects them in L_∞ has box counting dimension* $\dim_B(a(x, y)) = 2$.

Proof. Let $x \neq y$ be two distinct points in the fractal L_∞. From $a(x, y) \subset L_\infty$ it imediately follows that $\dim_B(a(x, y)) \leq \dim_B L_\infty = 2$. In order to prove the converse inequality, we proceed as follows.

For any $n \geq 1$ let $W_n(x)$ and $W_n(y)$ be white squares of level n, corresponding to the notation introduced above.

According to the construction of labyrinth sets and fractals it immediately follows that there exists a value $n_0 \geq 1$ such that the path in the tree $\mathcal{G}(\mathcal{W}_{n_0})$ that connects $W_{n_0}(x)$ and $W_{n_0}(y)$ consists of at least three white squares (vertices of $\mathcal{G}(\mathcal{W}_{n_0})$), i.e., there exists a square $W \in \mathcal{W}_{n_0}$ in this path, such that x, y do not lie in the interior of W.

Now, let us note that, due to the construction and structure of mixed labyrinth fractals, the intersection of the arc $a(x, y)$ that connects x and y in L_∞ with the square W is an arc $a' = a(x, y) \cap W$ that connects two exits of the fractal $L'_\infty = L_\infty \cap W$, which is just a scaled labyrinth fractal, obtained from the sequence of snake cross patterns $\{\mathcal{A}'_k\}_{k\geq 1}$, with $\mathcal{A}'_k = \mathcal{A}_{k+n_0}$, where n was chosen and fixed above. Thus, the arc a' is just the image of an arc between two exits of L_∞ through a similarity mapping of factor $1/m_1 m_2 \dots m_{n_0}$. Now we sum up as in formula (3), but for the integer k ranging from n_0+1 (instead of 1) to n. Since the limit of a sequence does not change by removing the first n_0 elements for some fixed n_0 we obtain, as in Proposition 3, that any arc that connects distinct exits of L'_∞ has box counting dimension two, and thus, in particular, $\dim_B(a') = 2$, and since $\dim_B(a(x,y)) \geq \dim_B(a') = 2$, we get $\dim_B(a(x,y)) = 2$. □

Remark 2. *The above results show that the snake cross patterns enable us to construct (mixed) labyrinth fractals based on horizontally and vertically blocked labyrinth patterns where the box counting dimension of every arc in the fractal as well as that of the fractal itself equals two. In the case of self-similar labyrinth fractals [2, 3] generated by both horizontally and vertically blocked patterns, all arcs in the fractal have the same box counting dimension (strictly less than* 2*) and this is in general different from the dimension of the fractal.*

5 "Omnidimensionality" of mixed labyrinth fractals

In this section we show that one can use the snake cross patterns in order to construct mixed labyrinth fractals of any dimension.

Theorem 5. *Let $\delta \in [1,2]$. Then there exists a (mixed) labyrinth fractal L_∞ with the property that every two exits in the fractal are connected in L_∞ by an arc whose box counting dimension equals δ.*

Before giving a proof of the theorem, we prove the following useful result.

Lemma 3. *Let a,b,c,d be positive real numbers and let $\frac{a}{b} < \alpha < \frac{c}{d}$. Then there are sequences $(p_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ of integers such that the sequence $(r_k)_{k\geq 1}$, with $r_k = \frac{p_k\cdot a + q_k \cdot c}{p_k \cdot b + q_k \cdot d}$, is increasing and $\lim_{k\to\infty} r_k = \alpha$.*

Proof. Note that from the lemma's assumptions we immediately have $\frac{c-\alpha d}{\alpha b - a} > 0$. Hence, there is an increasing sequence $\left(\frac{p_k}{q_k}\right)_{k\geq 1}$, with p_k, q_k positive integers, such that $\lim_{k\to\infty} \frac{p_k}{q_k} = \frac{c-\alpha d}{\alpha b - a}$. We denote $\varepsilon_k := \frac{p_k}{q_k} - \frac{c-\alpha d}{\alpha b - a}$, for $k \geq 1$. Observe, that $(\varepsilon_k)_{k\geq 1}$ is a decreasing sequence that converges to zero. Now, let us proceed to the following computations:

$$
\begin{aligned}
r_k &= \frac{p_k a + q_k c}{p_k b + q_k d} = \frac{\frac{p_k}{q_k} a + c}{\frac{p_k}{q_k} b + d} = \frac{\left(\frac{c-\alpha d}{\alpha b - a} + \varepsilon_k\right) a + c}{\left(\frac{c-\alpha d}{\alpha b - a} + \varepsilon_k\right) b + d} = \frac{(c-\alpha d)a + (\alpha b - a)a\varepsilon_k + c(\alpha b - a)}{(c-\alpha d)b + (\alpha b - a)b\varepsilon_k + d(\alpha b - a)} \\
&= \frac{(cb-da)\alpha + (\alpha b - a)a\varepsilon_k}{(cb-da) + (\alpha b - a)b\varepsilon_k} = \frac{(cb-da)\alpha + (\alpha b - a)b\alpha\varepsilon_k + [(\alpha b - a)a - (\alpha b - a)b\alpha]\varepsilon_k}{(cb-da) + (\alpha b - a)b\varepsilon_k} \\
&= \alpha - \frac{(\alpha b - a)^2}{(cb-da) + (\alpha b - a)b\varepsilon_k}\varepsilon_k.
\end{aligned}
$$

Thus, $\lim_{k\to\infty} r_k = \alpha$. In order to prove that the sequence $(r_k)_{k\geq 1}$ is increasing, let us consider the function

$$f(x) := \alpha - \frac{(\alpha b - a)^2}{(cb-da) + (\alpha b - a)bx} x.$$

Its first derivative is

$$f'(x) = -\frac{(\alpha b - a)^2 (cb-da)}{\left((cb-da) + (\alpha b - a)bx\right)^2} < 0.$$

Thus, the fact that the sequence $(\varepsilon_k)_{k\geq 1}$ is decreasing yields that the sequence $(r_k)_{k\geq 1}$ is increasing, which completes the proof. □

Proof of Theorem 5. The case $\delta = 2$ has been solved in Section 4.

For the case $\delta = 1$ it is enough to consider, e.g., the mixed labyrinth fractal generated by a sequence of labyrinth patterns $\{\mathcal{A}_k\}_{k\geq 1}$ defined as follows: Figure 7 shows the pattern $\mathcal{A}_1$. In general, for $k \geq 1$, $\mathcal{A}_k$ has width $2k+1$ and has the shape of a cross, consisting only of the $(k+1)$-th row and $(k+1)$-th column. Since the arcs connecting the exits of the resulting mixed labyrinth fractal are either straight line segments of length 1 or the union of two straigh line segements of length 1/2 (each), we are done. In this case it easily follows $\dim_B(L_\infty) = \dim_H(L_\infty) = 1$ and box counting dimension of any arc in the dendrite is 1.

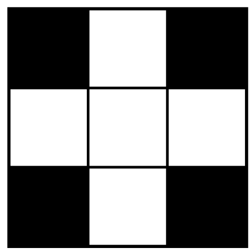

Figure 7: The first pattern in the sequence $\{\mathcal{A}_k\}_{k\geq 1}$ that generates a mixed labyrinth fractal having all arcs of box counting dimension one.

Now, we assume $\delta \in (1,2)$. Let $\mathcal{A}_0$ denote the (unblocked) labyrinth pattern in Figure 7. For $k \geq 1$ let $\mathcal{A}_k$ be the k-th snake cross pattern, as defined in Section 3. For every $k \geq 1$ we introduce the *k-th dimension quotient* d_k as follows. By definition,

$$d_0 := 1 \text{ and } d_k := \frac{\log(4k^2+4k+7)}{\log(4k+1)}, \text{ for all } k \geq 1.$$

Note that the quotients d_k, for $k \geq 1$ occur in the computations in formula (3). Since $\lim_{k\to\infty} d_k = 2$, it follows that there is an integer $k \geq 1$ such that

$$d_k \leq \delta \leq d_{k+1}.$$

If $\delta = d_k$ for some $k \geq 1$, then the (self similar) labyrinth fractal constructed based on the snake cross pattern $\mathcal{A}_k$ is a solution, and if $\delta = d_{k+1}$ then the (self similar) labyrinth fractal constructed based on the snake cross pattern $\mathcal{A}_{k+1}$ is a solution. Let us now assume that $d_k < \delta < d_{k+1}$. More precisely, we then have for this value of k,

$$\frac{\log(4k^2+4k+7)}{\log(4k+1)} < \delta < \frac{\log\left((4(k+1)^2+4(k+1)+7)\right)}{\log\left(4(k+1)+1\right)}. \tag{4}$$

We use the following notation:

$$a = \log(4k^2+4k+7),\ b = \log(4k+1),\ c = \log\left(4(k+1)^2+4(k+1)+7\right),\ d = \log\left(4(k+1)+1\right).$$

Moreover, let $\alpha = \delta$ and let $(p_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ be the sequences occurring in Lemma 3.

We introduce a new sequence of labyrinth patterns $(\mathcal{A}_k^*)_{k\geq 1}$ in the following way. For $k \geq 1$, let $\mathcal{A}_k^*(\delta)$, in short, when δ is known, $\mathcal{A}_k^*$, be the labyrinth pattern of width $m_k^* = m_k^{p_k} m_{k+1}^{q_k}$ that is identical to the mixed labyrinth set (of level $p_k + q_k$) constructed by first applying p_k times the pattern $\mathcal{A}_k$ and then applying q_k the pattern $\mathcal{A}_{k+1}$. Let $L_\infty(\mathcal{A}^*(\delta))$ be the mixed labyrinth fractal defined by the sequence of patterns $(\mathcal{A}_k^*)_{k\geq 1}$. Let $a(e_1,e_2)$ be an arc in $L_\infty(\mathcal{A}^*(\delta))$ that connects two exits $e_1 \neq e_2$ of this mixed labyrinth fractal. Then its box counting dimension is

$$\dim_B(a(e_1,e_2)) = \lim_{n\to\infty} \frac{\log\left(\prod_{k=1}^n \left((4k^2+4k+7)^{p_k}\left(4(k+1)^2+4(k+1)+7\right)^{q_k}\right)\right)}{\log\left(\prod_{k=1}^n \left((4k+1)^{p_k}\left(4(k+1)+1\right)^{q_k}\right)\right)}$$
$$= \lim_{n\to\infty} \frac{\sum_{k=1}^n p_k a + q_k c}{\sum_{k=1}^n p_k b + q_k d}.$$

Thus, with the notation from Lemma 3 we obtain

$$\dim_B(a(e_1,e_2)) = \lim_{n\to\infty} \frac{\sum_{k=1}^n (p_k b + q_k d)\left(\alpha - \frac{(\alpha b - a)^2}{(cb-da)+(\alpha b - a)b\varepsilon_k}\varepsilon_k\right)}{\sum_{k=1}^n (p_k b + q_k d)}$$
$$= \lim_{n\to\infty}\left(\alpha - \frac{\sum_{k=1}^n (p_k b + q_k d)\cdot\frac{(\alpha b - a)^2}{(cb-da)+(\alpha b - a)b\varepsilon_k}\varepsilon_k}{\sum_{k=1}^n (p_k b + q_k d)}\right).$$

Hence, $\dim_B(a(e_1,e_2)) \le \alpha$. It remains to prove that for every $\varepsilon > 0$, we have $\dim_B(a(e_1,e_2)) > \alpha - \varepsilon$, i.e., that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n (p_k b + q_k d)\cdot\frac{(\alpha b - a)^2}{(cb-da)+(\alpha b - a)b\varepsilon_k}\varepsilon_k}{\sum_{k=1}^n (p_k b + q_k d)} < \varepsilon. \tag{5}$$

By taking into consideration that $\lim_{k\to\infty} \frac{(\alpha b - a)^2}{(cb-da)+(\alpha b - a)b\varepsilon_k}\varepsilon_k = 0$, and that the sequence $\left(\frac{(\alpha b - a)^2}{(cb-da)+(\alpha b - a)b\varepsilon_k}\varepsilon_k\right)_{k\ge1}$ is monotonically decreasing, it follows that there is a k_0 such that for all $k \ge n_0$ the inequality $\frac{(\alpha b - a)^2}{(cb-da)+(\alpha b - a)b\varepsilon_k}\varepsilon_k < \frac{\varepsilon}{2}$ holds, and therefore we obtain that the left hand term in (5) is less than

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n_0} (p_k b + q_k d)\cdot\frac{(\alpha b - a)^2}{(cb-da)+(\alpha b - a)b\varepsilon_k}\varepsilon_k + \frac{\varepsilon}{2}\cdot\sum_{k=n_0+1}^n (p_k b + q_k d)}{\sum_{k=1}^{n_0} (p_k b + q_k d) + \sum_{k=n_0+1}^n (p_k b + q_k d)} < \frac{\varepsilon}{2},$$

which yields $\dim_B(a(e_1,e_2)) = \alpha = \delta$.

□

Corollary 2. *Under the assumptions of Proposition 3 we have*

$$\dim_B\left(L_\infty(\mathcal{A}^*(\delta))\right) = \delta.$$

Theorem 6. *Let $\delta \in [1,2)$, and $L_\infty(\mathcal{A}^*(\delta))$ be be the corrensponding labyrinth fractal, constructed as in the proof of Theorem 5. Then, for any two distinct points $x, y \in L_\infty$ the arc $a(x,y)$ that connects the two points in the fractal has $\dim_B(a(x,y)) = \delta$.*

Proof. (Sketch.) Let δ and $L_\infty(\mathcal{A}^*(\delta))$ be as in the above assumptions. For any distinct points $x, y \in L_\infty(\mathcal{A}^*(\delta))$, we have for the corresponding arc $a(x,y) \subseteq L_\infty(\mathcal{A}^*(\delta))$ that $\dim_B(a(x,y)) \le \dim_B\left(L_\infty(\mathcal{A}^*(\delta))\right) = \delta$. In order to show that $\dim_B(a(x,y)) \ge \delta$ we proceed analogously to the proof of Theorem 4. Let again n_0 denote an integer such that the length of the path in the tree $\mathcal{G}(\mathcal{W}_{n_0})$ from $W_{n_0}(x) \in \mathcal{V}(\mathcal{G}(\mathcal{W}_n))$ to $W_{n_0}(y) \in \mathcal{V}(\mathcal{G}(\mathcal{W}_n))$ is at least three. Thus, the arc $a(x,y)$ crosses this square along an arc $a' = a \cap \left(W \cap L_\infty(\mathcal{A}^*(\delta))\right)$. The set $L_\infty \cap W$ is a scaled mixed labyrinth fractal, defined by the sequence of patterns $(\mathcal{A}'_k)_{k\ge1}$, where $\mathcal{A}'_k = \mathcal{A}^*_{n_0+k}$, for all $k \ge 1$. More precisely, this new sequence of labyrinth patterns defines the labyrinth fractal L'_∞, which is just the image of the dendrite $L_\infty \cap W$ through a similarity mapping with scaling factor $m_1^* m_2^* \dots m_{n_0}^*$. Thus, the arc a' is the image through a similarity mapping of an arc between two exits of the fractal L'_∞. We obtain $\dim_B(a') = \delta$ and since $a' \subset a(x,y)$ it follows that $\dim_B(a(x,y)) \ge \dim_B(a') = \delta$, which completes the proof.

□

6 A glimpse into a more general setting

Let us first recall (see [2]) that the *core* of a labyrinth pattern $\mathcal{A}$ is the intersection of all labyrinth patterns that are subsets of $\mathcal{A}$, in other words a core of a labyrinth pattern is minimal in the sense that colouring any of its white squares in black would lead to a pattern that is not a labyrinth pattern anymore. It is easy to see that the core of any snake cross pattern is the pattern itself.

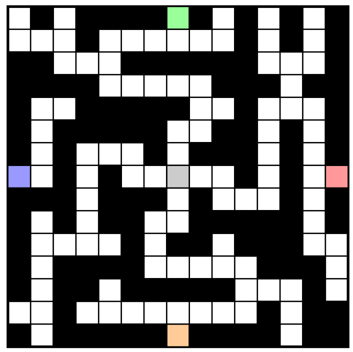

Figure 8: Example: a labyrinth pattern of width 15 whose core is the snake cross pattern $\mathcal{A}_2$ shown in Figures 4 and 6. The central square and the pattern's exits are marked in colour.

Let us denote by $\mathcal{A}_k(2)$, for $k \geq 1$, the k-th snake cross pattern that occurs in the construction used in the proof of Theorem 4 and $L_\infty(2)$ the corresponding mixed labyrinth fractal.

Corollary 3. *If $\mathcal{A}_k$ is a sequence of labyrinth patterns such that the core of $\mathcal{A}_k$ is $\mathcal{A}_k^*(2)$, then the box counting dimension of the resulting fractal L_∞ is $\dim_B(L_\infty) = 2$ and every arc that connects distinct points in this dendrite has box counting dimension 2.*

Proof. The fact that $\dim_B(L_\infty(2)) \leq 2$ is basic, since the box counting dimension of the unit square is 2. Furthermore, $L_\infty(2) \subset L_\infty$ implies $\dim_B(L_\infty) \geq 2$. For $x, y \in L_\infty$, $x \neq y$, by using the same ideas as in the proof of Theorem 4 we obtain that $\dim_B(a(x,y)) \geq 2$, which, together with the converse inequality, implies $\dim_B(a(x,y)) = 2$.

□

We conjecture that a more general statement is valid.

Conjecture 1. *Let $\delta \in [1,2]$. If $\{\mathcal{A}_k^*\}_{k\geq 1}$ is a sequence of labyrinth patterns such that the core of $\mathcal{A}_k$ is $\mathcal{A}_k^*(\delta)$, i.e., the k-th snake cross pattern used in the construction of $L_\infty(\mathcal{A}^*(\delta))$, then every arc that connects distinct points in this dendrite has box counting dimension δ.*

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The Golden Section's heptagonal connections

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Abstract

It is well known that the Golden Section plays an important role in the geometry of several polygons and polyhedra; the best known example is the length of a diagonal in the regular pentagon with unit side. In this contribution we show how the Golden Section appears as the solution of an enumerative problem connected with heptagons, more precisely, with heptagonal tilings of the hyperbolic plane. The results are then generalized by investigating whether it also appears in other types of hyperbolic tilings.

Keywords: Golden Section, polygonal cluster, heptagon, hyperbolic tiling

MSC: 05A15, 05A16, 52C20

1 Introduction and preliminaries

The Golden Section, $\phi = \frac{1+\sqrt{5}}{2}$, is, together with its reciprocal value $\varphi = \frac{\sqrt{5}-1}{2}$, one of the most fascinating numbers. Its extraordinary properties and numerous alleged appearances in many wildly different contexts keep attracting attention of mathematicians and laymen alike for more than two and a half millennia. It would go way beyond the scope of this paper to attempt even a far-from-exhaustive enumeration. Instead, we will mention just two of the best known appearances of this number. The first one is geometrical: ϕ is the length of a diagonal of a regular pentagon with unit side length. The

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second one is from enumerative combinatorics, where ϕ appears as the limit of ratios of two successive Fibonacci numbers F_{n+1}/F_n when n tends to infinity.

The Golden Section often appears in polygons and polyhedra with pentagonal symmetry, such as the icosahedron, sometimes connecting them with (golden) rectangles and triangles. Somewhat less known example is the rhombic triacontahedron all of whose 13 faces are golden rhombi. The number seven is not known for its connections with the Golden Sections. Yet, such connections exist and appear naturally in a context connecting geometry and enumerative combinatorics – that of regular tilings of the (hyperbolic) plane by regular heptagons.

A **tiling** or a **tessellation** of a surface is its covering with one or several shapes without gaps and without overlaps. If all shapes are polygons, the tilling is polygonal. A **regular tiling** is a polygonal tiling in which all polygons have the same number of sides and each vertex is shared by the same number of polygons. In regular tilings, if two tiles share more than one point, then they share a whole edge. The best known regular tilings are the tilings of the Euclidean plane by squares, by regular hexagons and by equilateral triangles shown in Fig. 1.

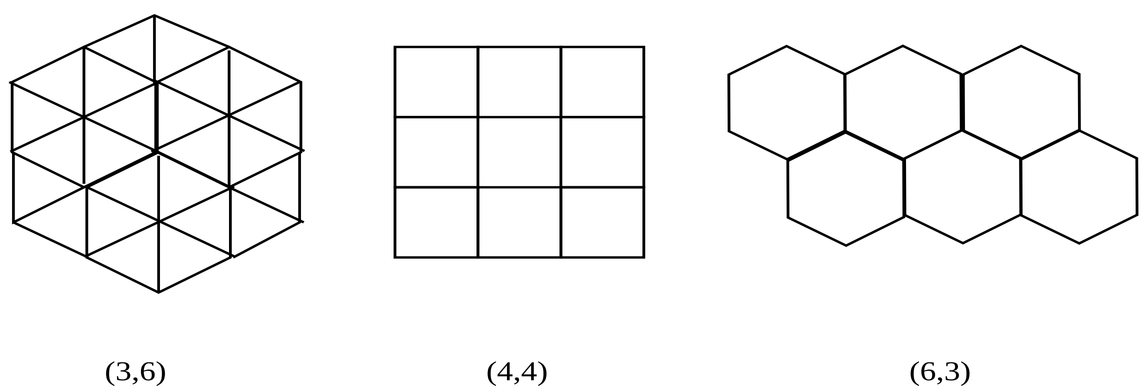

Figure 1: The three regular tilings of the Euclidean plane.

Any regular tiling is completely described by two integer parameters $r, s \geq 3$. An (r, s)-regular tiling, or simply an (r, s)-tiling is the tiling by regular r-gons such that exactly s of them meet at every vertex. The same two parameters suffice for classification of regular tilings.

The tilings with $1/r + 1/s > 1/2$ cover spherical surfaces. There are only five (r, s) pairs satisfying this condition, and they correspond to the five Platonic solids: $(3, 3)$ is the tetrahedron, $(4, 3)$ is the cube, $(3,4)$ the octahedron, $(5,3)$ is the dodecahedron and $(3,5)$ the icosahedron.

The tilings satisfying $1/r + 1/s = 1/2$ are tilings of the Euclidean plane. There are only three of them: $(4,4)$, $(6,3)$ and $(3,6)$ are the familiar tilings of the plane with squares, regular hexagons and equilateral triangles, respectively, mentioned above.

All other regular tilings, i.e., all tilings whose parameters satisfy $1/r + 1/s < 1/2$, are tilings of the hyperbolic plane. It is easily seen that there are infinitely many of them. The simplest among them are those with $s = 3$, and the smallest value of r yielding a hyperbolic tiling is $r = 7$. That brings us to the $(7,3)$-tiling of the hyperbolic plane, the main protagonist of this note.

Another way of looking at tilings is to look at their duals and to define a regular (r,s)-tiling as a planar graph in which every vertex is of degree s and every face has r sides. Such graphs can be drawn on the corresponding surface in a regular way, i.e., so that all edges have the same length and all faces the same area; here the lengths and areas are spherical, Euclidean or hyperbolic, depending on the value of $1/r + 1/s$. We prefer here the face-centered approach since it is easier to count faces than the vertices; see [13] for a vertex-oriented approach. For more on tilings and related matter we refer the reader to the classical monograph by Grünbaum and Shephard [10].

2 The simplest heptagonal tiling

In this section we consider the $(7,3)$-tiling of the hyperbolic plane. We build it iteratively by starting from a single central heptagon and adding successive layers of heptagons. In the first layer there are seven heptagons, each of them sharing an edge with the central heptagon. In the second layer there will be two types of heptagons: Some of them will share one edge with a heptagon in the first layer, while the others will share two adjacent edges, one with each of two adjacent hexagons in the first layer. The situation remains the same for any further layers. The two types of heptagons, denoted as type I and type II, respectively, are shown in Fig. 2. It is easily seen that there are 14 heptagons of type I and 7 heptagons of type II in the second layer, altogether 21 of them.

Let us denote by f_n the number of heptagons in the n-th layer for $n \geq 0$. Hence $f_0 = 1$, $f_1 = 7$ and

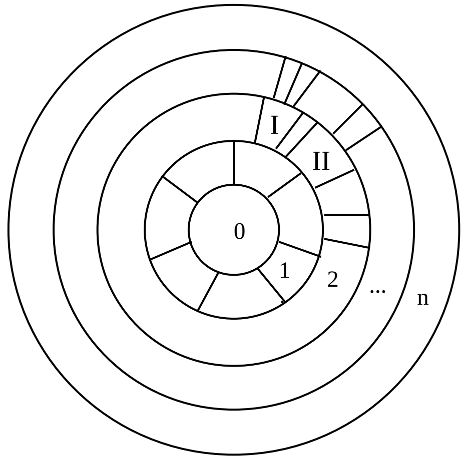

Figure 2: Layers of $(7, 3)$-tiling and heptagons of different types.

$f_2 = 21$.

Proposition 1. *The number of heptagons in the n-th layer of the $(7, 3)$-tiling satisfies the recurrence*

$$f_n = 3f_{n-1} - f_{n-2} \qquad \text{for} \qquad n \geq 3,$$

with the initial conditions $f_1 = 7,\ f_2 = 21$.

Proof. It is clear that each heptagon of the type I from layer $n - 1$ sends out three radial edges to the layer n, while each heptagon of type II sends out only two radial edges. Hence the number of outgoing radial edges can be obtained by multiplying the total number of heptagons in the $(n - 1)$-st layer by 3 and then subtracting the number of heptagons of type II. As there are exactly f_{n-2} heptagons of type II in the $(n - 1)$-st layer, the total number of outgoing radial edges is given by $3f_{n-1} - f_{n-2}$. But this is exactly the number of heptagons in the n-th layer, and that proves the claim. □

It follows immediately that the numbers f_n are multiples of even-indexed Fibonacci numbers.

Corollary 2. *$f_n = 7F_{2n}$ for $n \geq 1$. Here F_{2n} are the even-indexed Fibonacci numbers.*

Now we can easily compute the total number of heptagons in the first n layers of a $(7, 3)$-tiling. If we denote this quantity by g_n, we have $g_n = \sum_{k=0}^{n} f_k$. By using Corollary 2 and the well-known formula $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$, we immediately obtain

$$g_n = 1 + 7\sum_{k=1}^{n} F_{2k} = 7F_{2n+1} - 6.$$

So, the total number of heptagons in the first n layers of a $(7,3)$-tiling has a neat expression in terms of odd-indexed Fibonacci numbers.

The iterative constructions of the described type can serve as useful models of certain growth processes not only in the hyperbolic, but also in the Euclidean space. A nice example is presented by leaves and flowers of certain vegetables, which can be also mimicked by crocheting; see [14] for several nice illustrations. In such situations it might be of interest to consider the fraction of the periphery, whether expressed in terms of the number of cells, of the length of yarn, or of the mass of tissue, in the total bulk of the structure. It is known that in the Euclidean case this fraction tends to zero, since the total number of cells grows quadratically, while the number of peripheral cells grows only linearly with the number of generations (i.e., of layers). In the hyperbolic case the growth is exponential, and hence the fraction of peripheral cells becomes non-negligible. In the case of the $(7,3)$-tiling, the fraction of peripheral cells can be expressed in a particularly simple and elegant way.

Corollary 3. *The fraction of peripheral heptagons in a* $(7,3)$*-tiling tends to* $\varphi = \frac{\sqrt{5}-1}{2}$ *for large values of* n.

This is a direct consequence of Corollary 2 and of the already mentioned fact that the ratio of two successive Fibonacci numbers tends to the Golden Section for large values of n. So, there is the connection promised in the title of this note.

It remains to examine whether heptagons are in some respect special or the same fraction of periphery can also appear in some other tilings.

3 $(r,3)$-tilings

In this section we look at regular tilings of the hyperbolic plane with regular r-gons, such that exactly 3 of them meet at every vertex. Again, we start with a single r-gon in the center surrounded by r r-gons in the first layer. The number of r-gons in the n-th layer we denote by p_n. As the cases $r = 3, 4, 5$ lead to the Platonic solids, we consider only $r \geq 6$.

Proposition 4. *The numbers* p_n *satisfy the recurrence* $p_n = (r-4)p_{n-1} - p_{n-2}$ *for* $n \geq 3$ *with the initial conditions* $p_1 = r$, $p_2 = (r-4)r$.

The proof follows along the same lines as the proof for the heptagonal case in the previous Section, so we omit the details.

It is a routine exercise to compute the generating functions for sequences p_n and for sequences of their partial sums $q_n = \sum_{k=0}^{n} p_k$. Again, we omit the details and state only the final expressions.

$$P(x) = \sum_{n=0}^{\infty} p_n x^n = \frac{1 + 4x + x^2}{1 - (q-4)x + x^2}, \qquad Q(x) = \sum_{n=0}^{\infty} q_n x^n = \frac{1 + 4x + x^2}{(1-x)(1 - (q-4)x + x^2)}.$$

From the generating functions we can determine the asymptotic behavior of the enumerating sequences and then use it to compute the fraction of r-gons in the peripheral layer. We refer the reader to [1,15] for more information on obtaining the asymptotics of a sequence from its generating function. Here we rely on the following version of Darboux's theorem [1].

Theorem A

If the generating function $f(x) = \sum_{n \geq 0} a_n x^n$ of a sequence (a_n) can be written in the form $f(x) = \left(1 - \frac{x}{w}\right)^{\alpha} h(x)$, where w is the smallest modulus singularity of f and function h is analytic at w, then $a_n \sim \frac{h(w) n^{-\alpha-1}}{\Gamma(-\alpha) w^n}$, where Γ denotes the gamma function. $\square$

Since both our generating functions $P(x)$ i $Q(x)$ have the same smallest modulus singularity $w = w(r) = \frac{1}{2}\left(r - 4 - \sqrt{(r-4)^2 - 4}\right)$, both p_n and q_n will asymptotically behave as $w(r)^n$ and there will be a lot of cancellation in the ratio p_n/q_n. In fact, the only surviving term is $1 - w(r)$. As $w(r)$ is a decreasing function of r, tending to zero for large values of r, it follows that the fraction of peripheral r-gons tends to one for large values of r and n.

Proposition 5. *The fraction of peripheral r-gons in an $(r,3)$-tiling is $1 - w(r)$ and it tends to one for large values of r and n.*

Hence for large values of r almost all polygons are on the periphery. That offers an explanation for patterns observed in some living organisms employing such arrangements as an efficient way to package many parts which must be in contact with each other and also with the surroundings.

For $r = 7$ we have $w(7) = \frac{1}{2}\left(3 - \sqrt{5}\right) = \phi^{-2} = \varphi^2$. Since $\varphi = 1 - \varphi^2$, the case $r = 7$ is the only one yielding the Golden Section as the fraction of peripheral r-gons.

We close this section by observation that in the case $r = 6$ we obtain $w(6) = 1$, thus confirming the non-exponential nature of the growth of p_n and q_n.

4 (r, s)-tilings

Finally we consider the most general case $s > 3$. The case $r = 3$ will need a separate treatment, so assume, for the beginning, $r \geq 4$.

Let $z_n = z_n(r, s)$ denote he number of r-gons in the n-th layer of a regular (r, s)-tiling. As before, we have one central r-gon, so $z_0 = 1$. In the first layer there will be one r-gon for each of r sides of the central r-gon and $s - 3$ r-gons for each of its r vertices. Hence, $z_1 = r(s - 2)$.

In the second layer, and in all further layers, there will be two types of r-gons – the ones sharing a whole edge with $(n - 1)$-st layer, and the ones sharing only one vertex. We call them type I and type II, respectively, and denote their respective numbers in the n-th layer by v_n and w_n.

Clearly, each of the v_n r-gons of type I has one of its sides on the interior border of the n-th layer; it shares two of its sides with other r-gons of the same layer, and hence it has $r - 3$ sides with $r - 4$ vertices of degree 2 on the outer border of the n-th layer. Similarly, each of the w_n r-gons of type II shares two sides with other r-gons of the n-th layer and has $r - 2$ sides with $r - 3$ vertices on the outer border. There are altogether $v_n + w_n$ vertices of degree 3 on the outer border of the n-th layer.

Now we can compute the number of r-gons of each type in the next layer, as each side on the outer border gives rise to one r-gon of type I, and each vertex on the outer border gives rise to $s - 3$ or $s - 4$ r-gons of type II, depending on its degree. See Fig. 3. Hence,
$v_{n+1} = (r - 3)v_n + (r - 2)w_n$
$w_{n+1} = (s - 3)\left[(r - 4)v_n + (r - 3)w_n\right] + (s - 4)(v_n + w_n)$.
After some rearrangement, we obtain a double linear recurrence

$$\begin{aligned} v_{n+1} &= (r - 3)v_n + (r - 2)w_n \\ w_{n+1} &= \left[(s - 3)(r - 4) + (s - 4)\right] v_n + \left[(s - 3)(r - 3) + (s - 4)\right] w_n \end{aligned}$$

for $n \geq 2$ with the initial conditions $v_1 = r$, $w_1 = r(s - 3)$. (Essentially the same recurrence appears

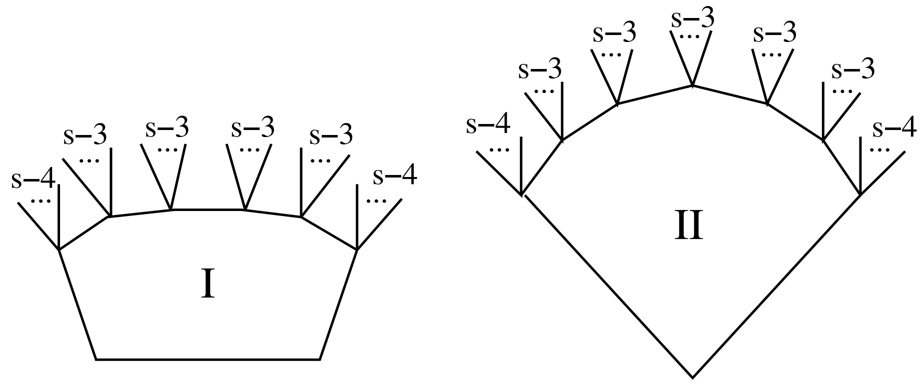

Figure 3: Two types of r-gons in a regular (r, s)-tiling.

in Lemma 1 of ref. [12] concerned with some probabilistic aspects of certain trees related to regular hyperbolic tilings.) One could proceed by expressing w_n from the first recurrence, plug it into the second one and solve the resulting second-order linear recurrence for w_n. Instead, we consider a general first-order linear double recurrence with constant coefficients

$$\begin{aligned} v_{n+1} &= Av_n + Bw_n \\ w_{n+1} &= Cv_n + Dw_n \end{aligned}$$

for $n \geq 2$ with the initial conditions $v_1 = E$, $w_1 = F$. Let $V(x)$ and $W(x)$ be the generating functions of sequences v_n andw_n, respectively. It is a straightforward exercise in generating function manipulation to convert the above recurrence into a linear system for $V(x)$ and $W(x)$,

$$\begin{aligned} V(x) &= Ax(V(x) - E) + Bx(W(x) - F) \\ W(x) &= Cx(V(x) - E) + Dx(W(x) - F) \end{aligned}$$

which can be easily solved by using Cramer's rule. The resulting expression for $Z(x) = V(x) + W(x)$ is

$$Z(x) = x\frac{(AD - BC)(E + F)x - [(A + C)E + (B + D)F]}{1 - (A + D)x + (AD - BC)x^2}.$$

By plugging in $AD - BC = 1$ and $A + D = rs - 2(r + s - 1)$ and by accounting for the index shift, we finally obtain

$$Z(x) = \frac{1 + 2(s - 1)x + x^2}{1 - [rs - 2(r + s - 1)]x + x^2},$$

corresponding to the recurrence

$$z_n = [rs - 2(r + s - 1)]z_{n-1} - z_{n-2}$$

for z_n with the initial conditions $z_0 = 1, z_1 = r(s-2)$, $z_2 = r(s-2)[rs - 2(r+s-1)]$.

It remains to consider the case $r = 3$. We may assume $s \geq 6$, since the smaller values of s lead to Platonic solids. Let t_n denote the number of triangles in the n-th layer. As before, $t_0 = 1$, $t_1 = 3(s-2)$ and $t_2 = 3(s-2)(s-4)$. Let u_n denote the number of vertices on the outer border of the n-th layer. Some of them are of degree 3, some are of degree 4. Those of degree 3 send $s-3$ radial edges out to the $(n+1)$-st layer; the ones of degree 4 send out $s-4$ radial edges. Clearly, the number of vertices of degree 4 is equal to the number of all vertices on the inner border of the considered layer, hence to u_{n-1}. Now the total number of outgoing radial edges can be computed as $(s-3)u_n - u_{n-1}$. Since each of u_n pairs of adjacent vertices sends out two edges that meet in a vertex on the outer border

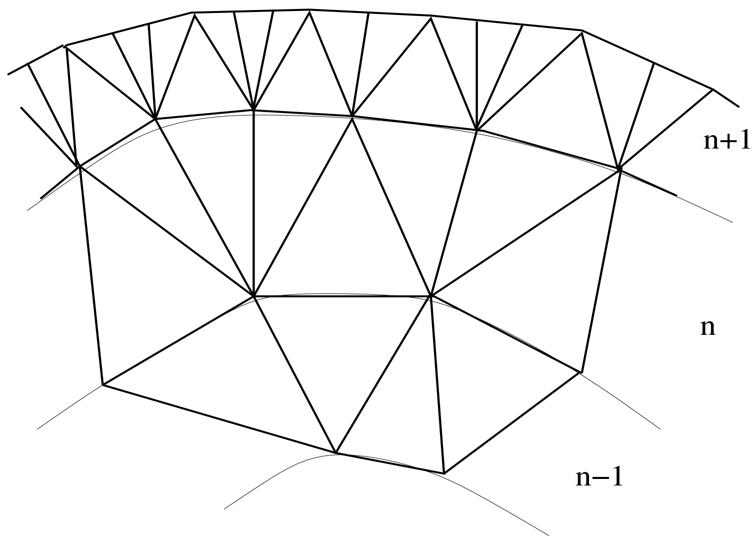

Figure 4: Fragments of three successive layers in a $(3, 7)$-tiling.

of the $(n+1)$-st layer, we must subtract u_n from the total number of outgoing radial edges in order to obtain u_{n+1}. Hence, $u_{n+1} = (s-4)u_n - u_{n-1}$. The situation is illustrated in Fig. 4. Now it immediately follows that $t_n = u_n + u_{n-1}$, since all triangles in the n-th layer share two of their sides with other triangles in the same layer. Those with the remaining side on the inner border are counted by u_{n-1}, while those with the third side on the outer border are counted by u_n. As both u_n and u_{n-1} satisfy the same recurrence, their sum must satisfy the same recurrence, with suitably modified initial conditions. Hence, $t_n = (s-4)t_{n-1} - t_{n-2}$ for $n \geq 3$ with $t_1 = 3(s-2)$ and $t_2 = 3(s-2)(s-4)$.

We see that the $(3, s)$-tilings follow the same recurrence as the tilings with polygons with larger number of sides. Further, for $s = 6$ we recover the polynomial growth of the tilings of Euclidean plane. It is interesting to note that our reasoning remains, in a sense, valid also for $s = 5$: from $t_1 = 9$ and $t_2 = 9$ one obtains $t_3 = 0$; that is wrong as the number of triangles in the third layer (equal to 1), but correct as the number of outgoing edges.

Now we can summarize our findings on (r, s)-tilings.

Proposition 6. *The number z_n of r-gons in the n-th layer of a regular (r, s)-tiling satisfies the recurrence*

$$z_n = [rs - 2(r + s - 1)]z_{n-1} - z_{n-2}$$

for $n \geq 3$ with the initial conditions $z_0 = 1, z_1 = r(s - 2)$, $z_2 = r(s - 2)[rs - 2(r + s - 1)]$.

It remains to check if any of (r, s)-tilings for $s > 3$ have the growth rate resulting in the Golden Section fraction of peripheral polygons. As φ^{-2} is the only $w = w(r, s)$ such that $1 - w = \varphi$, we must look for integer solutions of the equation $rs - 2(r + s - 1) = 3$, or, equivalently, $rs = 2(r + s) + 1$, satisfying the additional condition $r, s \geq 3$. Clearly, $r = \frac{2s+1}{s-1} < 3$ for all $r \geq 3, s \geq 8$. That leaves only the case $s = 7$, and it is easy to check that $r = 3$ is the only case satisfying the equation. Hence, we have identified all (r, s)-tilings in which the fraction of peripheral polygons tends to φ for large n.

Theorem 7. *There are only two regular (r, s)-tilings such that the fraction of peripheral polygons tends to the reciprocal value of the Golden Section. Those two tilings are the regular $(7, 3)$ and $(3, 7)$ tilings of the hyperbolic plane.*

We close this section by mentioning that many regular (r, s)-tilings have growth rates that are positive integer powers of ϕ. It has been known for a long time [2] that the polynomials $x^2 - L_n x + (-1)^n$ have the roots ϕ^n and $(-\varphi)^n$. (Here L_n denotes the n-th Lucas number.) For example, the case $n = 4$ yields the polynomial $x^2 - 7x + 1$ with the roots ϕ^4 and φ^4 that arises, rather trivially, as the characteristic polynomial of recurrences for the number of polygons in the n-th layer of a dual pair of regular $(11, 3)$ and $(3, 11)$ tilings, and also, somewhat less trivially, for the $(5, 5)$-tiling. However, only the case $n = 2$ results in the fraction of peripheral polygons which stands in a simple relation to the Golden Section. We leave to the interested reader to investigate for what other regular (r, s)-tilings their growth rates are positive integer powers of the Golden Section.

5 Concluding remarks

In this contribution we have explored some properties of regular hyperbolic tilings and identified all cases where the Golden Section appears as the fraction of peripheral cells in the bulk of a tiling. We found that there are only two such tilings, the dual pair $(7, 3)$ and $(3, 7)$. In all other possible cases the fraction of periphery in the total exceeds φ. Our intention was to point out an appearance of the Golden Section in a geometric context outside its usual habitat of polygons and polyhedra with pentagonal symmetry. Although it seems that heptagons and triangles play equally important roles in our findings, we have chosen to emphasize the heptagonal connections since the triangular ones are far better known. We leave to the interested reader to decide whether the fact that the number five lies halfway between three and seven has any relevance for our findings.

The results presented in this note are certainly not new. Most likely they were not new even back in 2003 when I first obtained them as a spin-off from a paper concerned with certain types of fullerene graphs. Since then, several papers appeared addressing some related but more general problems [4, 6, 13], and most of the present results can be obtained by specializing certain parameters in the main results of references [7–9]. However, the results are not widely known either – only a handful of the enumerating sequences are present in the *On-Line Encyclopedia of Integer Sequences*, and mostly without references to the corresponding tessellations. Croatian readers might find it interesting that the study of the objects considered here seems to be initiated by S. Bilinski in his Ph. D. thesis [3].

I am aware of one more heptagonal connection of the Golden Section: A closely related quantity, $\sqrt{5} = \phi + \varphi$, appears as the limit of ratios of certain quantities relevant for the study of sandpile models on the regular heptagonal tiling [11]. An interested reader might wish to explore whether the heptagonal tiling is the only regular hyperbolic tiling with this property.

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A VARIATION ON THE RUBIK'S CUBE

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Abstract. The Rubik's cube is a famous puzzle in which faces can be moved and the corresponding movement operations define a group. We consider here a generalization to any 3-valent map. We prove an upper bound on the size of the corresponding group which we conjecture to be tight.

1. Introduction

The Rubik's cube is a 3-dimensional toy in which each face of the cube is movable. There has been extensive study of its mathematics (see [7]). On the play side the Rubik's Cube has led to the creation of many different variants (Megaminx, Pyraminx, Tuttminx, Skewb diamond, etc.) The common feature of those variants is that they are all physical and built as toys.

Our idea is to extend the original rubik's cube to any 3-valent map M on any surface. To any face of the map we associate one transformation. The full group of such transformations is named $Rubik(M)$ and we study its size and its natural normal subgroups.

A well studied extension [5, 4, 3, 8] of the Rubik's cube is the $n \times n \times n$-cube where the 3-lanes of the cubes are extended to n. There has also been interest [10] in cryptographic applications of Rubik's Cube. Thus the large class of groups that we build could be of wide interest in computer science.

In Section 2, we construct the Rubik's cube transformation of the map. In Section 3, we explain how to use existing computer algebra systems such as `GAP` in order to work with the Rubik's groups considered. In Section 4 we define several subgroups of the Rubik group and prove a bound on the size of the Rubik group in the oriented case. In Section 5 several possible extensions and further works are mentioned.

2. The construction of the Rubik group from a 3-valent map M

In this section we use M to denote a 3-valent map. By $V(M)$, $E(M)$, respectively $Face(M)$, we denote the set of vertices, edges, respectively faces of M.

For any face F of M we define a *side movement* $sm(M, F)$ to be a movement in one direction of a face F. This is illustrated in Figure 1. If the face contains p edges then the element $sm(M, F)$ has order p.

For a set S of elements of a group G we define $Group(S)$ to be the group generated by S. Using this we can define the Rubik's group:

$$Rubik(M) = Group(\{sm(M, F) \text{ for } F \in Face(M)\})$$

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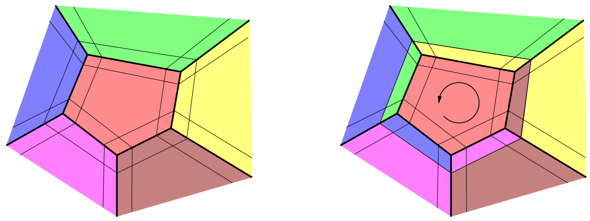

Figure 1. The action of the side movement

Rubik(*Cube*): Rubik's Cube.

Rubik(*Dodecahedron*): Megaminx

Figure 2. The Rubik construction for Cube and Dodecahedron

For the Cube and Dodecahedron this is known as Rubik's cube and Megaminx, see Figure 2 where the toy can be represented physically.

A *corner* of a map M is a pair (F, v) with v a vertex contained in a face F. A *side edge* of a map M is a pair (F, e) with e an edge contained in a face F. A 3-regular map M with v vertices has exactly $e = 3v/2$ edges. It will have $3v$ corners and $2e = 3v$ side edges. By $Corner(M)$, we define the set of corners of M.

The face movement $sm(M, F)$ acts on the set of corners and the set of side edges. This corresponds directly to the faces of the Rubik's cube.

3. The computer Algebra side of things

We present the group $Rubik(M)$ as a permutation group on the corners and side edges. This way of presenting the group follows [11] [1] where the standard Rubik's cube was considered.

For the example of $Prism_3$ plane graph, we can number the corners and side-edges according to Figure 3. The side-movements generators of $Rubik(Prism_3)$ can then be expressed as the following permutations:

```
gap> RubikPrism3:=Group([(31,35,33)(36,34,32)(3,19,11)(5,21,13)(8,24,16),
  (28,27,25)(30,28,26)(6,14,22)(4,12,20)(1,9,17),
  (1,6,8,3)(4,7,5,2)(17,35,16,25)(19,31,14,27)(18,36,15,26),
  (19,17,22,24)(18,20,23,21)(7,28,10,34)(8,27,9,33)(6,29,11,35),
  (9,14,16,11)(12,15,13,10)(1,31,24,29)(2,32,23,30)(3,33,22,25)]);
```

The advantage of this approach is that there are very efficient implementations of permutation group algorithms (GAP) which can address in an efficient way many questions one may have about a permutation group.

[1] See also an example in https://www.gap-system.org/Doc/Examples/rubik.html

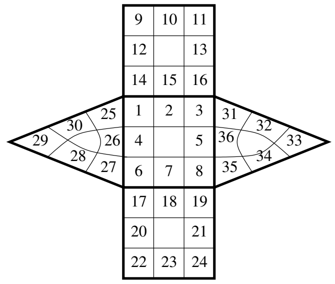

Figure 3. Numbering of corners and side-edge of $Prism_3$

One famous problem in Rubik's cube group theory is to find the minimal number of moves to go from one configuration to another, in other words to find the diameter of the Cayley graph of the group with the side moves as generators (see [9]). The `GAP` computer algebra software does not provide a solution to that problem, but it allows to express an element of the group in terms of the generators, that is find an expression, possibly not minimal.

More specific functionality for the computation is available in the GAP package [6].

4. Subgroups of $Rubik(M)$

The group $Rubik(M)$ acts not only of the corners and side edges, but also on the vertices and edges. To represent those actions we use subscript such as $sm(M,F)_{edge}$ and denote the corresponding group $Rubik(M)_{edge}$.

Lemma 1. *Given a map M and a face F we have:*

(i) The signature of $sm(M,F)_{side\,edge}$ is 1.

(ii) The signature of $sm(M,F)_{vertex}$ is equal to the signature of $sm(M,F)_{edge}$.

Proof. (i) If F is a face with p sides then F contains p side edges (F,e), each side edges is adjacent to another side edge (F_e,e). The side generator $sm(M,F)$ has one cycle formed by the side-edges (F,e) and another cycle formed by the side edges (F_e,e). Each of those cycles has the same length and so the signature of $sm(M,F)$ acting on the side-edges is 1.

(ii) If F is a face with p sides then it contains p edges and p vertices. The action on each of them is a cycle of length p. So their signature is the same. □

We now consider what additional properties can be obtained if the map M is oriented.

Definition 1. *Assume M is an oriented map. Every vertex v of M is contained in 3 corners $C_v = \{c_1, c_2, c_3\}$. Since M is orientable, we can assume that C_v is oriented in the direct way.*

We denote by $OrMap(M)$ the set of permutations of the corners that preserve the triples and their orientation.

Given two corners c, c' we call $sh(c,c')$ the number k of rotations r by $2\pi/3$ such that $r^k(c) = c'$.

The orientability of the map will allow us to restrict the possibilities of mapping the corners. We need the $OrMap(M)$ set of map in order to define some properties below for the Rubik's cube group. The shift is more complicated. If a vertex v has $C_v = \{c_1, c_2, c_3\}$ in direct orientation then we have e.g. $sh(c_i, c_i) = 0$, $sh(c_1, c_2) = 1$ and $sh(c_1, c_3) = 2$.

Theorem 1. *Let M be an oriented map. The following holds:*
(i) For c, c', c'' corners we have $sh(c, c'') = sh(c, c') + sh(c', c'') \pmod 3$
(ii) For a function $\phi : V(M) \to Corner(M)$ such that $\phi(v)$ is a corner of V and a function $f \in OrMap(M)$ the expression

$$\sum_{v\in V(M)} sh(f(\phi(v)), \phi(f(v)))$$

is independent of ϕ and denoted $sh(f)$.
(iii) If $f, g \in OrMap(M)$ then we have

$$sh(f \circ g) = sh(f) + sh(g) \pmod 3$$

(iv) sh is a non-trivial function on $OrMap(M)$.
(v) For $f \in Rubik(M)$ we have $sh(f) = 0 \pmod 3$.

Proof. (i) The equality follows by additivity of the rotational shift.
(ii) Let ϕ, ϕ' two such functions. Then we define $r(v) = sh(\phi(v), \phi'(v))$. Define first $a(\phi) = \sum_{v\in V(M)} sh(f(\phi(v)), \phi(f(v)))$, we then have:

$$\begin{array}{rcl}
a(\phi') & = & \sum_{v\in V(M)} sh(f(\phi'(v)), f(\phi(v)) + sh(f(\phi(v)), \phi(f(v)) \\
& & +sh(\phi(f(v)), \phi'(f(v))) \\
& = & a(\phi) + \sum_{v\in V(M)} sh(f(\phi'(v)), f(\phi(v)) + \sum_{v\in V(M)} sh(\phi(f(v)), \phi'(f(v))) \\
& = & a(\phi) + \sum_{v\in V(M)} sh(\phi'(v)), \phi(v)) + \sum_{v\in V(M)} sh(\phi(f(v)), \phi'(f(v))) \\
& = & a(\phi) + \sum_{v\in V(M)} sh(\phi'(v)), \phi(v)) + \sum_{v\in V(M)} sh(\phi(v)), \phi'(v)) \\
& = & a(\phi) + \sum_{v\in V(M)} sh(\phi'(v)), \phi(v)) + sh(\phi(v)), \phi'(v)) \\
& = & a(\phi) + \sum_{v\in V(M)} sh(\phi'(v)), \phi'(v)) \\
& = & a(\phi)
\end{array}$$

The equality $sh(f(\phi'(v)), f(\phi(v))) = sh(\phi'(v), \phi(v))$ comes from the fact that f preserves the orientation of the corners. The equality $\sum_{v\in V(M)} sh(\phi(f(v)), \phi'(f(v))) = \sum_{v\in V(M)} sh(\phi(v), \phi'(v))$ comes from the fact that f is permuting the vertices of M.
(iii) Let f and g be two such mappings. We then have

$$\begin{array}{rcl}
sh(f \circ g) & = & \sum_{v\in V(M)} sh(f(g(\phi(v))), \phi(f(g(v)))) \\
& = & \sum_{v\in V(M)} sh(f(g(\phi(v))), f(\phi(g(v)))) + sh(f(\phi(g(v))), \phi(f(g(v)))) \\
& = & \sum_{v\in V(M)} sh(g(\phi(v)), \phi(g(v))) + \sum_{v\in V(M)} sh(f(\phi(v)), \phi(f(v))) \\
& = & sh(g) + sh(f)
\end{array}$$

We have used that f preserves orientation and the fact that g permutes the vertices in our rewriting.
(iv) Let us select a vertex v of M with $C_v = \{c_1, c_2, c_3\}$. We set $f(c_i) = c_{i+1}$ for $c_i \in C_v$ and $f(c) = c$ otherwise. We find that $sh(f) = 1$.
(v) Let F be a face of M. We select a function ϕ such that $phi(v) = (F, v)$ for v and vertex contained in F and an arbitrary choice on other vertices.

Setting $f = sm(M, F)$ we have

$$sh(f) = \sum_{v \in F \cap V(M)} sh(f(\phi(v)), \phi(f(v)))$$

It is easy to see that $\phi(f(v)) = f(\phi(v))$ for $v \in F \cap V(M)$. Thus we get $sh(f)$. Since the side movement generate $Rubik(M)$, by (iii) we get that for all $f \in Rubik(M)$, $sh(f) = 0$. □

Theorem 2. *We have a sequence of group homomorphisms:*

$$\begin{aligned} Rubik(M) = Rubik(M)_{corner,side\ edge} &\to Rubik(M)_{corner,edge} \\ &\to Rubik(M)_{corner} \to Rubik(M)_{vertex} \end{aligned}$$

Proof. The equality $Rubik(M) = Rubik(M)_{corner,side\ edge}$ comes from the definition of the Rubik group. If the group acts on the side-edges, then it acts on the edges and so we get the map $Rubik(M)_{corner,side\ edge} \to Rubik(M)_{corner,edge}$. Dropping the edge action gets us the mapping to $Rubik(M)_{corner}$. Finally, a vertex is contained in 3 faces and so in 3 corners. This defines a mapping from $Rubik(M)_{corner}$ to $Rubik(M)_{vertex}$. □

We define following subgroups from those homomorphisms:

$$\begin{cases} G_1(M) &= Ker(Rubik(M)_{corner,side\ edge} \to Rubik(M)_{corner,edge}) \\ G_2(M) &= Ker(Rubik(M)_{corner,edge} \to Rubik(M)_{corner}) \\ G_3(M) &= Ker(Rubik(M)_{corner} \to Rubik(M)_{vertex}) \end{cases}$$

We can now use the above decomposition in order to get a conjectural description of $Rubik(M)$.

Theorem 3. *Given an oriented map M the following holds:*

(i) $G_1(M)$ is a subgroup of $\mathbb{Z}_2^{|E(M)|-1}$

(ii) $G_2(M)$ is a subgroup of the alternating group $A_{|E(M)|}$.

(iii) $G_3(M)$ is a subgroup of $\mathbb{Z}_3^{|V(M)|-1}$

(iv) $Rubik(M)_{vertex}$ is a subgroup of $A_{|V(M)|}$ if all faces of M have odd size and a subgroup of $S_{|V(M)|}$ otherwise.

Proof. $G_1(M)$ is formed by all the transformations that preserve all corners and edges but may switch a side edge (F_1, e) into another side edges (F_2, e). Since every edge is contained in two faces, this makes $G_1(M)$ a subgroup of the commutative subgroup $\mathbb{Z}_2^{|E(M)|}$. The group $\mathbb{Z}_2^{|E(M)|}$ contains some elements that switch just two side-edges and are thus of signature -1. Thus by Lemma 1.(i) $G_1(M)$ is isomorphic to a strict subgroup of $\mathbb{Z}_2^{|E(M)|}$.

$G_2(M)$ is formed by the transformations that preserves the corners but will permutes the edges. Thus we have that $G_2(M)$ is a subgroup of $S_{|E(M)|}$. Since elements of $G_2(M)$ preserves all vertices, the signature of their action is 1. By Lemma 1.(ii) the signature of their action on edges is also 1. Therefore $G_2(M)$ is a subgroup of $A_{|E(M)|}$.

$G_3(M)$ is formed by all transformations that preserves the vertices but may permutes the corners. Since the corners are oriented, the operation on each vertex v may be encoded by an element x_v of $\mathbb{Z}_3$. By Theorem 1.(v) we have $sh(f) = 0$ for each $f \in G_3(M)$. Thus we get $\sum_{v \in V(M)} x_v = 0$. This means that $G_3(M)$ is a subgroup of $\mathbb{Z}_3^{|V(M)|-1}$.

FIGURE 4. The Pyraminx toy

The side movements $sm(M, F)$ act on the vertices. For a face of size p the signature is $(-1)^{p-1}$. Thus if all faces of M have odd size then all side movements have signature 1 and $Rubik(M)_{vertex}$ is a subgroup of $A_{|V(M)|}$. If there is a face of even size then it is a subgroup of $S_{|V(M)|}$. □

Conjecture 1. *("First law of cubology") Given an oriented map M with v vertices and e edges the following holds:*
(i) $G_1(M)$ is isomorphic to $\mathbb{Z}_2^{|E(M)|-1}$
(ii) $G_2(M)$ is isomorphic to $A_{|E(M)|}$.
(iii) $G_3(M)$ is isomorphic to $\mathbb{Z}_3^{|V(M)|-1}$
(iv) $Rubik(M)_{vertex}$ is isomorphic to $A_{|V(M)|}$ if all faces of M have odd size and $S_{|V(M)|}$ otherwise.

This conjecture has been checked for many plane graphs e.g. the ones with at most 40 vertices and faces of size 6 or p with $3 \le p \le 5$. This Theorem is proved for the cube case in [7, Theorem 11.2.1], [1, Section 2.4] and [2, Theorem 1.3.24].

5. Possible extensions and open problems

There are many possible extensions of this work. First we could consider other constructions so as to generalize all the existing Rubik's cube variant toys to a combinatorial setting. For example the Pyraminx (Figure 4) does not belong to the family described here.

Proving the conjecture on the description of $Rubik(M)$ appears to be a difficult problem as the cube proofs seem difficult to generalize. Another research question is to consider the classification of elements of order 2 of $Rubik(M)$ as well as the determination of the maximal order of the elements of $Rubik(M)$ (see [2] for the Cube case).

Much time and energy has been spent on playing on Rubik's cube and this could be done as well for $Rubik(M)$. One would have to forget the physical part and accept playing on a smartphone, which could be programmed reasonably easily. For plane graph this could be represented by a Schlegel diagram and for toroidal maps, we could use a plane representation with a group of symmetries.

Classifying the 3-valent graphs for which $Rubik(M)$ may be constructed physically is also an interesting problem.

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MATRIX PRODUCTS OF BINOMIAL COEFFICIENTS AND UNSIGNED STIRLING NUMBERS

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Dedicated to Professor Dragutin Svrtan on the occasion of his 70^th birthday.

ABSTRACT. We study sums of the form $\sum_{k=m}^{n} a_{nk}b_{km}$, where a_{nk} and b_{km} are binomial coefficients or unsigned Stirling numbers. In a few cases they can be written in closed form. Failing that, the sums still share many common features: combinatorial interpretations, Pascal-like recurrences, inverse relations with their signed versions, and interpretations as coefficients of change between polynomial bases.

MSC: 05A10

1. Introduction

Let a_{nk} and b_{nk} be binomial coefficients $\binom{n}{k}$ or unsigned Stirling numbers of the first and second kind: $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ in Karamata-Knuth notation, respectively. We are interested in the sum $\sum_{k=m}^{n} a_{nk}b_{km}$. Denote by $A = [a_{nk}]$ and $B = [b_{nk}]$ the corresponding infinite lower-triangular matrices, indexed by the non-negative integers. The sum can be interpreted as the (n,m)-entry of the matrix product $A \cdot B$.

Our motivation are the following two sums, that can be written in closed form:

$$\sum_{k=m}^{n} \binom{n}{k} \left\{{k \atop m}\right\} = \left\{{n+1 \atop m+1}\right\}, \tag{1}$$

$$\sum_{k=m}^{n} \left[{n \atop k}\right] \binom{k}{m} = \left[{n+1 \atop m+1}\right]. \tag{2}$$

These are identities (6.15) and (6.16) in the book *Concrete Mathematics* [1]. Furthermore, the following sum are the Lah numbers [2, 3], denoted by $L(n,m)$:

$$\sum_{k=m}^{n} \left[{n \atop k}\right] \left\{{k \atop m}\right\} = \frac{n!}{m!}\binom{n-1}{m-1}. \tag{3}$$

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$\sum_k \binom{n}{k}\binom{k}{m}$	A038207	$\sum_k \binom{n}{k}\left[{k \atop m}\right]$	A094816	$\sum_k \binom{n}{k}\left\{{k \atop m}\right\}$	A008277
$\sum_k \left[{n \atop k}\right]\binom{k}{m}$	A130534	$\sum_k \left[{n \atop k}\right]\left[{k \atop m}\right]$	A325872*	$\sum_k \left[{n \atop k}\right]\left\{{k \atop m}\right\}$	A271703
$\sum_k \left\{{n \atop k}\right\}\binom{k}{m}$	A049020	$\sum_k \left\{{n \atop k}\right\}\left[{k \atop m}\right]$	A129062	$\sum_k \left\{{n \atop k}\right\}\left\{{k \atop m}\right\}$	A130191

TABLE 1. References for the sums in the On-line encyclopedia of integer sequences [7].

Closed-form expressions for two more sums are given in the next section. In the remaining cases we do not know closed forms, but the sums share many common features. Combinatorial interpretations are outlined in Section 2. Row sums of the matrix $A \cdot B$ are discussed in Section 3. Recurrences similar to Pascal's formula are proved in Section 4. Inverse relations with signed versions of the sums are given in Section 5, and in Section 6 the sums are interpreted as coefficients of change between various polynomial bases.

Our main focus are sums with a_{nk} and b_{nk} equal to $\binom{n}{k}$, $\left[{n \atop k}\right]$ or $\left\{{n \atop k}\right\}$, but we include some results on sums with $L(n,k)$ as well. Our nine main sums are listed in the On-line encyclopedia of integer sequences [7] as "triangles read by rows", see Table 1 (the entry marked by * is a signed version). Many properties of the sums are reported in [7]. We repeat some properties from [7], and reveal some new properties of the sums. We mainly deal with properties that can be proved by counting arguments.

2. Combinatorial interpretations and closed forms

Identity (1) follows from the usual combinatorial interpretation of Stirling numbers of the second kind. The right-hand side $\left\{{n+1 \atop m+1}\right\}$ counts partitions of an $(n+1)$-element set S into $m+1$ blocks. The left-hand side is obtained by distinguishing an element of S and assuming it is covered by a block containing $n-k$ other elements of S.

The Stirling number of the first kind $\left[{n \atop m}\right]$ is usually interpreted as the number of permutations of degree n with exactly m cycles. For the proof of identity (2), however, another combinatorial interpretation is more useful. Let Δ_n denote the board that remains after removing all tiles below and on the side diagonal of a $n \times n$ square board. Consider a new chess piece similar to a rook, but that only attacks tiles in its row. Loehr [4] calls this piece a weak rook or wrook. He goes on to show that the number of ways of placing k identical wrooks on Δ_n so that they don't attack each other is equal to $\left[{n \atop n-k}\right]$. The right-hand side of (2) can now be interpreted as the number of ways of placing $n-m$ non-attacking wrooks on Δ_{n+1}. The left-hand side is obtained by partitioning the set of placements based on the number of wrooks not in the first column, but on the remaining Δ_n subboard. If $n-k$ wrooks are placed on the Δ_n subboard, the remaining $k-m$ wrooks must be placed in k rows of the first column that are not attacked. This can be done in $\left[{n \atop k}\right]\binom{k}{k-m} = \left[{n \atop k}\right]\binom{k}{m}$ ways.

In [6], a nice combinatorial interpretation of the Lah numbers is given: $L(n,m)$ counts partitions of an n-element set into m lists, i.e. non-empty totally ordered subsets. From this, identity (3) follows easily. Two more closed-form identities are

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m} \tag{4}$$

and

$$\sum_{k=m}^{n} L(n,k)\, L(k,m) = 2^{n-m} L(n,m). \tag{5}$$

The left-hand side of (4) counts pairs (K,M), where $K \supseteq M$ are subsets of a fixed n-element set S, and $|M| = m$. The right-hand side is obtained by choosing M first. Identity (5) has an analogous proof using the combinatorial interpretation of Lah numbers from [6]. For two partitions Π_1, Π_2 of S into lists, we write $\Pi_1 \leq \Pi_2$ and say that Π_1 is a refinement of Π_2 if each list in Π_1 is a sublist of some list in Π_2. The left-hand side of (5) counts pairs (Π_1, Π_2) with $\Pi_1 \leq \Pi_2$ and $|\Pi_2| = m$. We first partition S into Π_1 with $|\Pi_1| = k \geq m$, then partition these k lists into m lists containing lists of Π_1 as elements, and finally concatenate the list of lists into elements of Π_2. We get the right-hand side by partitioning S into m lists of Π_2 first and splitting them up into smaller lists of Π_1. This can be done in 2^{n-m} ways: breaks can be made in $n-m$ places, before every element except at the beginnings of the lists of Π_2.

Similar combinatorial interpretations can be given for the double Stirling sums $\sum_{k=m}^{n} \left\{{n \atop k}\right\}\left\{{k \atop m}\right\}$ and $\sum_{k=m}^{n} \left[{n \atop k}\right]\left[{k \atop m}\right]$. The former is the number of pairs (Π_1, Π_2) of ordinary partitions of an n-element set into blocks (non-empty subsets), with Π_1 a refinement of Π_2 and $|\Pi_2| = m$. The latter is the number of pairs of permutations (π_1, π_2), where π_1 is of degree n, π_2 is of degree equal to the number of cycles of π_1, and π_2 has exactly m cycles. However, since we do not see an easy way to count the pairs if Π_2 or π_2 are chosen first, this does not lead to closed-form expressions for these two sums.

Combinatorial interpretations of the remaining sums in Table 1 are as follows. The sum $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \binom{k}{m}$ is the number of ordinary partitions of an n-element set with m blocks colored red, and the other blocks colored blue. The sum $\sum_{k=m}^{n} \binom{n}{k} \left[{k \atop m} \right]$ counts permutations with exactly m cycles of all subsets of an n-element set. Finally, $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left[{k \atop m} \right]$ is the number of pairs (Π, π), where Π is a partition of an n-element set into blocks, and π is a permutation with exactly m cycles of degree equal to the number of blocks of Π. Again, this does not lead to closed-form expressions, but the combinatorial interpretations of the sums will be used in the following sections to prove their properties.

3. Row sums

From the combinatorial interpretation of binomial coefficients and Stirling numbers, it is clear that $\sum_{m=0}^{n} \binom{n}{m} = 2^n$, $\sum_{m=0}^{n} \left[{n \atop m} \right] = n!$, and $\sum_{m=0}^{n} \left\{ {n \atop m} \right\} = B_n$. Here B_n is the n-th Bell number, i.e. the total number of partitions of an n-element set into blocks. In this section the goal is to determine

$$\sum_{m=0}^{n} \sum_{k=m}^{n} a_{nk} b_{km}.$$

This is the sum of the n-th row of the matrix $A \cdot B$.

From identities (1) and (2), we have

$$\sum_{m=0}^{n} \sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} = B_{n+1} \quad \text{and} \quad \sum_{m=0}^{n} \sum_{k=m}^{n} \left[{n \atop k} \right] \binom{k}{m} = (n+1)!.$$

Another row sum that can be written in closed form is

$$\sum_{m=0}^{n} \sum_{k=m}^{n} \binom{n}{k} \binom{k}{m} = 3^n.$$

The left-hand side counts the total number of pairs (K, M) of subsets of an n-set S with $K \supseteq M$. An alternative way of counting is to decide for each element $x \in S$ whether it is in M, in $K \setminus M$, or in $S \setminus K$, leading to the right-hand side.

The row sum of Lah numbers

$$\sum_{m=0}^{n} \sum_{k=m}^{n} \left[{n \atop k} \right] \left\{ {k \atop m} \right\} = \sum_{m=0}^{n} L(n, m)$$

can be interpreted as the total number of partitions of an n-set into lists. This is the "Lah equivalent" of the Bell number B_n and we will denote it by L_n. In [7], this is sequence number A000262.

Row sums of the remaining matrix products are given in the sequel. They have nice combinatorial interpretations and can be simplified to single sums.

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\left\{{n \atop k}\right\}\left[{k \atop m}\right]=\sum_{m=0}^{n}\left\{{n \atop m}\right\}m!.$$

These are the Fubini numbers, sequence A000670 in [7]. They count ordered partitions of an n-set, or weak orders on n elements.

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\left[{n \atop k}\right]\left[{k \atop m}\right]=\sum_{m=0}^{n}\left[{n \atop m}\right]m!.$$

This is the number of ordered factorizations of permutations of degree n into cycles, sequence A007840 in [7].

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\left\{{n \atop k}\right\}\binom{k}{m}=\sum_{m=0}^{n}\left\{{n \atop m}\right\}2^m.$$

The total number of partitions of an n-set with blocks colored red or blue. This is sequence A001861 in [7].

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\binom{n}{k}\left[{k \atop m}\right]=\sum_{m=0}^{n}\frac{n!}{m!}.$$

The total number of lists with elements from an n-set. Sequence number A000522 in [7].

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\left\{{n \atop k}\right\}\left\{{k \atop m}\right\}=\sum_{m=0}^{n}\left\{{n \atop m}\right\}B_m.$$

The total number of pairs (Π_1,Π_2) of partitions of an n-set with $\Pi_1 \leq \Pi_2$. In [7], this is sequence number A000258.

4. Pascal-like recurrences

The binomial coefficients can be computed by Pascal's formula

$$\binom{n}{m}=\binom{n-1}{m-1}+\binom{n-1}{m},\quad \binom{n}{0}=\binom{n}{n}=1.$$

Analogous recurrences for Stirling numbers are

$$\left[{n \atop m}\right]=\left[{n-1 \atop m-1}\right]+(n-1)\left[{n-1 \atop m}\right],\quad \left[{n \atop 0}\right]=\delta_{n0},\quad \left[{n \atop n}\right]=1,$$

$$\left\{ n \atop m \right\} = \left\{ n-1 \atop m-1 \right\} + m \left\{ n-1 \atop m \right\}, \quad \left\{ n \atop 0 \right\} = \delta_{n0}, \quad \left\{ n \atop n \right\} = 1,$$

and for Lah numbers

$$L(n,m) = L(n-1,m-1) + (n+m-1)L(n-1,m)$$

with boundary values $L(n,0) = \delta_{n0}$ (the Kronecker delta), $L(n,n) = 1$. See [6] for proofs of the formulae by distinguishing an element of the underlying n-set S and counting. Our sums satisfy similar recurrences that can also be established by counting arguments.

To make the formulae more readable, we denote the sum $\sum_{k=m}^{n} a_{nk}\, b_{km}$ by $\left| n \atop m \right|$. For example, the double binomial sum $\left| n \atop m \right| = \sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}$ can be computed from

$$\left| n \atop m \right| = \left| n-1 \atop m-1 \right| + 2 \left| n-1 \atop m \right|, \quad \left| n \atop 0 \right| = 2^n, \quad \left| n \atop n \right| = 1.$$

This sum also satisfies the absorption identity $\left| n \atop m \right| = \frac{n}{m} \left| n-1 \atop m-1 \right|$, just like the binomial coefficients.

By (1), the sum $\left| n \atop m \right| = \sum_{k=m}^{n} \binom{n}{k} \left\{ k \atop m \right\}$ are shifted Stirling numbers of the second kind. Therefore,

$$\left| n \atop m \right| = \left| n-1 \atop m-1 \right| + (m+1) \left| n-1 \atop m \right|, \quad \left| n \atop 0 \right| = \left| n \atop n \right| = 1.$$

Similarly, by (2), the sum $\left| n \atop m \right| = \sum_{k=m}^{n} \left[n \atop k \right] \binom{k}{m}$ satisfies

$$\left| n \atop m \right| = \left| n-1 \atop m-1 \right| + n \left| n-1 \atop m \right|, \quad \left| n \atop 0 \right| = n!, \quad \left| n \atop n \right| = 1.$$

The sum $\left| n \atop m \right| = \sum_{k=m}^{n} \binom{n}{k} L(k,m)$ is sequence A271705 in [7], where the following recurrence is given:

$$\left| n \atop m \right| = \frac{n}{m} \left| n-1 \atop m-1 \right| + n \left| n-1 \atop m \right|, \quad \left| n \atop 0 \right| = \left| n \atop n \right| = 1.$$

The sum $\left| n \atop m \right| = \sum_{k=m}^{n} L(n,k)\binom{k}{m}$ is sequence A059110. It satisfies the same recurrence, but with different boundary values $\left| n \atop 0 \right| = L_n$.

The double Lah sum $\left| n \atop m \right| = \sum_{k=m}^{n} L(n,k)L(k,m)$ satisfies

$$\left| n \atop m \right| = \frac{n}{m} \left| n-1 \atop m-1 \right| + 2n \left| n-1 \atop m \right|, \quad \left| n \atop 0 \right| = \delta_{n0}, \quad \left| n \atop n \right| = 1.$$

This sum also satisfies the absorption-like identity $\left| n \atop m \right| = \frac{n-m+1}{2m(m-1)} \left| n \atop m-1 \right|$, while the Lah numbers satisfy $L(n,m) = \frac{n-m+1}{m(m-1)} L(n,m-1)$.

Now let $\left|{n \atop m}\right| = \sum_{k=m}^{n} \binom{n}{k} \left[{k \atop m}\right]$. This sum cannot be computed from the two values $\left|{n-1 \atop m-1}\right|$ and $\left|{n-1 \atop m}\right|$ alone, but we can give a Pascal-like recurrence involving more previous values:

$$\left|\begin{matrix} n \\ m \end{matrix}\right| = \sum_{k=0}^{n-m} (n-1)^{\underline{k}} \left|\begin{matrix} n-1-k \\ m-1 \end{matrix}\right| + \left|\begin{matrix} n-1 \\ m \end{matrix}\right|, \quad \left|\begin{matrix} n \\ 0 \end{matrix}\right| = \left|\begin{matrix} n \\ n \end{matrix}\right| = 1.$$

Here $(n-1)^{\underline{k}} = (n-1)\cdot(n-2)\cdots(n-k)$ is the falling factorial. For the proof, recall that $\left|{n \atop m}\right|$ counts permutations with m cycles of subsets $T \subseteq S$, where S is a set of n elements. Fix an element $x \in S$ and divide the permutations according to whether they contain x or do not contain x. In the latter case there are clearly $\left|{n-1 \atop m}\right|$ permutations. In the former case, assume x is in a cycle with k other elements of S. This cycle can be chosen in $(n-1)^{\underline{k}}$ ways, and $m-1$ more cycles have to be chosen from the remaining $n-1-k$ elements. Thus, there are $\sum_{k=0}^{n-m} (n-1)^{\underline{k}} \left|{n-1-k \atop m-1}\right|$ permutations containing x.

The sum $\left|{n \atop m}\right| = \sum_{k=m}^{n} \left\{{n \atop k}\right\} \binom{k}{m}$ counts partitions of S with m blocks colored red, and the other blocks colored blue. Again, fix an element $x \in S$. If x is in a red block alone, there are $\left|{n-1 \atop m-1}\right|$ partitions. If x is in a red block with some other elements of S, there are $m\left|{n-1 \atop m}\right|$ partitions. Finally, if x is in a blue block with k other elements of S, there are $\binom{n-1}{k}\left|{n-1-k \atop m}\right|$ such partitions. Therefore, the following recursion holds:

$$\left|\begin{matrix} n \\ m \end{matrix}\right| = \left|\begin{matrix} n-1 \\ m-1 \end{matrix}\right| + m\left|\begin{matrix} n-1 \\ m \end{matrix}\right| + \sum_{k=0}^{n-m-1} \binom{n-1}{k} \left|\begin{matrix} n-1-k \\ m \end{matrix}\right|.$$

The boundary values are $\left|{n \atop 0}\right| = B_n$ and $\left|{n \atop n}\right| = 1$.

The sum $\left|{n \atop m}\right| = \sum_{k=m}^{n} \left\{{n \atop k}\right\} \left\{{k \atop m}\right\}$ counts pairs (Π_1, Π_2) of partitions of S, where Π_2 has m blocks and Π_1 is a refinement of Π_2. Now let the fixed element $x \in S$ be contained in a block of Π_2 of size k. We can choose this block in $\binom{n-1}{k-1}$ ways and partition it into blocks of Π_1 in B_k ways. The remaining blocks of Π_2 and Π_1 can be chosen in $\left|{n-k \atop m-1}\right|$ ways. Therefore,

$$\left|\begin{matrix} n \\ m \end{matrix}\right| = \sum_{k=1}^{n-m+1} \binom{n-1}{k-1} B_k \left|\begin{matrix} n-k \\ m-1 \end{matrix}\right|, \quad \left|\begin{matrix} n \\ 0 \end{matrix}\right| = \delta_{n0}, \quad \left|\begin{matrix} n \\ n \end{matrix}\right| = 1.$$

For the sum $\left|{n \atop m}\right| = \sum_{k=m}^{n} \left[{n \atop k}\right] \left[{k \atop m}\right]$ a similar argument leads to the recurrence

$$\left|\begin{matrix} n \\ m \end{matrix}\right| = \sum_{k=1}^{n-m+1} \binom{n-1}{k-1} \sum_{i=1}^{k} \left[\begin{matrix} k \\ i \end{matrix}\right] (i-1)! \left|\begin{matrix} n-k \\ m-1 \end{matrix}\right|, \quad \left|\begin{matrix} n \\ 0 \end{matrix}\right| = \delta_{n0}, \quad \left|\begin{matrix} n \\ n \end{matrix}\right| = 1,$$

and for the sum $\genfrac{|}{|}{0pt}{}{n}{m} = \sum_{k=m}^{n} \genfrac{\{}{\}}{0pt}{}{n}{k} \genfrac{[}{]}{0pt}{}{k}{m}$ to the recurrence

$$\genfrac{|}{|}{0pt}{}{n}{m} = \sum_{k=1}^{n-m+1} \binom{n-1}{k-1} \sum_{i=1}^{k} \genfrac{\{}{\}}{0pt}{}{k}{i} (i-1)! \genfrac{|}{|}{0pt}{}{n-k}{m-1}, \quad \genfrac{|}{|}{0pt}{}{n}{0} = \delta_{n0}, \quad \genfrac{|}{|}{0pt}{}{n}{n} = 1.$$

However, these increasingly complex recurrences become less useful as the coefficients are more difficult to evaluate than the sum $\genfrac{|}{|}{0pt}{}{n}{m}$ directly.

5. Inverses

We denote signed versions of the Stirling numbers and their relatives by an exponent $^\sigma$, e.g. $\genfrac{[}{]}{0pt}{}{n}{m}^\sigma = (-1)^{n-m} \genfrac{[}{]}{0pt}{}{n}{m}$. For the matrix $A = [a_{nm}]$, we denote $A^\sigma = [a^\sigma_{nm}] = [(-1)^{n-m} a_{nm}]$. To avoid excessive bracketing, we write $\genfrac{[}{]}{0pt}{}{n}{m}^{-1}$ for the inverse matrix A^{-1}.

It is well-known that $\binom{n}{m}^{-1} = \binom{n}{m}^\sigma$, $\genfrac{[}{]}{0pt}{}{n}{m}^{-1} = \genfrac{\{}{\}}{0pt}{}{n}{m}^\sigma$, and $\genfrac{\{}{\}}{0pt}{}{n}{m}^{-1} = \genfrac{[}{]}{0pt}{}{n}{m}^\sigma$. From this and the properties $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$ and $(A \cdot B)^\sigma = A^\sigma \cdot B^\sigma$, we can determine inverses of our sums. For example, let $a_{mn} = \genfrac{[}{]}{0pt}{}{n}{m}$ and $b_{mn} = \genfrac{\{}{\}}{0pt}{}{n}{m}$. Then $A \cdot B$ are the Lah numbers $L(n, m)$ and we have

$$L(n, m)^{-1} = (A \cdot B)^{-1} = B^{-1} \cdot A^{-1} = A^\sigma \cdot B^\sigma = (A \cdot B)^\sigma = L(n, m)^\sigma.$$

Similarly it follows

$$\left(\sum_{k=m}^{n} \binom{n}{k} \genfrac{\{}{\}}{0pt}{}{k}{m} \right)^{-1} = \left(\sum_{k=m}^{n} \genfrac{[}{]}{0pt}{}{n}{k} \binom{k}{m} \right)^{\sigma},$$

$$\left(\sum_{k=m}^{n} \genfrac{\{}{\}}{0pt}{}{n}{k} \binom{k}{m} \right)^{-1} = \left(\sum_{k=m}^{n} \binom{n}{k} \genfrac{[}{]}{0pt}{}{k}{m} \right)^{\sigma},$$

$$\left(\sum_{k=m}^{n} \genfrac{\{}{\}}{0pt}{}{n}{k} \genfrac{\{}{\}}{0pt}{}{k}{m} \right)^{-1} = \left(\sum_{k=m}^{n} \genfrac{[}{]}{0pt}{}{n}{k} \genfrac{[}{]}{0pt}{}{k}{m} \right)^{\sigma},$$

and

$$\left(\sum_{k=m}^{n} \genfrac{\{}{\}}{0pt}{}{n}{k} \genfrac{[}{]}{0pt}{}{k}{m} \right)^{-1} = \left(\sum_{k=m}^{n} \genfrac{\{}{\}}{0pt}{}{n}{k} \genfrac{[}{]}{0pt}{}{k}{m} \right)^{\sigma}.$$

The exponents $^{-1}$ and $^\sigma$ can be exchanged in the formulae above.

6. Polynomial bases

We denote the falling factorials by $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ and the rising factorials by $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$, following [1]. It is known that Stirling numbers of the second kind are coefficients of change from the standard polynomial basis of powers $(x^n \mid n \in \mathbb{N}_0)$ to the basis of falling factorials $(x^{\underline{m}} \mid m \in \mathbb{N}_0)$:

$$x^n = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} x^{\underline{m}}. \tag{6}$$

Stirling numbers of the first kind are coefficients of change from $(x^{\overline{n}})$ to (x^m):

$$x^{\overline{n}} = \sum_{m=0}^{n} \left[{n \atop m} \right] x^{m}. \tag{7}$$

From (6) and (7) it follows that coefficients of change from $(x^{\overline{n}})$ to $(x^{\underline{m}})$ are the Lah numbers $L(n,m) = \sum_{k=m}^{n} \left[{n \atop k} \right] \left\{ {k \atop m} \right\}$:

$$x^{\overline{n}} = \sum_{m=0}^{n} L(n,m) x^{\underline{m}}.$$

Ivo Lah's original definition of his numbers [2, 3] was a signed version of this relation. We concentrate on changes between polynomial bases with non-negative coefficients. The opposite changes have inverse coefficients, with alternating signs as shown in the previous section.

From the binomial theorem $(1+x)^n = \sum_{m=0}^{n} \binom{n}{m} x^m$ and (6) we see that the sums $\sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\}$ can be interpreted as coefficients of change from the polynomial basis $((1+x)^n)$ to the basis of falling factorials $(x^{\underline{m}})$:

$$(1+x)^n = \sum_{m=0}^{n} \left(\sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} \right) x^{\underline{m}}.$$

Similar interpretations can be given to other sums from Table 1. The double binomial sums $\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}$ are coefficients of change from the basis $((2+x)^n)$ to the standard basis (x^m). The sums $\sum_{k=m}^{n} \binom{n}{k} \left[{k \atop m} \right]$ are coefficients of change from the basis $(\sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}})$ to the standard basis. The former basis contains a special case of Charlier polynomials [7], a family of orthogonal polynomials that can be written in terms of the generalized hypergeometric function (see [5]). Double Stirling sums of the second kind $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left\{ {k \atop m} \right\}$ are coefficients of change from the basis of Bell polynomials $(B_n(x))$, $B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$ to the basis of falling factorials $(x^{\underline{m}})$.

The two families of sums $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left[{k \atop m} \right]$ and $\sum_{k=m}^{n} \left[{n \atop k} \right] \left[{k \atop m} \right]$ can be seen as coefficients of the polynomials $\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\overline{k}}$ and $\sum_{k=0}^{n} \left[{n \atop k} \right] x^{\overline{k}}$, i.e. coefficients of change to the standard basis (x^m). Similarly, the two families $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \binom{k}{m}$ and $\sum_{k=m}^{n} \left[{n \atop k} \right] \binom{k}{m}$ are coefficients of Bell polynomials $B_n(1+x)$ and polynomials $\sum_{k=0}^{n} \left[{n \atop k} \right] (1+x)^k$, respectively.

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New Class of Binomial Sums and Their Applications

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Abstract

We consider certain "S-sums" which are multiparametric generalizations $S(n, m; a_1, ..., a_l)$ of sums of powers of binomial coefficients. A new class of binomial sums, associated to the S-sums, is introduced. Also, some recursions for new sums are presented and proved.

We present three applications of our new sums. The first application is for the divisibility, by the central binomial coefficient, of the alternating sum of powers of binomial coefficients. Our proof is given more simply then originally by Calkin (1998) who used modular properties.

The second application is for the following binomial sum S: $\mathbb{Z}^{l+2}_{>=0} \to \mathbb{Z}$,

$$S(2n, m, a_1, a_2, \dots, a_l) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \prod_{i=1}^{l} \binom{a_i + k}{k} \binom{a_i + 2n - k}{2n - k};$$

which arises as a generalization of a known identity connected with a famous Dixon's formula. We claim that, if m is a positive integer, then $S(2n, m, a_1, a_2, \dots, a_l)$ is divisible by $\binom{2n}{n}$ and $\binom{a_i+n}{n}$ for all $i = 1, \dots, l$. We give a proof for the case $l = 1$.

As a third application, we present an interesting connection between our new sums and one Theorem by Guo, Jouhet and Zeng.

1 Introduction

Let m, n, and k be non-negative integers such that $m \geq 1$. We consider the sum

$$S(n, m) = \sum_{k=0}^{n} \binom{n}{k}^m F(n, k), \tag{1}$$

where $F(n, k)$ is an integer-valued function that depends only on n and k.

The aim is to examine some divisibility properties of sums of the form $S(n, m)$. To do this, we introduce the notion of "M sums".

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Definition 1. Let n, j, and t be non-negative integers such that $j \le \lfloor \frac{n}{2} \rfloor$. Then the M sums for $S(n,m)$ are as follows:

$$M_S(n,j,t) = \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} \binom{n}{j+v}^t F(n,j+v). \tag{2}$$

Obviously, for $m \ge 1$, the equation

$$S(n,m) = M_S(n,0,m-1) \tag{3}$$

holds.

Hence, we can see the sum $M_S(n,j,m-1)$ as a generalization of the sum $S(n,m)$.

Let n, j, and t be as in Definition 1. We present our main theorem:

Theorem 2.

$$M_S(n,j,t+1) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t).$$

Theorem 2 gives a recursive definition of a M_S sum.

For proving Theorem 2, we use a method of "D sums".

2 Background

Recently, in our previous paper [8], the notion of "D sums" was introduced.

Let n, j, and t be as in Definition 1; and let S be as in Eq. (1).

Then the D sums for $S(n,m)$ are

$$D_S(n,j,t) = \sum_{l=0}^{n-2j} \binom{n-j}{l} \binom{n-j}{j+l} \binom{n}{j+l}^t F(n,j+l). \tag{4}$$

For $m \ge 2$, by Eq. (4), it follows that

$$S(n,m) = D_S(n,0,m-2). \tag{5}$$

Furthermore, D sums satisfy the following two recurrence relations [8, Thm. 2, p. 2], [11, Eqns. (14) and (15), p. 4]:

$$D_S(n,j,t+1) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n}{j+u} \binom{n-j}{u} D_S(n,j+u,t), \tag{6}$$

$$D_S(n,j,0) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} \binom{n-j-u}{j+u} \sum_{v=0}^{n-2j-2u} \binom{n-2j-2u}{v} F(n,j+u+v). \tag{7}$$

In 1998, Calkin [1, Thm. 1] proved that the alternating binomial sum $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin's result [6, Thm. 1.2, Thm. 1.3, p. 2].

The first application of D sums [8, Section 8] was for proving Calkin's result [1, Thm. 1]. Also, by using D sums, it was proved [8] that $\sum_{k=0}^{2n}\binom{2n}{k}^m|n-k|$ is divisible by $n\binom{2n}{n}$ for all non-negative integers n and all positive integers m.

By the same method, it was proved [11, Thm. 1] that $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m\binom{2k}{k}\binom{4n-2k}{2n-k}$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. Furthermore, it was proved [11, Corollary 4] that $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m C_k C_{2n-k}$ is divisible by $\binom{2n}{n}$ for all non-negative integers n; and for all positive integers m.

Let $S(k,l)=\frac{\binom{2k}{k}\binom{2l}{l}}{\binom{k+l}{k}}$ denote the k-th super Catalan number of order l. Recently, it was proved [10], that $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m S(k,l)S(2n-k,l)$ is divisible by $S(n,l)$ for all non-negative integers n and l and all positive integers m.

We will show that there is a close relationship between M-sums and D-sums.

3 Proof of Theorem 2

We need one auxiliary result.

We will use a symmetry of binomial coefficients and the well-known binomial identity

$$\binom{a}{b}\binom{b}{c}=\binom{a}{c}\binom{a-c}{b-c}; \tag{8}$$

where a,b,c are non-negative integers such that $a\geq b\geq c$.

Lemma 3. *Let n and j be non-negative integers; and let t be a positive integer. Then the following equation*

$$M_S(n,j,t)=\binom{n}{j}D_S(n,j,t-1). \tag{9}$$

holds.

Proof. By Eq. (2), we know that

$$M_S(n,j,t)=\sum_{v=0}^{n-2j}\binom{n-j}{j}\binom{n-2j}{v}\binom{n}{j+v}^t F(n,j+v). \tag{10}$$

Starting from Eq. (10), by using Eq. (8) and a symmetry of binomial coefficients, it is not hard to prove that:

$$M_S(n,j,t) = \sum_{v=0}^{n-2j} \binom{n-j}{n-2j}\binom{n-2j}{v}\binom{n}{j+v}^t F(n,j+v)$$
$$= \binom{n}{j}\sum_{v=0}^{n-2j}\binom{n-j}{v}\binom{n-j}{j+v}\binom{n}{j+v}^{t-1} F(n,j+v).$$

By the definition of D sums (see Eq. (4)) and the last equation above, we conclude that

$$M_S(n,j,t) = \binom{n}{j} D_S(n,j,t-1).$$

This completes the proof of Lemma 3.

Now, we are ready for the proof of main theorem 2. We will use Eqns. (6), (7), and Lemma 3.

By Eq. (6), it follows that

$$D_S(n,j,t+1) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u}\binom{n}{j+u} D_S(n,j+u,t). \tag{11}$$

Let us recall that t in Eq. (11) is a non-negative integer.

By Lemma 3, it follows that

$$\binom{n}{j+u} D_S(n,j+u,t) = M_S(n,j+u,t+1). \tag{12}$$

By Eq. (12), Eq. (11) becomes as follows:

$$D_S(n,j,t+1) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t+1). \tag{13}$$

By multiplication of both sides of Eq. (13) with $\binom{n}{j}$, it follows that

$$\binom{n}{j} D_S(n,j,t+1) = \binom{n}{j}\sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t+1) \tag{14}$$

By Lemma 3, Eq. (14) becomes as follows:

$$M_S(n,j,t+2) = \binom{n}{j}\sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t+1)$$

By setting $t := t+1$, the last equation above becomes

$$M_S(n,j,t+1) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t); \tag{15}$$

where, now, t is a positive integer.

Therefore, Theorem 2 follows from Eq. (15), if t is a positive integer.

For $t=0$, we will use Eq. (7).

By Eq. (7), it follows that

$$\begin{aligned} D_S(n,j,0) &= \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} \binom{n-(j+u)}{j+u} \sum_{v=0}^{n-2(j+u)} \binom{n-2(j+u)}{v} F(n,(j+u)+v) \\ &= \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,0). \end{aligned}$$

By the last equation above, it follows that

$$D_S(n,j,0) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,0). \tag{16}$$

By multiplication of both sides of Eq. (16) with $\binom{n}{j}$, it follows that

$$\binom{n}{j} D_S(n,j,0) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,0). \tag{17}$$

By Lemma 3, we know that

$$M_S(n,j,1) = \binom{n}{j} D_S(n,j,0).$$

By using the equation above, Eq. (17) becomes, as follows:

$$M_S(n,j,1) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,0). \tag{18}$$

It is readily verified that Eq. (18) proves Theorem 2 for the case $t=0$.

Hence, Eqns. (15) and (18) complete the proof of Theorem 2.

□

Remark 4. By Eqns. (13) and (16), it follows that

$$D_S(n,j,t) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t); \tag{19}$$

where t is a non-negative integer.

Eqns. (9) and (19) give connections between M and D sums.

Note that Theorem 2 can be proved without using D sums. See [8, Proof of Theorem 3, p. 6].

4 How do we use "M" sums ?

In one particular situation, Theorem 2 implies a simple consequence which is important for us.

Let n be a fixed non-negative integer. Let t_0 be a non-negative integer, and let j be an arbitrary integer in the range $0 \le j \le \lfloor \frac{n}{2} \rfloor$. Suppose that $q = q(n)$ is a positive integer which divides $M_S(n,j,t_0)$ sums for all j in the given range. We want q to be as large as possible. Then it can be shown, by Theorem 2, that q divides $M_S(n,j,t_0+1)$ for all j in the given range.

By induction, it follows that q divides $M_S(n,j,t)$ for all t such that $t \ge t_0$ and for all j in the given range. By Eq. (3), it follows that q divides $S(n,t+1)$ for all t such that $t \ge t_0$.

See also [11, Section 5, p. 7].

5 The first application for the alternating sum

We give a proof of Calkin's result [1, Thm. 1] by using M sums.

Let n be a non-negative integer and let m be a positive integer. Let

$$S_1(2n,m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m.$$

Obviously, the sum $S_1(2n,m)$ is an instance of the sum (1), where $F_1(2n,k) = (-1)^k$.

We use the well-known identity [2, Eq. (1.25), p. 4]

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} = \begin{cases} 0, & \text{if } n > 0; \\ 1, & \text{if } n = 0, \end{cases} \tag{20}$$

where n is a non-negative integer.

By Eq. (20), we conclude that $S_1(2n,1)$ is divisible by $\binom{2n}{n}$.

Furthermore, it is known

$$S_1(2n,2)=(-1)^n\binom{2n}{n}, \qquad \text{(Kummer's formula)}$$
$$S_1(2n,3)=(-1)^n\binom{2n}{n}\binom{3n}{2n}. \qquad \text{(Dixon's formula)}$$

Therefore, it follows that $S_1(2n,m)$ is divisible by $\binom{2n}{n}$ for $1\le m\le 3$.
Let us calculate $M_{S_1}(2n,j,0)$ sum; where j is a non-negative integer such that $j\le n$.
Without loss of generality, let us assume that n is a positive integer.
By Def. (1), it follows that

$$\begin{aligned}M_{S_1}(2n,j,0)&=\binom{2n-j}{j}\sum_{v=0}^{2n-2j}\binom{2n-2j}{v}\binom{2n}{j+v}^0F_1(2n,j+v)\\&=\binom{2n-j}{j}\sum_{v=0}^{2n-2j}\binom{2n-2j}{v}(-1)^{j+v}.\end{aligned}$$

By the last equation above, it follows that

$$M_{S_1}(2n,j,0)=(-1)^j\binom{2n-j}{j}\sum_{v=0}^{2(n-j)}(-1)^v\binom{2(n-j)}{v}. \tag{21}$$

By Eqns. (20) and (21), it follows that

$$M_{S_1}(2n,j,0)=\begin{cases}0, & \text{if } 0\le j<n;\\ (-1)^n, & \text{if } j=n.\end{cases} \tag{22}$$

Let us calculate $M_{S_1}(2n,j,1)$ sum. We will use Theorem 2 and Eq. (22).
By setting $t=0$ in Theorem 2, it follows that

$$M_{S_1}(2n,j,1)=\binom{2n}{j}\sum_{u=0}^{n-j}\binom{2n-j}{u}M_{S_1}(2n,j+u,0). \tag{23}$$

By Eq. (22), it follows that

$$M_{S_1}(2n,j+u,0)=\begin{cases}0, & \text{if } 0\le u<n-j;\\ (-1)^n, & \text{if } u=n-j.\end{cases} \tag{24}$$

By Eq. (24), Eq. (23) becomes gradually

$$\begin{aligned} M_{S_1}(2n,j,1) &= (-1)^n \binom{2n}{j}\binom{2n-j}{n-j} \\ &= (-1)^n \binom{2n}{2n-j}\binom{2n-j}{n} && \text{(by symmetry)} \\ &= (-1)^n \binom{2n}{n}\binom{n}{n-j} && \text{(by Eq. (8)).} \end{aligned}$$

By the lst equation above, it follows that

$$M_{S_1}(2n,j,1) = (-1)^n \binom{2n}{n}\binom{n}{j}. \tag{25}$$

By setting $j = 0$ in Eq. (25) and by using Eq. (3), we obtain Kummer's formula.

Furthermore, Eq. (25) suggets setting $q_1(2n) = \binom{2n}{n}$.

By Theorem 2 and induction, it can be shown that $M_{S_1}(2n,j,t)$ is divisible by $\binom{2n}{n}$ for all positive integers t and all integers j such that $0 \le j \le n$.

Let $m \ge 2$. By Eq. (3), we know that

$$S_1(2n,m) = M_{S_1}(2n,0,m-1).$$

Since $m-1 \ge 1$, the sum $M_{S_1}(2n,0,m-1)$ is divisible by $\binom{2n}{n}$. By Eq. (3), $S_1(2n,m)$ is divisible by $\binom{2n}{n}$ for $m \ge 2$.

This proves Calkin's result.

Remark 5. By using M-sums, we also can prove Dixon's formula [5, 9]. Namely, by using Theorem 2, Eq. (25), and the Vandermonde identity, it can be shown that

$$M_{S_1}(2n,j,2) = (-1)^n \binom{2n}{n}\binom{2n}{j}\binom{3n-j}{2n}. \tag{26}$$

By setting $j = 0$ in Eq. (26) and by using Eq. (3), we obtain Dixon's formula.

6 A generalization of new sums

We begin with the following definition:

Definition 6. Let m, n, k, a_1, a_2, $\ldots$, a_l be non-negative integers such that $m \ge 1$. We consider the sum:

$$S(n,m;a_1,a_2,\ldots,a_l) = \sum_{k=0}^{n} \binom{n}{k}^m F(n,k;a_1,a_2,\ldots,a_l), \tag{27}$$

where $F(n,k;a_1,a_2,\ldots,a_l)$ is an integer-valued function.

The aim is to examine some divisibility properties of sums of the form $S(n, m, a_1, a_2, \dots, a_l)$. To do this, we introduce a natural generalization of a notion of M sums.

Definition 7. Let n, j, t, a_1, a_2, …, a_l be non-negative integers such that $j \le \lfloor \frac{n}{2} \rfloor$. Then the M sums for $S(n, m; a_1, a_2, \dots, a_l)$ are as follows:

$$M_S(n, j, t; a_1, a_2, \dots, a_l) = \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} \binom{n}{j+v}^t F(n, j+v; a_1, a_2, \dots, a_l). \quad (28)$$

Obviously, for $m \ge 1$, the equation

$$S(n, m; a_1, a_2, \dots, a_l) = M_S(n, 0, m-1; a_1, a_2, \dots, a_l) \quad (29)$$

holds.

Eqns. (28) and (29) are similar with Eqns. (2) and (3), respectively.

Let n, j, t, a_1, a_2, …, a_l be as in Definition 7. Then the following theorem is true:

Theorem 8.

$$M_S(n, j, t+1; a_1, a_2, \dots, a_l) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n, j+u, t; a_1, a_2, \dots, a_l).$$

The proof of Theorem 8 is similar with the proof of Theorem 2.

Remark 9. Similarly, we can introduce a natural generalization of a notion of D sums. Let n, j, t, a_1, a_2, …, a_l be non-negative integers such that $j \le \lfloor \frac{n}{2} \rfloor$. Then the D sums for $S(n, m; a_1, a_2, \dots, a_l)$ are as follows:

$$D_S(n, j, t; a_1, a_2, \dots, a_l) = \sum_{v=0}^{n-2j} \binom{n-j}{v} \binom{n-j}{j+v} \binom{n}{j+v}^t F(n, j+v; a_1, a_2, \dots, a_l). \quad (30)$$

Obviously, for $m \ge 2$, the equation

$$S(n, m; a_1, a_2, \dots, a_l) = D_S(n, 0, m-2; a_1, a_2, \dots, a_l) \quad (31)$$

holds.

Also, it can be readily verified that

$$D_S(n, j, t+1; a_1, a_2, \dots a_l) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n}{j+u} \binom{n-j}{u} D_S(n, j+u, t; a_1, a_2, \dots a_l), \quad (32)$$

$$M_S(n, j, t; a_1, a_2, \dots a_l) = \binom{n}{j} D_S(n, j, t-1; a_1, a_2, \dots a_l), \quad (33)$$

$$D_S(n, j, t; a_1, a_2, \dots a_l) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n, j+u, t; a_1, a_2, \dots a_l) \quad (34)$$

Eqns. (31), (32), (33), and (34) are similar with Eqns. (5), (6), (9), and (19), respectively.

7 The second application of new sums

Let us consider the following binomial sum S: $\mathbb{Z}^{l+2}_{>=0} \to \mathbb{Z}$

$$S(2n, m; a_1, a_2, \ldots, a_l) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \prod_{i=1}^{l} \binom{a_i + k}{k} \binom{a_i + 2n - k}{2n - k};$$

where l is a positive integer and n, m, a_1, a_2, …, a_l are non-negative integers.

For $m = 1$ and $l = 1$, the sum S reduces to an interesting combinatorial identity. It is known [3, Eq. (6.56), p. 29] that

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{a+k}{k} \binom{a+2n-k}{2n-k} = (-1)^n \binom{2n}{n} \binom{a+n}{2n}. \tag{35}$$

By using Eq. (30), it follows that $S(2n, 1, a_1) = (-1)^n \binom{2n}{n} \binom{a_1+n}{2n}$.

Moreover, the following formula

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{a+k}{k} \binom{a+2n-k}{2n-k} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{a}{k} \binom{a}{2n-k} \tag{36}$$

holds. Note that the right-side of Eq. (36) is an instance of the well-known Dixon's formula [9]. Furthermore, the right-side of Eq. (36) has the following generalization:

$$R(2n, m, a_1, a_2, \ldots, a_l) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \prod_{i=1}^{l} \binom{a_i}{k} \binom{a_i}{2n-k}. \tag{37}$$

Remark 10. By using the "upper negation" [4, Eq. (5.14), p. 164], it follows that $S(2n, m; a_1, a_2, \ldots, a_l) = R(2n, m; -a_1 - 1, \ldots, -a_l - 1)$. Hence, parameters a_1, …, a_l in the definition of the sum S need not to be positive.

In 1998, Calkin [1, Thm. 1] proved that the alternating binomial sum $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. It follows that $R(2n, m, 2n)$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and m.

In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin's result [6, Thm. 1.2, Thm. 1.3, p. 2]. As a special case of [6, Thm. 1.2, p. 2], they gave a direct generalization of Calkin's result [6, Thm. 1.4, p. 8] which is as follows:

Theorem 11. *Let s be a positive integer greater than 1, and let k, n_1, n_2, …,n_s be non-negative integers. Then the sum*

$$\sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^{s} \binom{n_i + n_{i+1}}{n_i + k}$$

is divisible by $\binom{n_1+n_s}{n_1}$, where $n_{s+1} = n_1$.

By Theorem 11, it follows that the sum $R(2n, m, a_1, a_2, \ldots, a_l)$ is divisible by $\binom{2n}{n}$ and by $\binom{a_i}{n}$ for all $i = 1, l$. Take $s = m + 2l$, $n_1 = n_2 = \ldots = n_m = n$ and $n_{m+1} = a - n$, $n_{m+2} = n$, $\ldots n_{m+2l-1} = a - n$, $n_{m+2l} = n$ in Theorem 11.

The sum $R(2n, m, a_1, a_2, \ldots, a_l)$ is a generalization of the right-side of Eq. (36), while the sum $S(2n, m, a_1, a_2, \ldots, a_l)$ is a generalization of the left-side of Eq. (36).

We claim that

Theorem 12. *If m is a positive integer, then $S(2n, m, a_1, a_2, \ldots, a_l)$ is divisible by $\binom{2n}{n}$ and $\binom{a_i+n}{n}$ for all $i = 1, \ldots, l$.*

Proof. We give a proof only for the case $l = 1$. For the sake of brevity and clarity, the rest of the proof of Theorem 12 is omitted.

We consider the following sum

$$S(2n, m; a) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \binom{a+k}{k} \binom{a+2n-k}{2n-k};$$

where a is a non-negative integer.

Obviously, the sum $S(2n, m; a)$ is an instance of Eq. (27); where $F(2n, k; a) = (-1)^k \binom{a+k}{k} \binom{a+2n-k}{2n-k}$.

By Definition (7) and Eq. (28), it follows that

$$M_S(2n, j, 0; a) = \binom{2n-j}{j} \sum_{v=0}^{2n-2j} (-1)^v \binom{2n-2j}{v} \binom{a+j+v}{j+v} \binom{a+2n-j-v}{2n-j-v}. \tag{38}$$

By using substitution $v = k - j$, Eq. (38) becomes

$$M_S(2n, j, 0; a) = (-1)^j \binom{2n-j}{j} \sum_{k=j}^{2n-j} (-1)^k \binom{2n-2j}{k-j} \binom{a+k}{k} \binom{a+2n-k}{2n-k}. \tag{39}$$

It is readily verified that (by Eq. (8))

$$\binom{2n-j}{j} \binom{2n-2j}{k-j} = \binom{2n-j}{k-j} \binom{2n-k}{2n-k-j}. \tag{40}$$

By using Eq. (40), Eq. (39) reduces to

$$M_S(2n, j, 0; a) = (-1)^j \sum_{k=j}^{2n-j} (-1)^k \binom{2n-j}{k-j} \binom{2n-k}{2n-k-j} \binom{a+k}{k} \binom{a+2n-k}{2n-k}. \tag{41}$$

It is readily verified that (by Eq. (8))

$$\binom{a+2n-k}{2n-k} \binom{2n-k}{2n-k-j} = \binom{a+2n-k}{a+j} \binom{a+j}{j}. \tag{42}$$

By using Eq. (42), Eq. (41) becomes

$$M_S(2n, j, 0; a) = (-1)^j \binom{a+j}{j} \sum_{k=j}^{2n-j} (-1)^k \binom{2n-j}{k-j} \binom{a+k}{a} \binom{a+2n-k}{a+j}. \tag{43}$$

Now we use a well-known Stanley's formula:

$$\sum_{l=0}^{min(m,n)} \binom{a}{m-l} \binom{b}{n-l} \binom{a+b+l}{l} = \binom{a+n}{m} \binom{b+m}{n}. \tag{44}$$

Remark 13. The identity 44 is equivalent to the triple binomial identity [4, Eq. (5.28), p. 171]; first proved by J. F. Pfaff [12]. See also [7, Ex. 31, p. 71] and [13, Problem 14, p. 4].

By Stanley's formula (44), it follows that

$$\sum_{l=0}^{a} \binom{2n-k}{a+j-l} \binom{k-j}{a-l} \binom{2n-j+l}{l} = \binom{a+2n-k}{a+j} \binom{a+k}{a}. \tag{45}$$

By using Eq. (45), Eq. (43) reduces to:

$$M_S(2n, j, 0; a) = (-1)^j \binom{a+j}{j} \sum_{k=j}^{2n-j} (-1)^k \binom{2n-j}{k-j} \sum_{l=0}^{a} \binom{2n-k}{a+j-l} \binom{k-j}{a-l} \binom{2n-j+l}{l}.$$

By changing the order of summation in the equation above, it follows that $M_S(2n, j, 0; a)$ is equal to

$$(-1)^j \binom{a+j}{j} \sum_{l=0}^{a} \binom{2n-j+l}{l} \sum_{k=j+a-l}^{2n-(j+a-l)} (-1)^k \binom{2n-j}{k-j} \binom{2n-k}{a+j-l} \binom{k-j}{a-l}. \tag{46}$$

It is readily verified that (by Eq. (8))

$$\binom{2n-j}{k-j} \binom{2n-k}{a+j-l} = \binom{2n-j}{a+j-l} \binom{2n-2j-a+l}{k-j}. \tag{47}$$

By using Eqns. (46) and (47), it follows that $M_S(2n, j, 0; a)$ is equal to

$$(-1)^j \binom{a+j}{j} \sum_{l=0}^{a} \binom{2n-j+l}{l} \binom{2n-j}{a+j-l} \sum_{k=j+a-l}^{2n-(j+a-l)} (-1)^k \binom{2n-2j-a+l}{k-j} \binom{k-j}{a-l}.$$

We know that

$$\binom{2n-2j-a+l}{k-j} \binom{k-j}{a-l} = \binom{2n-2j-a+l}{a-l} \binom{2n-2j-2a+2l}{k-j-a+l}.$$

By using the last two equation above, it follows that $M_S(2n, j, 0; a)$ is equal to

$$(-1)^j\binom{a+j}{j}\sum_{l=0}^{a}\binom{2n-j+l}{l}\binom{2n-j}{a+j-l}\binom{2n-2j-a+l}{a-l}\sum_{k=j+a-l}^{2n-(j+a-l)}(-1)^k\binom{2n-2j-2a+2l}{k-j-a+l}.$$

Note that (by Eq. (8))

$$\binom{2n-j}{a+j-l}\binom{2n-2j-a+l}{a-l}=\binom{2n-j}{a-l}\binom{2n-j-a+l}{j+a-l}.$$

By using the last two equation above, it follows that $M_S(2n, j, 0; a)$ is equal to

$$(-1)^j\binom{a+j}{j}\sum_{l=0}^{a}\binom{2n-j+l}{l}\binom{2n-j}{a-l}\binom{2n-j-a+l}{j+a-l}\sum_{k=j+a-l}^{2n-(j+a-l)}(-1)^k\binom{2n-2j-2a+2l}{k-j-a+l}. \tag{48}$$

By Eq. (20), it follows that

$$\sum_{k=j+a-l}^{2n-j-a+l}(-1)^k\binom{2n-2j-2a+2l}{k-j-a+l}=\begin{cases}0, & \text{if } n-j-a+l>0;\\ (-1)^{j+a-l}, & \text{if } n-j-a+l=0,\end{cases} \tag{49}$$

By Eq. (49), it follows that only non-vanishing term in Eq. (48) is for $l = a + j - n$. Therefore, by Eq. (48), it follows that $M_S(2n, j, 0; a)$ is equal to

$$(-1)^{n-j}\binom{a+j}{j}\binom{2n-j+(a+j-n)}{a+j-n}\binom{2n-j}{a-(a+j-n)}\binom{2n-j-a+(a+j-n)}{j+a-(a+j-n)}.$$

From the last equation above and symmetry of binomial coefficients, it follows that

$$M_S(2n, j, 0; a)=(-1)^{n-j}\binom{a+j}{j}\binom{a+n}{2n-j}\binom{2n-j}{n}. \tag{50}$$

Note that (by Eq. (8))

$$\binom{a+n}{2n-j}\binom{2n-j}{n}=\binom{a+n}{n}\binom{a}{n-j}. \tag{51}$$

Therefore, by using Eq. (51), Eq. (50) reduces to

$$M_S(2n, j, 0; a)=(-1)^{n-j}\binom{a+j}{j}\binom{a+n}{a}\binom{a}{n-j}. \tag{52}$$

Eq. (52) suggets setting $q_2(2n, a) = \binom{a+n}{a}$. Obviously, $q_2(2n, a)$ does not depend on j.
By Theorem 8 and induction, it can be shown that $M_S(2n, j, t; a)$ is divisible by $\binom{a+n}{a}$ for all non-negative integers t and all integers j such that $0 \le j \le n$.

Let $m \geq 1$. By Eq. (29), we know that

$$S(2n, m; a) = M_S(2n, 0, m-1; a).$$

Since $m \geq 1$, the sum $M_S(2n, 0, m-1; a)$ is divisible by $\binom{a+n}{a}$. By Eq. (29), $S(2n, m; a)$ is divisible by $\binom{a+n}{a}$ for all positive m.

Let us prove that $S(2n, m; a)$ is divisible by $\binom{2n}{n}$ for all positive m and for all non-negative integers n and a.

By setting $j = 0$ in Eq. (50) and by using Eq. (29), we obtain Eq. (35). It follows that $\binom{2n}{n}$ divides $S(2n, 1; a)$.

Let us calculate $M_S(2n, j, 1; a)$ sum. We will use Theorem 8 and Eq. (50).

By setting $t = 0$ in Theorem 8, we obtain that

$$M_S(2n, j, 1; a) = \binom{2n}{j} \sum_{u=0}^{n-j} \binom{2n-j}{u} M_S(2n, j+u, 0; a). \tag{53}$$

By using Eq. (50), it follows that

$$M_S(2n, j+u, 0; a) = (-1)^{n-j-u} \binom{a+j+u}{j+u} \binom{a+n}{2n-j-u} \binom{2n-j-u}{n}. \tag{54}$$

It is readily verified that (by Eq. (8))

$$\binom{2n}{j}\binom{2n-j}{u} = \binom{2n}{2n-j-u}\binom{j+u}{u}.$$

By the last equation above, Eq. (53) becomes

$$M_S(2n, j, 1; a) = \sum_{u=0}^{n-j} \binom{2n}{2n-j-u}\binom{j+u}{u} M_S(2n, j+u, 0; a). \tag{55}$$

By Eqns. (54) and (55), $M_S(2n, j, 1; a)$ is equal to

$$\sum_{u=0}^{n-j} \binom{2n}{2n-j-u}\binom{j+u}{u}(-1)^{n-j-u}\binom{a+j+u}{j+u}\binom{a+n}{2n-j-u}\binom{2n-j-u}{n}. \tag{56}$$

Note that (by Eq. (8))

$$\binom{2n}{2n-j-u}\binom{2n-j-u}{n} = \binom{2n}{n}\binom{n}{j+u}.$$

By using the last equation above, it follows that

$$M_S(2n, j, 1; a) = \binom{2n}{n} \sum_{u=0}^{n-j} \binom{n}{j+u}\binom{j+u}{u}(-1)^{n-j-u}\binom{a+j+u}{j+u}\binom{a+n}{2n-j-u}. \tag{57}$$

By Eq. (57), Theorem 8, and induction, it can be shown that $M_S(2n, j, t; a)$ is divisible by $\binom{2n}{n}$ for all positive integers t and all integers j such that $0 \leq j \leq n$.

Let $m \geq 2$. By Eq. (29), we know that

$$S(2n, m; a) = M_S(2n, 0, m-1; a).$$

Since $m \geq 2$, the sum $M_S(2n, 0, m-1; a)$ is divisible by $\binom{2n}{n}$. By Eq. (29), $S(2n, m; a)$ is divisible by $\binom{2n}{n}$ for all positive m and for all non-negative integers n and a.

□

8 The third application of new sums

There is an interesting connection between Theorem 11 and M-sums.

We assert that the following implication is true:

Theorem 14. *Let s be a fixed positive integer greater than 2, and let k, n_1, n_2, …,n_s be arbitrary non-negative integers. Let us suppose that the sum*

$$\sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^{s} \binom{n_i + n_{i+1}}{n_i + k}$$

is divisible by $\binom{n_1+n_s}{n_1}$, where $n_{s+1} = n_1$. Then, by using M sums and the assumption above, it follows that the following sum

$$\sum_{k=-n_1}^{n_1} (-1)^k \binom{2n_1}{k+n_1}^m \cdot \prod_{i=1}^{s-1} \binom{n_i + n_{i+1}}{n_i + k}$$

is divisible by $\binom{2n_1}{n_1}$, $\binom{n_1+n_2}{n_1}$, …, $\binom{n_{s-2}+n_{s-1}}{n_{s-2}}$, and $\binom{n_1+n_{s-1}}{n_1}$; where m is an arbitrary positive integer and $n_s = n_1$.

Proof. For $m = 1$, the proof of Theorem 14 directly follows from our assumption.

Let us suppose that $m \geq 2$.

Let $P(2n_1, m; n_2, \ldots, n_{s-1})$ denote $\sum_{k=-n_1}^{n_1} (-1)^k \binom{2n_1}{k+n_1}^m \cdot \prod_{i=1}^{s-1} \binom{n_i+n_{i+1}}{n_i+k}$.

By using substitution $u = k + n_1$, it follows that

$$P(2n_1, m; n_2, \ldots, n_{s-1}) = \sum_{u=0}^{2n_1} (-1)^{u-n_1} \binom{2n_1}{u}^m \cdot \prod_{i=1}^{s-1} \binom{n_i + n_{i+1}}{u + n_i - n_1}.$$

Obviously, the sum $P(2n_1, m; n_2, \ldots, n_{s-1})$ is an instance of the sum from Definition 6; where

$$F(2n_1, u; n_2, \ldots, n_{s-1}) = (-1)^{u-n_1} \prod_{i=1}^{s-1} \binom{n_i + n_{i+1}}{u + n_i - n_1}. \tag{58}$$

By using Definition 7 and Eq. (28), we obtain that

$$M_P(2n_1, j, 0; n_2, \dots n_{s-1}) = \binom{2n_1 - j}{j} \sum_{v=0}^{2n_1-2j} \binom{2n_1 - 2j}{v} F(2n_1, j + v; n_2, \dots n_{s-1}). \quad (59)$$

By using substitution $u = v + j$, the Eq. (59) reduces to

$$M_P(2n_1, j, 0; n_2, \dots n_{s-1}) = \binom{2n_1 - j}{j} \sum_{u=j}^{2n_1-j} \binom{2n_1 - 2j}{u - j} F(2n_1, u; n_2, \dots n_{s-1}). \quad (60)$$

By using Eq. (58), it follows that $M_P(2n_1, j, 0; n_2, \dots n_{s-1})$ is equal to the following sum

$$\binom{2n_1 - j}{j} \sum_{u=j}^{2n_1-j} \binom{2n_1 - 2j}{u - j} (-1)^{u-n_1} \prod_{i=1}^{s-1} \binom{n_i + n_{i+1}}{u + n_i - n_1}. \quad (61)$$

It is readily verified that (by Eq. (8))

$$\binom{2n_1 - j}{j} \binom{2n_1 - 2j}{u - j} = \binom{2n_1 - j}{u - j} \binom{2n_1 - u}{j}. \quad (62)$$

By using Eqns. (61) and (62), it follows that $M_P(2n_1, j, 0; n_2, \dots n_{s-1})$ is equal to

$$\sum_{u=j}^{2n_1-j} (-1)^{u-n_1} \binom{2n_1 - j}{u - j} \binom{2n_1 - u}{j} \prod_{i=1}^{s-1} \binom{n_i + n_{i+1}}{u + n_i - n_1}. \quad (63)$$

By a symmetry of binomial coefficients, we know that the last term in a product equals

$$\binom{n_{s-1} + n_1}{u + n_{s-1} - n_1} = \binom{n_{s-1} + n_1}{2n_1 - u}.$$

The right-side of Eq. (63) can be rewritten as

$$\sum_{u=j}^{2n_1-j} (-1)^{u-n_1} \binom{2n_1 - j}{u - j} \binom{n_{s-1} + n_1}{2n_1 - u} \binom{2n_1 - u}{j} \prod_{i=1}^{s-2} \binom{n_i + n_{i+1}}{u + n_i - n_1} \quad (64)$$

It is readily verified that (by Eq. (8))

$$\binom{n_{s-1} + n_1}{2n_1 - u} \binom{2n_1 - u}{j} = \binom{n_{s-1} + n_1}{j} \binom{n_{s-1} + n_1 - j}{u + n_{s-1} - n_1}. \quad (65)$$

By using Eqns. (64) and (65), it follows that $M_P(2n_1, j, 0; n_2, \dots n_{s-1})$ is equal to

$$\binom{n_{s-1} + n_1}{j} \sum_{u=j}^{2n_1-j} (-1)^{u-n_1} \binom{n_{s-1} + n_1 - j}{u + n_{s-1} - n_1} \binom{2n_1 - j}{u - j} \prod_{i=1}^{s-2} \binom{n_i + n_{i+1}}{u + n_i - n_1}. \quad (66)$$

By using substitution $p = u - n_1$ in Eq. (66), it follows that $M_P(2n_1, j, 0; n_2, \dots n_{s-1})$ is equal to

$$\binom{n_{s-1}+n_1}{j} \sum_{p=-(n_1-j)}^{n_1-j} (-1)^p \binom{(n_1-j)+n_1}{p+n_1-j} (\prod_{i=1}^{s-2} \binom{n_i+n_{i+1}}{p+n_i}) \binom{n_{s-1}+(n_1-j)}{p+n_{s-1}}. \tag{67}$$

By using our assumption in Theorem 14, it follows that inner sum in Eq. (67) is divisible by integers $\binom{2n_1-j}{n_1}$, $\binom{n_1+n_2}{n_1}$, $\dots$, $\binom{n_{s-2}+n_{s-1}}{n_{s-2}}$, and by $\binom{n_{s-1}+n_1-j}{n_{s-1}}$.

Therefore, we obtain that $M_P(2n_1, j, 0; n_2, \dots n_{s-1})$ is divisible by integers $\binom{2n_1-j}{n_1}$, $\binom{n_1+n_2}{n_1}$, $\dots$, $\binom{n_{s-2}+n_{s-1}}{n_{s-2}}$, and by $\binom{n_{s-1}+n_1-j}{n_{s-1}}$ for all non-negative integers j such that $j \le n_1$.

By using Theorem 8 and the induction principle, it can be shown that $M_P(2n_1, j, t; n_2, \dots n_{s-1})$ is divisible by integers $\binom{n_1+n_2}{n_1}$, $\dots$, $\binom{n_{s-2}+n_{s-1}}{n_{s-2}}$ for all non-negative integers t and j such that $j \le n_1$. By using Eq. (29), it follows that the sum $P(2n_1, m; n_2, \dots, n_{s-1})$ is divisible by integers $\binom{n_1+n_2}{n_1}$, $\dots$, $\binom{n_{s-2}+n_{s-1}}{n_{s-2}}$ for all positive integers m.

Let us prove that $P(2n_1, m; n_2, \dots, n_{s-1})$ is divisible by $\binom{2n_1}{n_1}$ for all positive integers $m \ge 2$.

By setting $t = 0$ in Theorem 8, we obtain that

$$M_S(2n_1, j, 1; n_2, \dots n_{s-1}) = \binom{2n_1}{j} \sum_{u=0}^{n_1-j} \binom{2n_1-j}{u} M_S(2n_1, j+u, 0; n_2, \dots n_{s-1}). \tag{68}$$

We know that $M_S(2n_1, j+u, 0; n_2, \dots n_{s-1}) = \binom{2n_1-(j+u)}{n_1} \cdot c(j, u, n_1, \dots, n_{s-1})$; where $c(j, u, n_1, \dots, n_{s-1})$ is an integer.

By using Eq. (68), we have gradually

$$M_S(2n_1, j, 1; n_2, \dots n_{s-1}) = \binom{2n_1}{j} \sum_{u=0}^{n_1-j} \binom{2n_1-j}{u} \binom{2n_1-j-u}{n_1} \cdot c(j, u, n_1, \dots, n_{s-1})$$

$$M_S(2n_1, j, 1; n_2, \dots n_{s-1}) = \sum_{u=0}^{n_1-j} \binom{2n_1}{j} (\binom{2n_1-j}{2n_1-j-u} \binom{2n_1-j-u}{n_1}) \cdot c(j, u, n_1, \dots, n_{s-1})$$

$$M_S(2n_1, j, 1; n_2, \dots n_{s-1}) = \sum_{u=0}^{n_1-j} \binom{2n_1}{j} (\binom{2n_1-j}{n_1} \binom{n_1-j}{n_1-j-u}) \cdot c(j, u, n_1, \dots, n_{s-1})$$

$$M_S(2n_1, j, 1; n_2, \dots n_{s-1}) = \sum_{u=0}^{n_1-j} (\binom{2n_1}{2n_1-j} \binom{2n_1-j}{n_1}) \binom{n_1-j}{n_1-j-u} \cdot c(j, u, n_1, \dots, n_{s-1})$$

$$M_S(2n_1, j, 1; n_2, \dots n_{s-1}) = \binom{2n_1}{n_1} \sum_{u=0}^{n_1-j} \binom{n_1}{j} \binom{n_1-j}{u} \cdot c(j, u, n_1, \dots, n_{s-1}). \tag{69}$$

By using the Eq. (69), it follows that $M_S(2n_1, j, 1; n_2, \dots n_{s-1})$ is divisible by $\binom{2n_1}{n_1}$ for all non-negative j such that $j \le n_1$.

By using Eq. (29), it follows that the sum $P(2n_1, m; n_2, \ldots, n_{s-1})$ is divisible by $\binom{2n_1}{n_1}$ for all positive integers m such that $m \geq 2$.

Finally, let us prove that $P(2n_1, m; n_2, \ldots, n_{s-1})$ is divisible by $\binom{n_1+n_{s-1}}{n_1}$ for all positive integers m.

By Eq. (67) and our assumption, it follows that

$$M_P(2n_1, j, 0; n_2, \ldots n_{s-1}) = \binom{n_{s-1}+n_1}{j}\binom{n_1-j+n_{s-1}}{n_{s-1}} \cdot d(n_1-j, n_2, \ldots, n_{s-1}); \quad (70)$$

where $d(n_1-j, n_2, \ldots, n_{s-1})$ is an integer.

It is readily verified that (by Eq. (8))

$$\binom{n_{s-1}+n_1}{j}\binom{n_1-j+n_{s-1}}{n_{s-1}} = \binom{n_1+n_{s-1}}{n_{s-1}}\binom{n_1}{j}. \quad (71)$$

By using Eq. (71), Eq. (70) becomes

$$M_P(2n_1, j, 0; n_2, \ldots n_{s-1}) = \binom{n_1+n_{s-1}}{n_{s-1}}\binom{n_1}{j} \cdot d(n_1-j, n_2, \ldots, n_{s-1}). \quad (72)$$

By using Theorem 8 and the induction principle, it can be shown that $M_P(2n_1, j, t; n_2, \ldots n_{s-1})$ is divisible by an integer $\binom{n_1+n_{s-1}}{n_{s-1}}$ for all non-negative integers t and j such that $j \leq n_1$. By using Eq. (29), it follows that the sum $P(2n_1, m; n_2, \ldots, n_{s-1})$ is divisible by an integer $\binom{n_1+n_{s-1}}{n_{s-1}}$ for all positive integers m.

This completes the proof of Theorem 14.

□

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On a Family of Triangular Arrays of Natural Numbers and the Tower of Hanoi with Four Pegs

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Abstract

Several arrangements of the natural numbers $\mathbb{N}$ in triangular arrays $\big(T_j(n,k)\big)_{j\in\mathbb{N}}$, where $n \in \mathbb{N}$ denotes the row and $k \in \mathbb{N}$ the column, will be considered. These arrays are defined as follows: The first row has one element and each row has $j \in \mathbb{N}$ elements more than the preceding one, that is, the nth row has $jn-(j-1)$ elements. At first, a formula for the number appearing in the nth row and kth column will be given and, secondly, we consider the inverse problem, i.e. given a number $N \in \mathbb{N}$ determine in which row n and in which column k one can find this number. Then some sequences appearing in these arrays together with the sequences of their partial sums will be studied. Finally, a surprising connection of $T_1(n,k)$ with an iterative minimal solution of the Tower of Hanoi with four pegs and $N \in \mathbb{N}$ discs will be shown. As a result it is possible to determine in advance how many subtowers with how many discs have to be built in the iterative minimal solution of the Tower of Hanoi with four pegs and $N \in \mathbb{N}$ discs.

Key words: m-dimensional figurate numbers, polygonal numbers, pyramidal numbers, figured numbers, Tower of Hanoi with four pegs

AMS subject classification (2000): 05A10, 05A15, 11B37

1 Introduction

Consider the sequence of the natural numbers arranged as follows in the triangular arrays $\big(T_j(n,k)\big)_{j\in\mathbb{N}}$, where $n \in \mathbb{N}$ denotes the row and $k \in \mathbb{N}$ the column: The first row has one element and each row has $j \in \mathbb{N}$ elements more than the preceding one, i.e. the nth row has $jn-(j-1)$ elements. This means that the number of elements in the array builds an arithmetic progression with difference j and initial value 1 (see Tables 1, 2, and 3 for the cases $j = 1, 2, 3$, where the last column gives the row sum sequence $s_j^{(0)}(n)$).

In this paper we intend to study these arrays and some special subsequences of these arrays together with their partial sums. The paper is organized as follows: First of all we

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$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	...	$s_1^{(0)}(n)$
1	1													1
2	2	3												5
3	4	5	6											15
4	7	8	9	10										34
5	11	12	13	14	15									65
6	16	17	18	19	20	21								111
7	22	23	24	25	26	27	28							175
8	29	30	31	32	33	34	35	36						260
9	37	38	39	40	41	42	43	44	45					369
10	46	47	48	49	50	51	52	53	54	55				505
11	56	57	58	59	60	61	62	63	64	65	66			671
12	67	68	69	70	71	72	73	74	75	76	77	78		870

Table 1: Triangular array $T_1(n,k), 1 \leq n \leq 12,\ 1 \leq k \leq n$

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	...	$s_2^{(0)}(n)$
1	1														1
2	2	3	4												9
3	5	6	7	8	9										35
4	10	11	12	13	14	15	16								91
5	17	18	19	20	21	22	23	24	25						189
6	26	27	28	29	30	31	32	33	34	35	36				341
7	37	38	39	40	41	42	43	44	45	46	47	48	49		559

Table 2: Triangular array $T_2(n,k), 1 \leq n \leq 7,\ 1 \leq k \leq 2n-1$

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	...	$s_3^{(0)}(n)$
1	1														1
2	2	3	4	5											14
3	6	7	8	9	10	11	12								63
4	13	14	15	16	17	18	19	20	21	22					175
5	23	24	25	26	27	28	29	30	31	32	33	34	35		377

Table 3: Triangular array $T_3(n,k), 1 \leq n \leq 5,\ 1 \leq k \leq 3n-2$

define the triangular arrays and give a formula for the number appearing in the nth row and kth column, then we consider the inverse problem, that is given a number $N \in \mathbb{N}$ we determine in which row n and in which column k one can find this number. After that we study four different sequences and their partial sums arising in these arrays and, finally, we will show a surprising connection of $T_1(n,k)$ with an iterative minimal solution of the Tower of Hanoi with four pegs and $N \in \mathbb{N}$ discs given by Hinz [8].

2 Determination of $T_j(n,k)$

We recall that the *m-dimensional figurate numbers* are defined as

$$f_{m,j}(n) := \binom{n+m-2}{m-1} + j\binom{n+m-2}{m} = \frac{jn-(j-m)}{n+m-1}\binom{n+m-1}{m} \qquad (2.1)$$

for all $m, j, n \in \mathbb{N}$, where j is called the *set number*, n the *rank* and m the *dimension* of the figurate number. The number $s := j+2 \geq 3$ is the *number of sides* of the geometric figure of $f_{m,j}(n)$ (cf. [1, pp. 185–199]).

Remark 2.1 *The first few m-dimensional figurate numbers are for $j, n \in \mathbb{N}$*

- $m = 1$: $f_{1,j}(n) = \binom{n-1}{0} + j\binom{n-1}{1} = jn - (j-1)$.
  These are called the linear group of figurate numbers*. They are simply the terms of an arithmetic progression of difference $j \in \mathbb{N}$ starting with 1. Special cases of the linear group of figurate numbers are:*
  $j = 1 : f_{1,1}(n) = n$. *These are the* natural numbers*, the sequence A000027 in the On-Line Encyclopedia of Integer Sequences (OEIS ®) [16].*
  $j = 2 : f_{1,2}(n) = 2n - 1$. *These are the* odd numbers *(or* gnomic numbers*) (A005408).*

- $m = 2$: $f_{2,j}(n) = \binom{n}{1} + j\binom{n}{2} = \frac{1}{2}n\big(jn - (j-2)\big)$.
  These are the plane figurate numbers *(also called* polygonal numbers *or* s-gonal numbers*) ($s := j+2 \geq 3$). They are also denoted by p_s^n, that is $p_s^n = \frac{1}{2}n\big((s-2)n - (s-4)\big)$. Special cases of the polygonal numbers are:*
  $j = 1 : p_3^n = f_{2,1}(n) = \frac{1}{2}n(n+1) = \binom{n+1}{2}$. *These are the* triangular numbers *(or* 3-gonal numbers*) (A000217), the sum of the first natural numbers from 1 to n.*
  $j = 2 : p_4^n = f_{2,2}(n) = n^2$. *These are the* square numbers *(or* 4-gonal numbers*) (A000290), the sum of the first odd numbers from 1 to $2n-1$.*
  $j = 3 : p_5^n = f_{2,3}(n) = \frac{1}{2}n(3n-1)$. *These are the* pentagonal numbers *(or* 5-gonal numbers*) (A000326). Note that $3f_{2,3}(n) = \frac{1}{2} \cdot 3n(3n-1) = f_{2,1}(3n-1)$ or $p_5^n = f_{2,3}(n) = \frac{1}{3}f_{2,1}(3n-1) = \frac{1}{3}p_3^{3n-1}$, that is every pentagonal number is one-third of some triangular number (cf. [4, p. 41]).*
  $j = 4 : p_6^n = f_{2,4}(n) = n(2n-1)$. *These are the* hexagonal numbers *(or* 6-gonal numbers*) (A000384). Note that this is a subsequence of $f_{2,1}(n)$, since $f_{2,4}(n) = f_{2,1}(2n-1) = f_{2,1}(f_{1,2}(n))$, that is every hexagonal number is also a triangular number (cf. [4, p. 39].*

- $m = 3 : f_{3,j}(n) = \binom{n+1}{2} + j\binom{n+1}{3} = \frac{1}{6}n(n+1)\big(jn-(j-3)\big)$.
These are the solid figurate numbers *(also called* pyramidal numbers*). They are also denoted by P_s^n ($s := j+2 \geq 3$), that is $P_s^n = \frac{1}{6}n(n+1)\big((s-2)n-(s-5)\big)$. Special cases of the pyramidal numbers are:*
$j = 1 : P_3^n = f_{3,1}(n) = \frac{1}{6}n(n+1)(n+2) = \binom{n+2}{3}$. *These are the* tetrahedral numbers *(A000292).*
$j = 2 : P_4^n = f_{3,2}(n) = \frac{1}{6}n(n+1)(2n+1)$. *These are the* square pyramidal numbers *(A000330), the sum of the first square numbers from 1 to n^2.*
$j = 3 : P_5^n = f_{3,3}(n) = \frac{1}{2}n^2(n+1)$. *These are the* pentagonal pyramidal numbers *(A002411).*
$j = 4 : P_6^n = f_{3,4}(n) = \frac{1}{6}n(n+1)(4n-1)$. *These are the* hexagonal pyramidal numbers *(A002412).*

Finally, the numbers $f_{m,1}(n) = \binom{n+m-2}{m-1} + \binom{n+m-2}{m} = \binom{n+m-1}{m} = f_{m+1,0}(n)$ have also been called figured numbers *with the alternative notation $\langle {n \atop m} \rangle$ (cf. [3, pp. 15–17]) or nth* regular m-topic number *with the notation $P_n^{(m)}$ (cf. [14, p. 211]). Note that we have used the recurrence relation for the binomial coefficients known as* Pascal's rule *or* addition formula *(cf. [7, Eq.(5.8), p.158]. With these notations we have $f_{m,j}(n) = \langle {n \atop m-1} \rangle + j\langle {n-1 \atop m} \rangle$ and $f_{m,j}(n) = P_n^{(m-1)} + jP_{n-1}^{(m)}$, respectively. Combinatorially, there are $\binom{n+m-1}{m}$ ways to choose m elements from a set of n elements ($m \leq n$) where order does not count and repetitions are allowed. Note also that $f_{0,1}(n) = 1$ for all $n \in \mathbb{N}$ and that by Eq.(2.1) the m-dimensional figurate numbers are generated by the figured numbers.*

Theorem 2.2 *Let $j, n, k \in \mathbb{N}$ and $k \in [jn-(j-1)]$, where $[p] := \{1, 2, \dots, p\}$ for all natural numbers $p \geq 1$, then $T_j(n,k)$ is given by*

$$T_j(n,k) = \frac{n-1}{2}\Big(j(n-1)-(j-2)\Big)+k = f_{2,j}(n)-f_{1,j}(n)+k = \binom{n-1}{1}+j\binom{n-1}{2}+k. \quad (2.2)$$

Proof. The difference between two consecutive numbers in the nth row of $T_j(n,k)$ is by definition equal to 1, that is

$$\forall j, n, k \in \mathbb{N},\ k \in [jn-j] : \quad T_j(n,k+1) = T_j(n,k) + 1.$$

The solution of this arithmetic progression (in $k \in \mathbb{N}$) with difference 1 is given by

$$T_j(n,k) = T_j(n,1) + k - 1, \quad k \in [jn-(j-1)], \qquad (2.3)$$

where $T_j(n,1)$ is the initial value of the nth row. This value can be determined as follows: Since the nth row has $jn-(j-1)$ elements we obtain the recurrence relation in $n \geq 2$

$$T_j\big(n, jn-(j-1)+1\big) = T_j\big(n-1, jn-(j-1)+1\big) + jn-(j-1)+1, \quad T_j(1,1) = 1,$$

with the solution

$$T_j\big(n, jn-(j-1)+1\big) = \sum_{k=1}^{n}\big(jk-(j-1)\big) = \frac{1}{2}jn(n+1)-(j-1)n = \frac{1}{2}n\big(jn-(j-2)\big) = f_{2,j}(n)$$

by using the formula for the sum of the first n natural numbers $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. Hence,

$$\begin{aligned} T_j(n,1) &= T_j\big(n-1, j(n-1)-(j-1)\big)+1 = \frac{1}{2}(n-1)\big(j(n-1)-(j-2)\big)+1 \\ &= \frac{1}{2}n\big(jn-(j-2)\big)-\big(jn-(j-1)\big)+1 = f_{2,j}(n)-f_{1,j}(n)+1 \\ &= f_{2,j}(n-1) = \tbinom{n-1}{1}+j\tbinom{n-1}{2}. \end{aligned}$$

Substituting this value into Eq.(2.3) we obtain Eq.(2.2). □

The first three special cases of Eq.(2.2) are for all $n \geq 1$

$$\begin{aligned} T_1(n,k) &= \frac{1}{2}n(n-1)+k, \quad k \in [n], \\ T_2(n,k) &= (n-1)^2+k, \quad k \in [2n-1] \\ T_3(n,k) &= \frac{1}{2}(n-1)(3n-4)+k, \quad k \in [3n-2]. \end{aligned}$$

For example $T_3(5,6) = \frac{1}{2}\cdot 4 \cdot (3\cdot 5-4)+6 = 28$ as one can see in Table 3.
Note that $f_{1,j}(n)$ gives the number of elements in the nth row of T_j and $f_{2,j}(n)$ counts the total number of elements from the first row up to and including the nth row.

An immediate consequence of Theorem 2.2 is given in the following corollary.

Corollary 2.3 *For all $j, n \in \mathbb{N}$ and for all $k \in [jn-(j-1)]$:*

$$T_j(n,k)+T_j(n+2,k) = 2T_j(n+1,k)+j \tag{2.4}$$

and

$$T_j(n,k)+T_j(n+2,k+1) = 2T_j(n+1,k+1)+j-1 \tag{2.5}$$

Proof. For all $k \in [jn-(j-1)]$ we have by Eq.(2.2) : $T_j(n,k) = \frac{1}{2}(n-1)\big(j(n-1)-(j-2)\big)+k$ and $T_j(n+2,k) = \frac{1}{2}(n+1)\big(j(n+1)-(j-2)\big)+k$. Adding these two equations we obtain

$$\begin{aligned} T_j(n,k)+T_j(n+2,k) &= \frac{j}{2}\big((n-1)^2+(n+1)^2\big)-\frac{1}{2}(j-2)(n-1+n+1)+2k \\ &= \frac{j}{2}(2n^2+2)-(j-2)n+2k \\ &= 2\cdot\left(\frac{n}{2}\cdot\big(jn-(j-2)\big)+k\right)+j = 2T_j(n+1,k)+j. \end{aligned}$$

Similarly, using Eq.(2.4) we obtain $T_j(n,k)+T_j(n+2,k+1) = T_j(n,k)+T_j(n+2,k)+1 = 2T_j(n+1,k)+j+1 = 2T_j(n+1,k+1)+j-1$. □

Conversely, we now determine in which row and in which column a given $N \in \mathbb{N}$ can be found in $T_j(n,k)$.

Theorem 2.4 *Let $N \in \mathbb{N}$, then $N = T_j(n,k)$, $(j,n,k \in \mathbb{N})$, where*

$$n = n_j(N) = \left\lfloor \frac{3j-2+\sqrt{8j(N-1)+(j-2)^2}}{2j} \right\rfloor \tag{2.6}$$

and

$$k = k_j(N) = N - \frac{1}{2}\big(n_j(N)-1\big)\Big(j\big(n_j(N)-1\big)-(j-2)\Big). \tag{2.7}$$

Proof. N is in the nth row of $T_j(n,k)$ if and only if

$$T_j(n,1) \le N \le T_j\big(n, jn-(j-1)\big) = T_j(n+1,1)-1,$$

or by (2.2)

$$\frac{1}{2}(n-1)\big(j(n-1)-(j-2)\big)+1 \le N \le \frac{1}{2}n\big(j(n+1)-2(j-1)\big)+1-1,$$

that is

$$(n-1)\big(jn-2(j-1)\big)+2 \le 2N \le n\big(jn-(j-2)\big)$$

or, equivalently,

$$j\Big(n^2-\frac{3j-2}{j}n+2\Big) \le 2N \le j\Big(n^2-\frac{j-2}{j}n\Big).$$

Dividing these inequalities by $j \neq 0$ and completing the square of the first and third inequality we obtain

$$n^2-\frac{3j-2}{j}n+2+\Big(\frac{3j-2}{2j}\Big)^2-\Big(\frac{3j-2}{2j}\Big)^2 \le \frac{2N}{j} \le n^2-\frac{j-2}{j}n+\Big(\frac{j-2}{2j}\Big)^2-\Big(\frac{j-2}{2j}\Big)^2$$

or adding $\left(\frac{j-2}{2j}\right)^2$ and simplifying

$$\Big(n-\frac{3j-2}{2j}\Big)^2+\frac{2}{j} \le \frac{2N}{j}+\Big(\frac{j-2}{2j}\Big)^2 \le \Big(n-\frac{j-2}{2j}\Big)^2.$$

Further, subtracting $\frac{2}{j}$ we get

$$\Big(n-\frac{3j-2}{2j}\Big)^2 \le \frac{2(N-1)}{j}+\Big(\frac{j-2}{2j}\Big)^2 \le \Big(n-\frac{j-2}{2j}\Big)^2-\frac{2}{j} < \Big(n-\frac{j-2}{2j}\Big)^2$$

and these inequalities are equivalent to

$$n-\frac{3j-2}{2j} \le \frac{\sqrt{8j(N-1)+(j-2)^2}}{2j} < n-\frac{j-2}{2j}$$

or, finally,

$$n \le \frac{3j-2+\sqrt{8j(N-1)+(j-2)^2}}{2j} < n+1.$$

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$n_1(N)$	1	2	2	3	3	3	4	4	4	4	5	5	5	5	5	6
$n_2(N)$	1	2	2	2	3	3	3	3	3	4	4	4	4	4	4	4
$n_3(N)$	1	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4

Table 4: $n_j(N)$, $1 \le j \le 3$, $1 \le N \le 16$

This is exactly the assertion (2.6) by the definition of the floor function (cf. [7, Eq.(3.5) (a), p. 69]). Finally, Eq.(2.7) follows from Eqs.(2.6) and (2.2). $\square$

For example, let $j = 2$ and $N = 19$, then by Eqs.(2.6) and (2.7) we obtain $n = \lfloor 1 + 3\sqrt{2} \rfloor = \lfloor 5.2426\ldots \rfloor = 5$ and $k = 19 - \frac{1}{2} \cdot 4 \cdot \big(2 \cdot (5-1) - 0\big) = 19 - 2 \cdot 8 = 3$ as one can see in Table 2.

Remark 2.5 *By Theorem 2.4 every number $N \in \mathbb{N}$ can be written as*

$$N = f_{2,j}\big(n_j(N) - 1\big) + k_j(N),$$

where $n_j(N)$ and $k_j(N)$ are given by Eqs.(2.6) and (2.7), respectively.

The family of sequences $(n_j(N))_{N\in\mathbb{N}}$ with the parameter $j \in \mathbb{N}$ means "*the number N appears $f_{1,j}(N) = jN - (j-1)$ times*".

The first three sequences are:

$$\begin{aligned}
n_1(N) &= \left\lfloor \frac{1}{2}\big(1 + \sqrt{8N-7}\big) \right\rfloor, && (\text{"}N \textit{ appears } N \textit{ times"}) \quad (\underline{A002024}),\\
n_2(N) &= \big\lfloor 1 + \sqrt{N-1} \big\rfloor, && (\text{"}N \textit{ appears } 2N-1 \textit{ times"}) \quad (\underline{A003059}),\\
n_3(N) &= \left\lfloor \frac{1}{6}\big(7 + \sqrt{24N-23}\big) \right\rfloor, && (\text{"}N \textit{ appears } 3N-2 \textit{ times"}) \quad (\underline{A083277}).
\end{aligned}$$

In Table 4 one can find the first few values of these sequences.

The family of sequences $(k_j(N))_{N\in\mathbb{N}}$ with the parameter $j \in \mathbb{N}$ means "*start counting the integers 1 to $jN - (j-1)$ followed by the integers 1 to $jN + 1, N \in \mathbb{N}$, and so on*".

The first three sequences are:

$$\begin{aligned}
k_1(N) &= N - \frac{1}{2} n_1(N)\big(n_1(N) - 1\big), (\text{"\textit{Restart counting after each new integer}"}) \, (\underline{A002260}),\\
k_2(N) &= N - \Big(\big\lfloor \sqrt{N-1} \big\rfloor\Big)^2, (\text{"\textit{Restart counting after each new odd integer}"}) \, (\underline{A071797}),\\
k_3(N) &= N - \frac{1}{2}\big(n_3(N) - 1\big)\big(3n_3(N) - 4\big), (\text{not available in the OEIS [16]}).
\end{aligned}$$

In Table 5 one can find the first few values of these sequences.

Obviously, for $j \to \infty$, we obtain the sequences

$$\big(n_\infty(N)\big)_{N\in\mathbb{N}} = (1, 2, 2, 2, 2\ldots) = \big(2 - (N = 1)\big)_{N\in\mathbb{N}}, \; (\underline{A040000})$$

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$k_1(N)$	1	1	2	1	2	3	1	2	3	4	1	2	3	4	5	1
$k_2(N)$	1	1	2	3	1	2	3	4	5	1	2	3	4	5	6	7
$k_3(N)$	1	1	2	3	4	1	2	3	4	5	6	7	1	2	3	4

Table 5: $k_j(N)$, $1 \leq j \leq 3$, $1 \leq N \leq 16$

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$n_1^{(1)}(N)$	1	3	5	8	11	14	18	22	26	30	35	40	45	50	55	61
$n_2^{(1)}(N)$	1	3	5	7	10	13	16	19	22	26	30	34	38	42	46	50
$n_3^{(1)}(N)$	1	3	5	7	9	12	15	18	21	24	27	30	34	38	42	46
$k_1^{(1)}(N)$	1	2	4	5	7	10	11	13	16	20	21	23	26	30	35	36
$k_2^{(1)}(N)$	1	2	4	7	8	10	13	17	22	23	25	28	32	37	43	50
$k_3^{(1)}(N)$	1	2	4	7	11	12	14	17	21	26	32	39	40	42	45	49

Table 6: $n_j^{(1)}(N)$, $k_j^{(1)}(N)$, $1 \leq j \leq 3$, $1 \leq N \leq 16$

and

$$\big(k_\infty(N)\big)_{N\in\mathbb{N}} = (1, 1, 2, 3, 4 \ldots) = \big(N - (N > 1)\big)_{N\in\mathbb{N}}, \text{ (\textit{A028310})},$$

where (A) is *Iverson's bracket convention*, meaning 1 if the statement A is true and 0 otherwise.

The sequences of the partial sums of these two families of sequences are by definition $n_j^{(1)}(N) := \sum_{i=1}^{N} n_j(i)$ and $k_j^{(1)}(N) := \sum_{i=1}^{N} k_j(i), N \in \mathbb{N}$.

In Table 6 one can find the first few values of the first three sequences of both families. The first sequence is (A060432), the second is (A327672), the fourth is (A014370), whereas the other three are not available in the OEIS [16].

Note that for $j \to \infty$ we obtain the sequences $n_\infty^{(1)}(N) = f_{1,2}(N) = 2N - 1$ and $k_\infty^{(1)}(N) = T_1(N, 1) = \frac{1}{2}N(N-1) + 1$, $N \in \mathbb{N}$.

3 Four Particular Sequences

In this section we shall study four special subsequences of the arrays $\big(T_j(n,k)\big)_{j\in\mathbb{N}}$ and the sequences of their partial sums.
The first sequence is obtained by choosing $k := f_{1,j}(n) = jn - (j-1)$. We get

$$d_j^{(0)}(n) := T_j(n, jn - (j-1)) = f_{2,j}(n) = \frac{1}{2}n\big(jn - (j-2)\big) \tag{3.1}$$

and these are the polygonal numbers.
Choosing $k := 1$ we obtain by Theorem 2.2 the second sequence

$$c_j^{(0)}(n) := T_j(n,1) = f_{2,j}(n) - f_{1,j}(n) + 1 = n + j\tbinom{n-1}{2} \tag{3.2}$$

and this is the first column in Tables 1,2 and 3.
Choosing $k := n$ we obtain again by Theorem 2.2 the third sequence

$$h_j^{(0)}(n) := T_j(n,n) = f_{2,j}(n) - f_{1,j}(n) + n = 2n - 1 + j\tbinom{n-1}{2}. \tag{3.3}$$

Notice that $h_j^{(0)}(n) = c_j^{(0)}(n) + n - 1$.
Summing up the values in a row (see the last columns in Tables 1,2 and 3 for the cases $j \in [3]$) we get the fourth sequence

$$s_j^{(0)}(n) := \sum_{k=1}^{jn-(j-1)} T_j(n,k). \tag{3.4}$$

In order to determine the partial sums of the first three sequences we need a lemma on figurate numbers.

Lemma 3.1 *Let $j, m, n \in \mathbb{N}$, then*

$$f_{m+1,j}(n) = \sum_{k=1}^{n} f_{m,j}(k) \tag{3.5}$$

$$(m+1)f_{m+1,j}(n) = (n+m-1)f_{m,j}(n) + f_{m,1}(n) \tag{3.6}$$

or, alternatively,

$$(m+1)f_{m+1,j}(n) = nf_{m,j}(n+1) + (1-j)f_{m,1}(n) \tag{3.7}$$

and for $n \geq 2$

$$f_{m,j+1}(n) = f_{m,j}(n) + f_{m,1}(n-1) \tag{3.8}$$

$$f_{m+1,j}(n) = f_{m,j}(n) + f_{m+1,j}(n-1) \tag{3.9}$$

$$f_{m,2j}(n) = f_{m,j}(n) + jf_{m,1}(n-1) \tag{3.10}$$

$$f_{m,2j+1}(n) = f_{m,j}(n) + (j+1)f_{m,1}(n-1). \tag{3.11}$$

Proof. By Eq.(2.1) and applying the formula of *Chu-Shih-Chieh (1303)*, also called *upper summation* (cf. [7, Table 174, p.174]),

$$\sum_{k=1}^{n} \binom{k+m-1}{m} = \binom{n+m}{m+1} \tag{3.12}$$

we obtain

$$\begin{aligned}\sum_{k=1}^{n} f_{m,j}(k) &= \sum_{k=1}^{n}\Big(\tbinom{k+m-2}{m-1} + j\tbinom{k+m-2}{m}\Big)\\ &= \sum_{k=1}^{n}\tbinom{k+m-2}{m-1} + j\sum_{k=1}^{n}\tbinom{k+m-2}{m}\\ &= \tbinom{n+m-1}{m} + j\tbinom{n+m-1}{m+1} = f_{m+1,j}(n)\end{aligned}$$

and this proves Eq.(3.5).

Again by Eq.(2.1) we have

$$\begin{aligned}(n+m-1)f_{m+1,j}(n) &= (n+m-1)\cdot\Big(\tbinom{n+m-2}{m-1} + j\tbinom{n+m-2}{m}\Big)\\ &= \frac{(n+m-1)!}{(m-1)!(n-1)!}\cdot\frac{m}{m} + j\cdot\frac{(n+m-1)!}{m!(n-2)!}\cdot\frac{m+1}{m+1}\\ &= \tbinom{n+m-1}{m}m + j\tbinom{n+m-1}{m+1}(m+1)\end{aligned}$$

and therefore

$$(n+m-1)f_{m,j}(n) + f_{m,1}(n) = (m+1)f_{m+1,j}(n),$$

thus proving Eq.(3.6).

To prove the alternative Eq.(3.7) we note that for $n+1$ instead of n we obtain from Eq.(2.1) $f_{m,j}(n+1) = \binom{n+m-1}{m-1}m + j\binom{n+m-1}{m}$, hence

$$\begin{aligned}(n+m-1)f_{m,j}(n) &= (n+m-1)\cdot\Big(\tbinom{n+m-2}{m-1} + j\tbinom{n+m-2}{m}\Big)\\ &= \frac{(n+m-1)!}{(m-1)!(n-1)!}\cdot\frac{n}{n} + j\cdot\frac{(n+m-1)!}{m!(n-2)!}\cdot\frac{n-1}{n-1}\\ &= n\tbinom{n+m-1}{m-1} + (n-1)j\tbinom{n+m-1}{m}\\ &= nf_{m,j}(n+1) - jf_{m,1}(n).\end{aligned}$$

Eq.(3.7) follows by inserting this equation into Eq.(3.6).

Now let $n \geq 2$, then by Eq.(2.1) we have

$$\begin{aligned}f_{m,j+1}(n) - f_{m,j}(n) &= \tbinom{n+m-2}{m-1} + (j+1)\tbinom{n+m-2}{m} - \Big(\tbinom{n+m-2}{m-1} + j\tbinom{n+m-2}{m}\Big)\\ &= \tbinom{n+m-2}{m} = \tbinom{(n-1)+m-1}{m} = f_{m,1}(n-1)\end{aligned}$$

and this proves Eq.(3.8).

Again by Eq.(2.1) it follows

$$\begin{aligned}f_{m+1,j}(n) - f_{m,j}(n) &= \tbinom{n+m-1}{m} + j\tbinom{n+m-1}{m+1} - \Big(\tbinom{n+m-2}{m-1} + j\tbinom{n+m-2}{m}\Big)\\ &= \tbinom{n+m-1}{m} - \tbinom{n+m-2}{m-1} + j\Big(\tbinom{n+m-1}{m+1} - \tbinom{n+m-2}{m}\Big)\\ &= \tbinom{n+m-2}{m} + j\tbinom{n+m-2}{m+1} = f_{m+1,j}(n-1)\end{aligned}$$

by using repeatedly Pascal's rule and this proves Eq.(3.9).

By definition we have $f_{m,2j}(n) = \binom{n+m-2}{m-1} + 2j\binom{n+m-2}{m}$ and $f_{m,j}(n) + jf_{m,1}(n-1) = \binom{n+m-2}{m-1} + j\binom{n+m-2}{m} + j\binom{n+m-2}{m} = \binom{n+m-2}{m-1} + 2j\binom{n+m-2}{m}$ and this proves Eq.(3.10).

Finally, again by definition and applying Pascal's rule we have $f_{m,2j+1}(n) = \binom{n+m-2}{m-1} + (2j+1)\binom{n+m-2}{m} = \binom{n+m-1}{m} + 2j\binom{n+m-2}{m}$ and $f_{m,j}(n) + (j+1)f_{m,1}(n-1) = \binom{n+m-2}{m-1} + j\binom{n+m-2}{m} + (j+1)\binom{n+m-2}{m} = \binom{n+m-1}{m} + 2j\binom{n+m-2}{m}$. This proves Eq.(3.11) and therefore the lemma is proved. □

We now make some remarks to Lemma 3.1.

Remark 3.2 *Formula (3.5) shows that for $m \geq 1$ the $(m+1)$-dimensional figurate numbers are the partial sums of the m-dimensional figurate numbers. In particular, for $m = 2$ and $s = j + 2 \geq 3$, we obtain*

$$P_s^n = \sum_{k=1}^{n} p_s^k. \tag{3.13}$$

Remark 3.3 *For $m = 1$ we obtain from Eq.(3.6)*

$$f_{2,j}(n) = \frac{1}{2}\big(nf_{1,j}(n) + f_{1,1}(n)\big) = \frac{1}{2}n\big(f_{1,j}(n) + 1\big). \tag{3.14}$$

This formula is a generalization of the sum of the first n natural numbers. (Choose in Eq.(3.14) $j = 1$, then $f_{1,1}(n) = n$ and by Eq.(3.5) $f_{2,1}(n) = \sum_{k=1}^{n} f_{1,1}(k) = \sum_{k=1}^{n} k$, hence $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.)
For $m = 2$ we get the formula of the Roman geometers and pupils of Heron of Alexandria Epaphroditus and Vitrius Rufus (about 100 A.D.), since $f_{3,j}(n) = \frac{1}{3}\big((n+1)f_{2,j}(n) + f_{2,1}(n)\big) = \frac{1}{6}(n+1)\big(2f_{2,j}(n) + n\big)$ or, equivalently, for $s \geq 3$

$$P_s^n = \frac{1}{6}(n+1)(2p_s^n + n), \tag{3.15}$$

which shows how to obtain a pyramidal number P_s^n from the corresponding polygonal number p_s^n. Note also that Eq.(3.15) is a generalization of the sum of the first n square numbers. (Choose in Eq.(3.15) $j = 2$, then $f_{2,2}(n) = n^2$ and by Eq.(3.5) $f_{3,2}(n) = \sum_{k=1}^{n} f_{2,2}(k) = \sum_{k=1}^{n} k^2$, thus obtaining $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.)

Moreover, the formula (3.6) shows that for all $j, n \in \mathbb{N}$ the m-dimensional figurate numbers $f_{m,j}(n)$ are the solution of the recurrence relation in $m \in \mathbb{N}$

$$a_{m+1,j}(n) = \frac{n+m-1}{m+1} \cdot a_{m,j}(n) + \frac{1}{m+1}\binom{n+m-1}{m} \tag{3.16}$$

with the initial value $a_{1,j}(n) = jn - (j-1)$.

Remark 3.4 *For $j = 1$ Eq.(3.7) gives $(m+1)f_{m+1,1}(n) = nf_{m,1}(n+1)$ (P.de Fermat(1636)), since $(m+1)\binom{n+m}{m+1} = \frac{(m+1)(n+m)!}{(m+1)!(n-1)!} = \frac{n(n+m)!}{m!n!} = n\binom{n+m}{m} = nf_{m,1}(n+1)$.*

Remark 3.5 *The special cases of Eq.(3.8) for* $m = 2 : p^n_{s+1} = p^n_s + p^{n-1}_3$ *and for* $m = 3 : P^n_{s+1} = P^n_s + P^{n-1}_3$ $(n \geq 2$ *and* $s \geq 3)$ *have been given by Nicomachus of Gerasa (about 100 A.D.). Note that from these formulae one can obtain by induction* $p^n_s = p^n_3 + (s-3)p^{n-1}_3$ *(C.G. Bachet (1621)) and* $P^n_s = P^n_3 + (s-3)P^{n-1}_3$ *(F. Maurolico (1575))* $(n \geq 2$ *and* $s \geq 3)$.

Remark 3.6 *For* $m = 2$ *and* $j = 2$ *Eq.(3.9) gives* $p^n_6 = f_{2,4}(n) = 2f_{2,1}(n-1) + f_{2,2}(n) = 2p^{n-1}_3 + n^2$, *(F. Maurolico (1575)), whereas for* $m = 2$ *and* $j = 1$ *Eq.(3.10) gives* $p^n_5 = f_{2,3}(n) = 2f_{2,1}(n-1) + f_{2,1}(n) = 2p^{n-1}_3 + p^n_3$. *Since* $p^n_3 = p^{n-1}_3 + n$ *we obtain in this way* $p^n_5 = 3p^{n-1}_3 + n$ *(Nicomachus of Gerasa (about 100 A.D.)). In general, we have for* $n \geq 2$ *and* $s \geq 3 : p^n_s = p^n_3 + (s-3)p^{n-1}_3 = p^{n-1}_3 + n + (s-3)p^{n-1}_3 = (s-2)p^{n-1}_3 + n$ *and* $p^n_s = p^n_3 + (s-3)p^{n-1}_3 = p^n_3 + p^{n-1}_3 + (s-4)p^{n-1}_3 = (s-4)p^{n-1}_3 + n^2$, *since* $p^n_3 + p^{n-1}_3 = \binom{n+1}{2} + \binom{n}{2} = n^2$.

Before proceeding, we recall that the *hypersequence of the mth generation*, $m \in \mathbb{N}$, of an arbitrary sequence $(a_n)_{n\in\mathbb{N}}$ is defined as $a^{(m)}_n := \sum_{k=1}^n a^{(m-1)}_k, a^{(0)}_n := a_n$.

Proposition 3.7 *Let* $m \geq 1$ *and* $d^{(m)}_j(n) := \sum_{k=1}^n d^{(m-1)}_j(k), c^{(m)}_j(n) := \sum_{k=1}^n c^{(m-1)}_j(k)$, $h^{(m)}_j(n) := \sum_{k=1}^n h^{(m-1)}_j(k)$ *be the hypersequences of the mth generation of* $\left(d^{(0)}_j(n)\right)_{n\in\mathbb{N}}$, $\left(c^{(0)}_j(n)\right)_{n\in\mathbb{N}}$ *and* $\left(h^{(0)}_j(n)\right)_{n\in\mathbb{N}}$, *respectively. Then for all* $m \in \mathbb{N}_0$ *and all* $n, j \in \mathbb{N}$:

$$d^{(m)}_j(n) = f_{m+2,j}(n) = \binom{n+m}{m+1} + j\binom{n+m}{m+2} \tag{3.17}$$

$$c^{(m)}_j(n) = f_{m+1,1}(n) + jf_{m+2,1}(n-2) = \binom{n+m}{m+1} + j\binom{n+m-1}{m+2} \tag{3.18}$$

$$h^{(m)}_j(n) = f_{m+1,2}(n) + jf_{m+2,1}(n-2) = \binom{n+m+1}{m+2} + (j-1)\binom{n+m-1}{m+2}. \tag{3.19}$$

Proof. By definition (3.1) $d^{(0)}_j(n) = f_{2,j}(n)$ and, therefore, applying Eq.(3.5) m times we obtain the first equation. The second one follows immediately from Eq.(2.1) and this proves (3.17).

Next, we obtain by Eq.(3.2) $c^{(0)}_j(n) = f_{1,1}(n) + jf_{2,1}(n-2)$, since $f_{1,1}(n) = n$ and $f_{2,1}(n) = \binom{n+1}{2}$. Hence, applying Eq.(3.5) m times we get $c^{(m)}_j(n) = f_{m+1,1}(n) + jf_{m+2,1}(n-2)$. The second equation now follows from Eq.(2.1) and this proves (3.18).

Finally, since $f_{1,2}(n) = 2n-1$ we have by (3.3) $h^{(0)}_j(n) = f_{1,2}(n) + jf_{2,1}(n-2)$. Applying Eq.(3.5) m times we obtain $h^{(m)}_j(n) = f_{m+1,2}(n) + jf_{m+2,1}(n-2)$ and this proves the first equation of (3.19). The second equation follows from (2.1) by noting that

$$\begin{aligned} f_{m+1,2}(n) &= \binom{n+m-1}{m} + 2\binom{n+m-1}{m+1} \\ &= \binom{n+m}{m+1} + \binom{n+m-1}{m+1} + \binom{n+m-1}{m+2} - \binom{n+m-1}{m+2} \\ &= \binom{n+m}{m+1} + \binom{n+m}{m+2} - \binom{n+m-1}{m+2} = \binom{n+m+1}{m+2} - \binom{n+m-1}{m+2} \end{aligned}$$

and using Pascal's rule several times. □

We notice that $h_j^{(m)}(n) = c_j^{(m)}(n) + f_{m+1,1}(n) - f_{m,1}(n) = c_j^{(m)}(n) + \binom{n+m+1}{m+1}$ by Pascal's rule.
Special cases of $c_j^{(m)}(n)$ are

- $m = 0$: $c_j^{(0)}(n) = \binom{n}{1} + j\binom{n-1}{2}$.
  $j = 1 : c_1^{(0)}(n) = \binom{n}{1} + \binom{n-1}{2} = \binom{n}{2} + 1$. These are the *centered polygonal numbers* (A000124). They give the maximal number of regions into which $n \in \mathbb{N}$ lines separate the plane (cf. [15, pp. 124–126]).
  $j = 2 : c_2^{(0)}(n) = n + 2\binom{n-1}{2} = n^2 - 2n + 2 = (n-1)^2 + 1$ (A002522). Note that $c_2^{(0)}(n)$ is one less than the arithmetic mean of its neighbours.
  $j = 3 : c_3^{(0)}(n) = n + 3\binom{n-1}{2} = \frac{1}{2}(3n^2 - 7n + 6)$ (A143689).

- $m = 1$: $c_j^{(1)}(n) = \binom{n+1}{2} + j\binom{n}{3}$.
  $j = 1 : c_1^{(1)}(n) = \frac{1}{6}n(n^2+5) = \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$. These numbers are the *3-dimensional analog of centered polygonal numbers* (A004006).
  $j = 2 : c_2^{(1)}(n) = \frac{1}{6}n(2n^2 - 3n + 7) = \binom{n}{1} + \binom{n}{2} + 2\binom{n}{3}$ (A081489).
  $j = 3 : c_3^{(1)}(n) = \frac{1}{2}n(n^2 - 2n + 3) = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3}$ (A064808). This is the sequence of the nth $(n+1)$-gonal number.

- $m = 2$: $c_j^{(2)}(n) = \binom{n+2}{3} + j\binom{n+1}{4}$.
  $j = 1 : c_1^{(2)}(n) = \frac{1}{24}n(n+1)(n^2 + n + 10)$ (A006522).
  $j = 2 : c_2^{(2)}(n) = \frac{1}{12}n(n+1)(n^2 - n + 6)$. This sequence is not available in the OEIS [16].
  $j = 3 : c_3^{(2)}(n) = \frac{1}{24}n(n+1)(3n^2 - 5n + 14)$. This sequence is not available in the OEIS [16].

and special cases of $h_j^{(m)}(n)$ are

- $m = 0$: $h_j^{(0)}(n) = \binom{n+1}{2} + (j-1)\binom{n-1}{2} = 2n - 1 + j\binom{n-1}{2}$.
  $j = 1 : h_1^{(0)}(n) = \binom{n+1}{2}$. These are the *triangular numbers* (A000217).
  $j = 2 : h_2^{(0)}(n) = n^2 - n + 1$. These are the *Hogben's centered polygonal numbers* (A002061).
  $j = 3 : h_3^{(0)}(n) = \frac{1}{2}(3n^2 - 5n + 4)$ (A104249).

- $m = 1$: $h_j^{(1)}(n) = \binom{n+2}{3} + (j-1)\binom{n}{3} = n^2 + j\binom{n}{3}$.
  $j = 1 : h_1^{(1)}(n) = \binom{n+2}{3}$. These are the *tetrahedral numbers* (A000292).
  $j = 2 : h_2^{(1)}(n) = \frac{1}{3}n(n^2 + 2)$ (A006527).
  $j = 3 : h_3^{(1)}(n) = \frac{1}{2}n(n^2 - n + 2)$ (A006000).

- $m = 2$: $h_j^{(2)}(n) = \binom{n+3}{4} + (j-1)\binom{n+1}{4} = \frac{1}{6}n(n+1)(2n+1) + j\binom{n+1}{4}$.
  $j = 1 : h_1^{(2)}(n) = \binom{n+3}{4}$. These are the *pentatope numbers* (A000332).
  $j = 2 : h_2^{(2)}(n) = \frac{1}{12}n(n+1)(n^2+n+4)$ (A006007).
  $j = 3 : h_3^{(2)}(n) = \frac{1}{24}n(n+1)(3n^2-n+10)$ (A133252)

In order to determine the hypersequence of the mth generation of the fourth sequence (3.4) we need a general formula which gives the hypersequence of the mth generation of an arbitrary sequence as the partial sums of another sequence.

Theorem 3.8 *Let $(a_n)_{n\in\mathbb{N}}$ be an arbitrary sequence (of real numbers) and $a_n^{(m)} := \sum_{k=1}^{n} a_k^{(m-1)}$, $m \in \mathbb{N}$, $a_n^{(0)} := a_n$, be the hypersequence of the mth generation of $(a_n)_{n\in\mathbb{N}}$. Then for all $m \in \mathbb{N}_0$:*

$$a_n^{(m)} = \sum_{k=1}^{n} \binom{n+m-1-k}{m-1} a_k = \sum_{k=1}^{n} \binom{m+k-2}{m-1} a_{n-k+1}. \tag{3.20}$$

Proof. By induction on $m \geq 0$. For $m = 0$ we have $\binom{n-1-k}{-1} = 0$ for $k \in [n-1]$ and for $k = n$ it is $\binom{-1}{-1} = 1$, hence $a_n^{(0)} = a_n$.
For $m = 1$ we have $\binom{n+1-1-k}{0} = 1$ for all $k \in [n]$, hence $a_n^{(1)} := \sum_{k=1}^{n} a_k$.
Let us now assume that the formula is true for some $m \geq 0$, then

$$\begin{aligned}
a_n^{(m+1)} &= \sum_{k=1}^{n} a_k^{(m)} = \sum_{k=1}^{n}\Big(\sum_{i=1}^{k} \binom{k+m-1-i}{m-1} a_i\Big) \\
&= \binom{m-1}{m-1}a_1 + \Big[\binom{2+m-1-1}{m-1}a_1 + \binom{2+m-1-2}{m-1}a_2\Big] + \\
&+ \Big[\binom{3+m-1-1}{m-1}a_1 + \binom{3+m-1-2}{m-1}a_2 + \binom{3+m-1-3}{m-1}a_3\Big] + \dots \\
&\dots + \Big[\binom{n+m-1-1}{m-1}a_1 + \binom{n+m-1-2}{m-1}a_2 + \dots + \binom{n+m-1-n}{m-1}a_n\Big].
\end{aligned}$$

Collecting the coefficients of $a_1, a_2, \dots, a_n$ we obtain

$$\begin{aligned}
a_n^{(m+1)} = &\Big[\binom{m-1}{m-1} + \binom{m}{m-1} + \binom{m+1}{m-1} + \dots + \binom{n+m-2}{m-1}\Big]a_1 + \\
&+ \Big[\binom{m-1}{m-1} + \binom{m}{m-1} + \binom{m+1}{m-1} + \dots + \binom{n+m-3}{m-1}\Big]a_2 + \dots + \binom{m-1}{m-1}a_n
\end{aligned}$$

Making use of the formula (3.12) for $m-1$ instead of m and noting that $\binom{m-1}{m-1} = \binom{m}{m}$ we finally obtain

$$\begin{aligned}
a_n^{(m+1)} &= \binom{n+m-1}{m}a_1 + \binom{n+m-2}{m}a_2 + \dots + \binom{m}{m}a_n \\
&= \sum_{k=1}^{n} \binom{n+m-k}{m} a_k = \sum_{k=1}^{n} \binom{n+(m+1)-1-k}{(m+1)-1} a_k
\end{aligned}$$

and this verifies Eq.(3.20) for $m+1$ and in conclusion we have proved the theorem for all $m \in \mathbb{N}_0$. The second equation follows from the fact that $k \in [n]$ if and only if $n-k+1 \in [n]$. □

Remark 3.9 *Formula (3.20) seems to be the discrete analogon of the formula of Cauchy*

$$\int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{n-1}} f(x_n)dx_n\cdots dx_2dx_1 = \frac{1}{(n-1)!}\int_{x_0}^{x} f(t)(x-t)^{n-1}dt, \tag{3.21}$$

according to which an n-fold repeated integral of the function $f(t)$ over the interval $[x_0, x]$ is expressed by a single integral with the integrand $\frac{(x-t)^{n-1}}{(n-1)!}f(t)$ over the same interval. Note that formula (3.21) says that $y(x) := \frac{1}{(n-1)!}\int_{x_0}^{x} f(t)(x-t)^{n-1}dt$ is the solution of the nth order differential equation $y^{(n)}(x) = f(x)$ with the initial values $y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0$.

Before applying Theorem (3.8) to the sequences $(1)_{n\in\mathbb{N}}$, $(n)_{n\in\mathbb{N}}$, $(n^2)_{n\in\mathbb{N}}$ and $(n^3)_{n\in\mathbb{N}}$ we need the following lemma.

Lemma 3.10 *For all $k, m, n \in \mathbb{N}$:*

$$f_{m+k,k}(n) = kf_{m+k,1}(n) - (k-1)f_{m+k-1,1}(n) \tag{3.22}$$

Proof. Since $f_{m,1}(n) = \binom{n+m-1}{m}$ we obtain by Pascal's rule

$$kf_{m+k,1}(n) = k\binom{n+m+k-1}{m+k} = k\left(\binom{n+m+k-2}{m+k} + \binom{n+m+k-2}{m+k-1}\right)$$

and

$$(k-1)f_{m+k,1}(n) = (k-1)\binom{n+m+k-2}{m+k-1}.$$

Hence,

$$\begin{aligned} kf_{m+k,1}(n) - (k-1)f_{m+k-1,1}(n) &= k\left(\binom{n+m+k-2}{m+k} + \binom{n+m+k-2}{m+k-1}\right) - (k-1)\binom{n+m+k-2}{m+k-1} \\ &= \binom{n+m+k-2}{m+k-1} + k\binom{n+m+k-2}{m+k} \end{aligned}$$

and the last term on the right-hand side is by definition $f_{m+k,k}(n)$. □

Alternatively, for $m-2$ instead of m, we obtain for all $k, m, n \in \mathbb{N}$

$$f_{m+k-2,k}(n) = kf_{m+k-2,1}(n) - (k-1)f_{m+k-3,1}(n). \tag{3.23}$$

In particular, for $k = 3$ and $m = 1$, $p_5^n = f_{2,3}(n) = 3f_{2,1}(n) - 2f_{1,1}(n) = 3p_3^n - 2n$.

Lemma 3.11 *For all $m, n \in \mathbb{N}$:*

$$\sum_{k=1}^{n}\binom{n+m-1-k}{m-1} = \binom{n+m-1}{m} = f_{m,1}(n) \tag{3.24}$$

$$\sum_{k=1}^{n}k\binom{n+m-1-k}{m-1} = \binom{n+m}{m+1} = f_{m+1,1}(n) \tag{3.25}$$

$$\sum_{k=1}^{n} k^2 \binom{n+m-1-k}{m-1} = 2\binom{n+m+1}{m+2} - \binom{n+m}{m+1} = f_{m+2,2}(n) \tag{3.26}$$

$$\sum_{k=1}^{n} k^3 \binom{n+m-1-k}{m-1} = 6\binom{n+m+1}{m+3} + \binom{n+m}{m+1} = 2f_{m+3,3}(n) - f_{m+2,2}(n). \tag{3.27}$$

Proof. Formula (3.24) follows from (3.12) for $m-1$ instead of m.
Further, using repeatedly (3.24), the definition (2.1) and the identity $\binom{m-1}{m-1} = \binom{m}{m}$ we obtain

$$\begin{aligned}
\sum_{k=1}^{n} k \binom{n+m-1-k}{m-1} &= 1\binom{n+m-2}{m-1} + 2\binom{n+m-3}{m-1} + 3\binom{n+m-4}{m-1} + \cdots + n\binom{m-1}{m-1} \\
&= \sum_{k=1}^{n} \binom{n+m-1-k}{m-1} + \sum_{k=2}^{n} \binom{n+m-1-k}{m-1} + \cdots + \sum_{k=n}^{n} \binom{n+m-1-k}{m-1} \\
&= \binom{n+m-1}{m} + \binom{n+m-2}{m} + \cdots + \binom{m-1}{m-1} = \binom{n+m}{m+1} = f_{m+1,1}(n).
\end{aligned}$$

To prove (3.26) note that by Eqs. (3.24) and (3.25) and since $\binom{m-1}{m-1} = \binom{m}{m}$ we have

$$\begin{aligned}
\sum_{k=1}^{n} k^2 \binom{n+m-1-k}{m-1} &= 1^2\binom{n+m-2}{m-1} + 2^2\binom{n+m-3}{m-1} + 3^2\binom{n+m-4}{m-1} + \cdots + n^2\binom{m-1}{m-1} \\
&= 1\sum_{k=1}^{n} \binom{n+m-1-k}{m-1} + 3\sum_{k=2}^{n} \binom{n+m-1-k}{m-1} + 5\sum_{k=3}^{n} \binom{n+m-1-k}{m-1} + \cdots \\
&\quad \cdots + (2n-1)\sum_{k=n}^{n} \binom{n+m-1-k}{m-1} \\
&= 1\binom{n+m-1}{m} + 3\binom{n+m-2}{m} + 5\binom{n+m-3}{m} + \cdots + (2n-1)\binom{m}{m} \\
&= \sum_{k=1}^{n} (2k-1)\binom{n+m-k}{m} = 2\sum_{k=1}^{n} k\binom{n+m-k}{m} - \sum_{k=1}^{n} \binom{n+m-k}{m} \\
&= 2\binom{n+m+1}{m+2} - \binom{n+m}{m+1} = 2f_{m+2,1}(n) - f_{m+1,1}(n) = f_{m+2,2}(n),
\end{aligned}$$

where the last equation follows from Lemma 3.10 for $k = 2$.

Finally, to prove (3.27) note that by Eqs. (3.24), (3.25) and (3.26) we have

$$\begin{aligned}
\sum_{k=1}^{n} k^3\binom{n+m-1-k}{m-1} &= 1^3\binom{n+m-2}{m-1} + 2^3\binom{n+m-3}{m-1} + 3^3\binom{n+m-4}{m-1} + \cdots + n^3\binom{m-1}{m-1} \\
&= 1\sum_{k=1}^{n}\binom{n+m-1-k}{m-1} + 7\sum_{k=2}^{n}\binom{n+m-1-k}{m-1} + 19\sum_{k=3}^{n}\binom{n+m-1-k}{m-1} + \cdots \\
&\quad \cdots + \left(n^3-(n-1)^3\right)\sum_{k=n}^{n}\binom{n+m-1-k}{m-1} \\
&= 1\binom{n+m-1}{m} + 7\binom{n+m-2}{m} + 19\binom{n+m-3}{m} + \cdots + (3n^2-3n+1)\binom{m}{m} \\
&= \sum_{k=1}^{n}(3k^2-3k+1)\binom{n+m-k}{m} \\
&= 3\sum_{k=1}^{n}k^2\binom{n+m-k}{m} - 3\sum_{k=1}^{n}k\binom{n+m-k}{m} + \sum_{k=1}^{n}\binom{n+m-k}{m} \\
&= 6\binom{n+m+1}{m+3} - 3\binom{n+m+1}{m+2} - 3\binom{n+m+1}{m+2} + \binom{n+m}{m+1} \\
&= 6f_{m+3,1}(n) - 6f_{m+2,1}(n) + f_{m+1,1}(n) \\
&= 2\big(3f_{m+3,1}(n) - 2f_{m+2,1}(n)\big) - \big(2f_{m+2,1}(n) - f_{m+1,1}(n)\big) \\
&= 2f_{m+3,3}(n) - f_{m+2,2}(n),
\end{aligned}$$

where the last equation follows from Lemma 3.10 for $k = 2$ and $k = 3$ noting that $\binom{m-1}{m-1} = \binom{m}{m}$. □

Corollary 3.12 *Let $(a_n)_{n\in\mathbb{N}}$ be an arithmetic progression with the initial value a_1 and the constant difference $d \in \mathbb{N}$, that is $a_n = a_1 + (n-1)d$. Then for all $m \in \mathbb{N}$ ($a_n^{(0)} := a_n$)*

$$a_n^{(m)} = \sum_{k=1}^{n} a_k^{(m-1)} = \frac{ma_1 + a_n}{m+1}\binom{n+m-1}{m}. \tag{3.28}$$

Proof. By Theorem 3.8 and applying Eqs. (3.24) and (3.25) it follows

$$\begin{aligned}
a_n^{(m)} &= \sum_{k=1}^{n}\binom{n+m-1-k}{m-1}\big(a_1+(k-1)d\big) \\
&= a_1\sum_{k=1}^{n}\binom{n+m-1-k}{m-1} + d\sum_{k=1}^{n}\binom{n+m-1-k}{m-1}(k-1) \\
&= a_1 f_{m,1}(n) - d\cdot f_{m,1}(n) + d\sum_{k=1}^{n}k\binom{n+m-1-k}{m-1} \\
&= (a_1-d)f_{m,1}(n) + d\cdot f_{m+1,1}(n) = \left(a_1 - d + d\cdot\frac{n+m}{m+1}\right)f_{m,1}(n) \\
&= \frac{ma_1 + a_1 + (n-1)d}{m+1}f_{m,1}(n) = \frac{ma_1+a_n}{m+1}f_{m,1}(n) = \frac{ma_1+a_n}{m+1}\binom{n+m-1}{m},
\end{aligned}$$

since $f_{m+1,1}(n) = \binom{n+m}{m+1} = \frac{n+m}{m+1}\binom{n+m-1}{m} = \frac{n+m}{m+1} f_{m,1}(n)$. $\square$

Note that the special case $m = 1$ gives the well-known formula $a_n^{(1)} = \frac{1}{2}n(a_1 + a_n)$ for the sum of the first n terms of an arithmetic progression.

We are now in a position to determine the fourth sequence $\big(s_j^{(0)}(n)\big)_{n\in\mathbb{N}}$ and all its hypersequences of the mth generation $s_j^{(m)}(n) := \sum_{k=1}^{n} s_j^{(m-1)}(k)$, $m \in \mathbb{N}$.

Proposition 3.13 *For all $j, n \in \mathbb{N}$:*

$$\begin{aligned} s_j^{(0)}(n) &= \frac{1}{2} f_{1,j}(n)\big(2f_{2,j}(n) - f_{1,j}(n) + 1\big) \\ &= \frac{1}{2}\big(jn - (j-1)\big)\big(jn^2 - 2(j-1)n + j\big) \end{aligned} \tag{3.29}$$

and for all $m \in \mathbb{N}_0$

$$\begin{aligned} s_j^{(m)}(n) &= j^2 f_{m+3,3}(n) - \frac{j(4j-3)}{2} f_{m+2,2}(n) + \frac{3j^2-4j+2}{2} f_{m+1,1}(n) - \frac{j(j-1)}{2} f_{m,1}(n) \\ &= 3j^2 f_{m+3,1}(n) - 3j(2j-1) f_{m+2,1}(n) + \frac{7j^2-7j+2}{2} f_{m+1,1}(n) - \frac{j(j-1)}{2} f_{m,1}(n). \end{aligned} \tag{3.30}$$

Proof. By (3.4) and (2.2) we have

$$\begin{aligned} s_j^{(0)}(n) &= \sum_{k=1}^{f_{1,j}(n)} T_j(n,k) = \sum_{k=1}^{f_{1,j}(n)} \big(f_{2,j}(n) - f_{1,j}(n) + k\big) \\ &= f_{1,j}(n)\big(f_{2,j}(n) - f_{1,j}(n)\big) + \sum_{k=1}^{f_{1,j}(n)} k \\ &= f_{1,j}(n)\big(f_{2,j}(n) - f_{1,j}(n)\big) + \frac{1}{2} f_{1,j}(n)\big(f_{1,j}(n) + 1\big) \\ &= \frac{1}{2} f_{1,j}(n)\big(2f_{2,j}(n) - f_{1,j}(n) + 1\big). \end{aligned}$$

This proves the first expression in (3.29). The second expression can be obtained by substituting $f_{1,j}(n) = jn - (j-1)$ and $f_{2,j}(n) = \frac{1}{2}n\big(jn - (j-2)\big)$ into this equation.

To prove the formula (3.30) we apply Theorem 3.8 to the sequence $\big(s_j^{(0)}(n)\big)_{n\in\mathbb{N}}$

$$s_j^{(m)}(n) = \sum_{k=1}^{n} \binom{n+m-1-k}{m-1} s_j^{(0)}(k). \tag{3.31}$$

Expanding (3.29) we obtain

$$s_j^{(0)}(n) = \frac{1}{2}\big(jn - (j-1)\big)\big(jn^2 - 2(j-1)n + j\big) = \alpha_j n^3 + \beta_j n^2 + \gamma_j n + \delta_j,$$

where $\alpha_j = \frac{1}{2}j^2$, $\beta_j = -\frac{3}{2}j(j-1)$, $\gamma_j = \frac{1}{2}(3j^2-4j+2)$ and $\delta_j = -\frac{1}{2}j(j-1)$. Substituting the last equation with these expressions into Eq.(3.31) we obtain

$$s_j^{(0)}(n) = \alpha_j S_3(n) + \beta_j S_2(n) + \gamma_j S_1(n) + \delta_j S_0(n),$$

where $S_0(n) := \sum_{k=1}^{n} \binom{n+m-1-k}{m-1}$, $S_1(n) := \sum_{k=1}^{n} k\binom{n+m-1-k}{m-1}$, $S_2(n) := \sum_{k=1}^{n} k^2\binom{n+m-1-k}{m-1}$, and $S_3(n) := \sum_{k=1}^{n} k^3\binom{n+m-1-k}{m-1}$. By Lemma 3.11 we have $S_0(n) = f_{m,1}(n)$, $S_1(n) = f_{m+1,1}(n)$, $S_2(n) = f_{m+2,2}(n)$, and $S_3(n) = 2f_{m+3,3}(n) - f_{m+2,2}(n)$. Hence,

$$\begin{aligned}
s_j^{(m)}(n) &= \alpha_j\big(2f_{m+3,3}(n) - f_{m+2,2}(n)\big) + \beta_j f_{m+2,2}(n) + \gamma_j f_{m+1,1}(n) + \delta_j f_{m,1}(n)\\
&= 2\alpha_j f_{m+3,3}(n) + (\beta_j - \alpha_j) f_{m+2,2}(n) + \gamma_j f_{m+1,1}(n) + \delta_j f_{m,1}(n)\\
&= j^2 f_{m+3,3}(n) - \frac{j(4j-3)}{2} f_{m+2,2}(n) + \frac{3j^2-4j+2}{2} f_{m+1,1}(n) - \frac{j(j-1)}{2} f_{m,1}(n).
\end{aligned}$$

This is the first formula given in Eq.(3.30), the second is obtained by using Lemma 3.10 for $k = 2$ and $k = 3$ and this proves the proposition. □

Remark 3.14 *Note that formula (3.29) can also be derived by the fact that $s_j^{(0)}(n)$ is by definition the sum of the arithmetic progression with the initial value $T_j(n,1) = c_j^{(0)}(n) = f_{2,j}(n) - f_{1,j}(n) + 1$, the constant difference $d = 1$ and $f_{1,j}(n) = jn - (j-1)$ terms. The last term is then $f_{2,j}(n)$. Hence, by Corollary 3.12 for $m = 1$, we have*

$$\begin{aligned}
s_j^{(0)}(n) &= \frac{1}{2} f_{1,j}(n)\big(f_{2,j}(n) - f_{1,j}(n) + 1 + f_{2,1}(n)\big)\\
&= \frac{1}{2}\big(jn-(j-1)\big)\Big(2\cdot\frac{n}{2}\big(jn-(j-1)\big) - \big(jn-(j-1)\big) + 1\Big)\\
&= \frac{1}{2}\big(jn-(j-1)\big)\big(jn^2 - 2(j-1)n + j\big)
\end{aligned}$$

and this is formula (3.29).

The first few sequences $s_j^{(m)}(n)$ for $j, n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ are:

- $m = 0$: $s_j^{(0)}(n) = \frac{1}{2}\big(jn-(j-1)\big)\big(jn^2-2(j-1)n+j\big)$.
  Note that formula (3.30) for $m = 0$ gives another representation of this sequence. Indeed, since $f_{3,1}(n) = \binom{n+2}{3}$, $f_{2,1}(n) = \binom{n+1}{2}$, $f_{1,1}(n) = n$ and $f_{0,1}(n) = 1$, we obtain

$$\begin{aligned}
s_j^{(0)}(n) &= 3j^2 f_{3,1}(n) - 3j(2j-1) f_{2,1}(n) + \frac{7j^2-7j+2}{2} f_{1,1}(n) - \frac{j(j-1)}{2} f_{0,1}(n)\\
&= 3j^2\frac{n}{6}(n+1)(n+2) - 3j(2j-1)\frac{n}{2}(n+1) + \frac{7j^2-7j+2}{2}n - \frac{j(j-1)}{2}\\
&= \frac{1}{2}j^2n^3 - \frac{3}{2}j(j-1)n^2 + \frac{1}{2}(3j^2-4j+2)n - \frac{1}{2}j(j-1).
\end{aligned}$$

This cubic polynomial in n has the simple root $\frac{j-1}{j}$ and therefore it can be factorized obtaining in this way formula (3.29).
Special cases are:
$j = 1 : s_1^{(0)}(n) = \frac{1}{2}n(n^2+1)$. These are the *centered triangular pyramidal numbers* (A006003). It is also the *row, column and diagonal magic sum* in the magic square of order n, since $\frac{1}{n}\sum_{k=1}^{n^2} = \frac{1}{2}n(n^2+1)$.
$j = 2 : s_2^{(0)}(n) = (2n-1)(n^2-n+1) = n^3+(n-1)^3$. These are the *centered cube numbers* (A005898).
$j = 3 : s_3^{(0)}(n) = \frac{1}{2}(3n-2)(3n^2-4n+3)$. This sequence is not available in the OEIS [16].

- $m = 1$: $s_j^{(1)}(n) = \frac{1}{8}n\big(jn-(j-2)\big)\big(jn^2-(j-2)n+2\big)$.
This formula can be derived from (3.30) for $m = 1$ noting that $f_{4,1}(n) = \binom{n+3}{4}$, $f_{3,1}(n) = \binom{n+2}{3}$, $f_{2,1}(n) = \binom{n+1}{2}$ and $f_{1,1}(n) = n$. We obtain after some algebraic simplifications
$$s_j^{(1)}(n) = \frac{1}{8}n\big(j^2n^3 - 2j(j-2)n^2 + (j^2-2j+4)n - 2(j-2)\big).$$
This cubic polynomial in n has the simple root $\frac{j-2}{j}$ and therefore it can be factorized obtaining in this way the asserted formula.
Since $s_j^{(1)}(n) = f_{2,1}\big(f_{2,j}(n)\big) = f_{2,1}\Big(\frac{1}{2}\big(jn-(j-1)\big)\Big)$ we see that $s_j^{(1)}(n)$ are (for all $j \in \mathbb{N}$) subsequences of the triangular numbers. Moreover, by Eq.(2.2)
$$\begin{aligned} s_j^{(1)}(n) &= T_1\big(f_{2,j}(n), f_{2,j}(n)\big) = \frac{1}{2}f_{2,j}(n)\big(f_{2,j}(n)-1\big) + f_{2,j}(n) \\ &= \frac{1}{2}f_{2,j}(n)\big(f_{2,j}(n)+1\big). \end{aligned} \tag{3.32}$$
Special cases are:
$j = 1 : s_1^{(1)}(n) = \frac{1}{8}n(n+1)(n^2+n+2)$. These are the *doubly triangularl numbers* (A002817). Since $f_{2,1}\big(f_{2,1}(n)\big) = f_{2,1}\big(\frac{1}{2}n(n+1)\big) = \frac{1}{2}\big(\frac{1}{2}n(n+1)\big)\big(\frac{1}{2}n(n+1)\big)+1\big) = \frac{1}{8}n(n+1)(n^2+n+2)$ this sequence is the second iterated function of $f_{2,1}(n)$.
$j = 2 : s_2^{(1)}(n) = \frac{1}{2}n^2(n^2+1)$ This is the sum of the first n^2 natural numbers (A037270).
$j = 3 : s_3^{(1)}(n) = \frac{1}{8}n(3n-1)(3n^2-n+2)$. This sequence is not available in the OEIS [16].

- $m = 2$: $s_j^{(2)}(n) = \frac{1}{120}n(n+1)\big(3j^2n^3-3j(j-5)n^2-(2j^2-5j-40)n+2(j^2-10j+20)\big)$.
This formula can be derived from (3.30) for $m = 2$ noting that $f_{5,1}(n) = \binom{n+4}{5}$, $f_{4,1}(n) = \binom{n+3}{4}$, $f_{3,1}(n) = \binom{n+2}{3}$ and $f_{2,1}(n) = \binom{n+1}{2}$. We obtain after some lengthy algebraic simplifications the asserted formula.

Special cases are:
$j=1: s_1^{(2)}(n)=\frac{1}{120}n(n+1)(n+2)(3n^2+6n+11)$. This sequence is not available in the OEIS [16].
$j=2: s_2^{(2)}(n)=\frac{1}{60}n(n+1)(2n+1)(3n^2+3n+4)=\frac{\big((n+1)^6-n^6\big)-\big((n+1)^2-n^2\big)}{60}$. These are the *polynexus numbers* of order 6 (A079547).
$j=3: s_3^{(2)}(n)=\frac{1}{120}n(n+1)(27n^3+18n^2+17n-2)$. This sequence is not available in the OEIS [16].

4 The Case $T_1(n,k)$

Recall that by Theorem 2.2 $T_1(n,k)=\frac{1}{2}n(n-1)+k, k\in[n]$ and that, conversely, by Theorem 2.4, the number N is in the nth row and kth column of the array $T_1(n,k)$, where $n=\left\lfloor\frac{1}{2}(1+\sqrt{8N-7})\right\rfloor$ and $k=N-\frac{1}{2}n(n-1)$. Note that in this and in the next chapter we will use repeatedly the rules $\lceil -x\rceil=-\lfloor x\rfloor$ for all $x\in\mathbb{R}$ (cf. [7, Eq.(3.4), p. 68]) and $\lfloor x\rfloor+n=\lfloor x+n\rfloor$ (resp. $\lceil x\rceil+n=\lceil x+n\rceil$ for all $x\in\mathbb{R}$ and $n\in\mathbb{Z}$ (cf. [7, Eq.(3.6), p. 69]).

Remark 4.1 *If $N\in\mathbb{N}$ is in the nth row of $T_1(n,k)$, where $n=\left\lfloor\frac{1}{2}(1+\sqrt{8N-7})\right\rfloor$, then $N+1$ is also in the same row if and only if N is not a triangular number, whereas $N+1$ is in the $(n+1)$st row if and only if N is a triangular number. This means that $\left\lfloor\frac{1}{2}(1+\sqrt{8N-7})\right\rfloor=\left\lfloor\frac{1}{2}(1+\sqrt{8(N+1)-7})\right\rfloor=\left\lfloor\frac{1}{2}(1+\sqrt{8N+1})\right\rfloor\Leftrightarrow N$ is not a triangular number and $\left\lfloor\frac{1}{2}(1+\sqrt{8(N+1)-7})\right\rfloor=\left\lfloor\frac{1}{2}(1+\sqrt{8N+1})\right\rfloor=\left\lfloor\frac{1}{2}(1+\sqrt{8N-7})\right\rfloor+1\Leftrightarrow N$ is a triangular number. Both assertions can be expressed more concisely as follows:*

$$\left\lfloor\frac{1}{2}(1+\sqrt{8N+1})\right\rfloor-\left\lfloor\frac{1}{2}(1+\sqrt{8N-7})\right\rfloor=(N\text{ is a triangular number})\tag{4.1}$$

or, equivalently,

$$\left\lfloor\frac{1}{2}(-1+\sqrt{8N+1})\right\rfloor-\left\lfloor\frac{1}{2}(-1+\sqrt{8N-7})\right\rfloor=(N\text{ is a triangular number})\tag{4.2}$$

Theorem 4.2 *Let $\varphi(n):=\frac{1}{2}(2n+1-\sqrt{8n+1})$, $n\in\mathbb{N}$. Then for all $n\geq 2$:*

$$T_1(n-1,n-1)=\big\lceil\varphi\big(T_1(n,n)\big)\big\rceil\tag{4.3}$$

and for all $k\in\mathbb{N}$ and for all $n\geq k$:

$$T_1(n-1,k)=\big\lceil\varphi\big(T_1(n,k)\big)\big\rceil.\tag{4.4}$$

Proof. Let N be a triangular number, that is $N=\frac{1}{2}n(n+1)$, $n\in\mathbb{N}$, then by Theorem 2.4 and by (4.1) N is the $n_1(N)$th row of $T_1(n,k)$, where

$$\begin{aligned}n_1(N)&=\left\lfloor\frac{1+\sqrt{8N-7}}{2}\right\rfloor=\left\lfloor\frac{1+\sqrt{8N+1}}{2}\right\rfloor-1=\left\lfloor\frac{1+\sqrt{4n(n+1)+1}}{2}\right\rfloor-1\\&=\left\lfloor\frac{1+\sqrt{(2n+1)^2}}{2}\right\rfloor-1=\frac{1+2n+1}{2}-1=n\end{aligned}$$

and in the kth column, where $k = N - \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1) - \frac{1}{2}n(n-1) = n$. Hence, by Theorem 2.2, $N = T_1(n,n)$ and $\varphi\big(T_1(n,n)\big) = \varphi(N) = \frac{1}{2}(2N+1-\sqrt{8N+1})$. Substituting $N = \frac{1}{2}n(n+1)$ into the last equation and simplifying we obtain $\varphi\big(T_1(n,n)\big) = \frac{1}{2}n(n-1) = T_1(n-1,n-1) \in \mathbb{N}$. Therefore, $\lceil\varphi\big(T_1(n,n)\big)\rceil = T_1(n-1,n-1)$ and this is Eq.(4.3).

Now let $N := T_1(n,k)$ be a non-triangular number, i.e., for all $n \in \mathbb{N} : N = T_1(n-1,k) \neq \frac{1}{2}n(n+1)$, then by Theorem 2.4, N is in the nth row and in the kth column of $T_1(n,k)$, where $n = \left\lfloor \frac{1}{2}(1+\sqrt{8N-7})\right\rfloor$ and $k = N - \frac{1}{2}n(n-1)$. Hence, by Remark 4.1 and using Eq.(4.1) we get

$$\begin{aligned} T_1(n-1,k) = T_1(n,k) - (n-1) = N - n + 1 &= N - \left\lfloor \frac{1+\sqrt{8N-7}}{2} \right\rfloor + 1 \\ = N - \left\lfloor \frac{1+\sqrt{8N+1}}{2} \right\rfloor + 1 &= N - \left\lfloor \frac{-1+\sqrt{8N+1}}{2} \right\rfloor \\ = \left\lceil \frac{2N+1-\sqrt{8N+1}}{2} \right\rceil &= \lceil\varphi(N)\rceil = \lceil\varphi\big(T_1(n,k)\big)\rceil. \end{aligned}$$

The proof of the theorem is complete. □

Remark 4.3 *In the proof of Theorem 4.2 we have used the well-known property of the triangular numbers*

$$8 \cdot \frac{n(n+1)}{2} + 1 = (2n+1)^2 \tag{4.5}$$

according to which N is a triangular number if and only if $8N+1$ is a perfect odd square number (cf. [4, p. 32]). Alternatively, for odd numbers x : $x^2 \equiv 1 \,(\mathrm{mod}\ 8)$ or 8 is a divisor of x^2-1. This result was already known to Plutarch (about 100 A.D.).

Applying Theorem 4.2 $n_1(N)-1$ times (noting that $N = T_1(n,k)$, $n \in \mathbb{N}$, $k \in [n]$) we obtain

Corollary 4.4 *Let φ defined as in Theorem 4.2, $N \in \mathbb{N}$, and*

$$\psi(N) := \lceil\varphi(N)\rceil, \quad \ell(N) := n_1(N) - 1 = \left\lfloor \frac{-1+\sqrt{8N-7}}{2} \right\rfloor.$$

Then the ℓth iterated function of ψ is given by $\psi^{\ell(N)}(N) = 1$, where $\psi^1(N) := \psi(N)$ and $\psi^{k+1}(N) := \psi(\psi^k(N))$, $k < \ell(N)$.

The meaning of Theorem 4.2 and Corollary 4.4 is as follows: Firstly, if $N \in \mathbb{N}$ is a triangular number, i.e. $N = T_1(n,n)$, $n \in \mathbb{N}$, then $\psi(N) = \psi\big(T_1(n,n)\big) = \lceil\varphi\big(T_1(n,n)\big)\rceil$ is by 4.3 equal to the preceding triangular number $T_1(n-1,n-1)$ and secondly, if $N \in \mathbb{N}$ is not a triangular number, that is $N = T_1(n,k)$, $n \in \mathbb{N}$, $k \in [n-1]$, then $\psi(N) = \psi\big(T_1(n,k)\big) = \lceil\varphi\big(T_1(n,k)\big)\rceil = T_1(n-1,k)$ is by 4.4 exactly the number above

$T_1(n,k)$ in the same column k. Hence, starting with an arbitrary $N \in \mathbb{N}$ in Table 1 we go a row upward in the same column until we reach a triangular number (in case N is a triangular number there is nothing to do). Then we continue going upward in Table 1 along the principal diagonal until we reach after $\ell(N) = n_1(N) - 1$ steps the first row with the value 1.

5 Application to the Tower of Hanoi with Four Pegs

In 1989 Hinz [8] presented an iterative algorithm for the minimal solution of the Tower of Hanoi with four pegs (also known as *the Reve's puzzle*) and called it the *presumed minimal solution* of the Tower of Hanoi with four pegs. Meanwhile, in 2014 T. Bousch [2] proved in a brilliant article that the *Frame-Stewart conjecture* is true for the Reve's puzzle. Since the number of moves of the iterative solution by Hinz is exactly the same as those given by the Frame-Stewart conjecture we can speak of an iterative minimal solution of the Reve's puzzle. (Other different iterative minimal solutions for the Reve's puzzle have been given by van de Liefvoort [12] and Lu [13].) The minimum number of moves is given as the solution of the recurrence relation

$$\nu(0) = 0, \quad \forall n \in \mathbb{N}: \quad \nu(n) = \min_{0 \le f < n} \{2\nu(f) + 2^{n-f} - 1\}, \tag{5.1}$$

where any $f \in [n]_0 := \{0, 1, 2, \ldots, n-1\}$ for which $\nu(n) = 2\nu(f) + 2^{n-f} - 1$ holds, is called a *minimum partition number* for n. In general, this number is not unique. In [8, Corollary, pp. 136–137] it is shown that for φ as defined in Theorem 4.2

$$f(n) = \lfloor \varphi(N) \rfloor \tag{5.2}$$

is a minimum partition number for n.
In this chapter we shall show that $\lceil \varphi(N) \rceil$ is a minimum partition number for n, too. To this end we recall the following recurrence relation (of first order) for $\nu(n)$ (cf. [11, Lemma 4.5 for $p = 4$, p. 151] and [10, p. 159 for $p = 4$])

$$\nu(0) = 0, \quad \forall n \in \mathbb{N}_0: \quad \nu(n+1) = \nu(n) + 2^{\lfloor \frac{1}{2}(-1+\sqrt{8n+1}) \rfloor} = \nu(n) + 2^{n - \lceil \varphi(n) \rceil}. \tag{5.3}$$

By summation one gets from (5.3) the formula (cf. [11, Theorem 4.6 for $p = 4$, p. 152])

$$\forall n \in \mathbb{N}_0: \quad \nu(n) = \sum_{k=0}^{n-1} 2^{\lfloor \frac{1}{2}(-1+\sqrt{8k+1}) \rfloor} = \sum_{k=0}^{n-1} 2^{k - \lceil \varphi(k) \rceil}. \tag{5.4}$$

Alternatively, from Eq.(5.3) we obtain a second recurrence relation (of second-order) for $\nu(n)$.

Corollary 5.1 *For all $n \in \mathbb{N}$:*

$$\nu(n+1) = \begin{cases} 3\nu(n) - 2\nu(n-1), & \textit{if } n \textit{ is a triangular number}, \\ 2\nu(n) - \nu(n-1), & \textit{otherwise}, \end{cases} \tag{5.5}$$

or, equivalently,

$$\nu(n+1) = 2\nu(n) - \nu(n-1) + \big(\nu(n) - \nu(n-1)\big) \cdot (n \textit{ is a triangular number}) \tag{5.6}$$

with the initial values $\nu(0) = 0$, $\nu(1) = 1$.

Proof. By Eq.(5.3) it is $\nu(n+1) - \nu(n) = 2^{\lfloor \frac{1}{2}(-1+\sqrt{8n+1}) \rfloor}$ and for $n-1$ instead of n we have $\nu(n) - \nu(n-1) = 2^{\lfloor \frac{1}{2}(-1+\sqrt{8n-7}) \rfloor}$. Hence, by subtraction $\nu(n+1) - \nu(n) - \big(\nu(n) - \nu(n-1)\big) = 2^{\lfloor \frac{1}{2}(-1+\sqrt{8n+1}) \rfloor} - 2^{\lfloor \frac{1}{2}(-1+\sqrt{8n-7}) \rfloor}$ or, equivalently, $\nu(n+1) = 2\nu(n) - \nu(n-1) + 2^{\lfloor \frac{1}{2}(-1+\sqrt{8n+1}) \rfloor} - 2^{\lfloor \frac{1}{2}(-1+\sqrt{8n-7}) \rfloor}$. By Eq.(4.2) one sees that when n is not a triangular number, the two powers of 2 in the last equation are equal. Therefore, $\nu(n+1) = 2\nu(n) - \nu(n-1)$. However, if n is a triangular number, then again by Eq.(4.2) and noting that $2^{\lfloor \frac{1}{2}(-1+\sqrt{8n-7}) \rfloor} = \nu(n) - \nu(n-1)$ we obtain

$$\begin{aligned} \nu(n+1) &= 2\nu(n) - \nu(n-1) + 2^{\lfloor \frac{1}{2}(-1+\sqrt{8n-7}) \rfloor} \\ &= 2\nu(n) - \nu(n-1) + \big(\nu(n) - \nu(n-1)\big) \\ &= 3\nu(n) - 2\nu(n-1) \end{aligned}$$

and this proves (5.5). The assertion (5.6) follows immediately from (5.5). $\square$

Finally, if n is a triangular number, then by Eq.(5.5) for $n-1$ instead of n we have $\nu(n) = 2\nu(n-1) - \nu(n-2)$ and therefore

$$\begin{aligned} \nu(n+1) &= 2\nu(n) + \nu(n) - 2\nu(n-1) \\ &= 2\nu(n) + 2\nu(n-1) - \nu(n-2) - 2\nu(n-1) = 2\nu(n) - \nu(n-2). \end{aligned}$$

Hence, by Corollary 5.1 we obtain a third recurrence relation (of third-order) for $\nu(n)$.

Corollary 5.2 *For all* $n \geq 2$:

$$\nu(n+1) = 2\nu(n) - \begin{cases} \nu(n-2), & \textit{if } n \textit{ is a triangular number,} \\ \nu(n-1), & \textit{otherwise,} \end{cases} \tag{5.7}$$

or, equivalently,

$$\nu(n+1) = 2\nu(n) - \nu(n-1) + \big(\nu(n-1) - \nu(n-2)\big) \cdot (n \textit{ is a triangular number}) \tag{5.8}$$

with the initial values $\nu(0) = 0$, $\nu(1) = 1$, $\nu(2) = 3$.

The first few values of the sequence $\big(\nu(n)\big)_{n \in \mathbb{N}_0}$ (A007664) are

$$(0, 1, 3, 5, 9, 13, 17, 25, 33, 41, 49, 65, 81, 97, 113, 129, 161, 193, 225, 257, 289, 321, 385, \ldots),$$

whereas the first few values of the difference $\big(\nu(n+1) - \nu(n)\big)_{n \in \mathbb{N}_0}$ (A137688) are

$$(1, 2, 2, 4, 4, 4, 8, 8, 8, 8, 16, 16, 16, 16, 16, 32, 32, 32, 32, 32, 32, 64, 64, 64, 64, 64, 64, 64, \ldots)$$

(*" 2^n appears $n+1$ times, $n \geq 0$"*). This is the non-decreasing sequence of numbers of the form $2^j \cdot 2^k$, $j, k \in \mathbb{N}_0$ (also denoted by $s^{(2)} := (s_i^{(2)})_{j \in \mathbb{N}}$). Note also that Eq.(5.4) can be written as $\nu(n) = \sum_{k=1}^{n} s_i^{(2)}$ (cf. [9, Corollary 5.6, p. 214]).

Proposition 5.3 *Let φ be defined as in Theorem (4.2) and $\tilde f(n) = \lceil\varphi(N)\rceil$. Then*

$$\forall n \in \mathbb{N}: \quad \nu(n) = 2\nu\big(\tilde f(n)\big) + 2^{n-\tilde f(n)} - 1, \tag{5.9}$$

i.e., together with $\lfloor\varphi(N)\rfloor$ is $\lceil\varphi(N)\rceil$ a minimum partition number for n, too. In particular, these two values are coincident for a triangular number (cf. [5, Theorem 2, pp. 235–236] and [12, p. 91]).

Proof. Since $\tilde f(n) - f(n) = \lceil\varphi(n)\rceil - \lfloor\varphi(n)\rfloor = (n$ is <u>not</u> a triangular number$)$, we distinguish two cases:
1) n is a triangular number, then $\tilde f(n) = f(n)$ and so Eq.(5.9) is established.
2) n is <u>not</u> a triangular number, then $\tilde f(n) = f(n) + 1$. Consider now the difference

$$\Delta(n) := 2\nu\big(\tilde f(n)\big) + 2^{n-\tilde f(n)} - 1 - \nu(n)$$

Inserting $\tilde f(n) = f(n)+1$ into the above equation and using Eqs.(5.2) and (5.3) for $f(n)$ instead of n we obtain

$$\begin{aligned}\Delta(n) &= 2\nu\big(f(n)+1\big) + 2^{n-f(n)-1} - \Big(2\nu\big(f(n)\big) + 2^{n-f(n)} - 1\Big)\\ &= 2\Big(\nu\big(f(n)+1\big) - \nu\big(f(n)\big)\Big) + 2^{n-f(n)-1} - 2^{n-f(n)}\\ &= 2^{1+f(n)-\lceil\varphi(f(n))\rceil} - 2^{n-f(n)-1}.\end{aligned}$$

The assertion (5.9) is proved if and only if the two exponents in the last equation are equal for all non-triangular numbers $n \in \mathbb{N}$. Hence, it remains to show that for all non-triangular numbers $N \in \mathbb{N}$ the following term

$$\begin{aligned}\delta(N) :&= 1 + f(N) - \lceil\varphi(f(N))\rceil - \big(N - f(N) - 1\big)\\ &= 2\big(f(N)+1\big) - \lceil\varphi(f(N))\rceil - N\\ &= 2\lceil\varphi(N)\rceil - \lceil\varphi\big(\lceil\varphi(N)\rceil - 1\big)\rceil - N\end{aligned}$$

is exactly 0. (Note that $\lceil\varphi(N)\rceil = f(N) + 1$.)

In order to prove this, assume that $\lceil\varphi(N)\rceil$ is a triangular number. Then, by Theorem 4.2 there is a number $n \in \mathbb{N}$ such that $N = T_1(n+1,n)$ and by Eq.(4.4) $\lceil\varphi(N)\rceil = \lceil\varphi\big(T_1(n+1,n)\big)\rceil = T_1(n,n)$ and $\lceil\varphi(N)\rceil - 1 = T_1(n,n) - 1 = T_1(n,n-1)$. Hence, by Eq.(4.3) $\lceil\varphi\big(\lceil\varphi(N)\rceil - 1\big)\rceil = \lceil\varphi\big(T_1(n,n-1)\big)\rceil = T_1(n-1,n-1)$. By Eq.(2.5) for $n-1$ instead of n and for $k = n-1$ and $j = 1$ we obtain $\delta(N) = 2\cdot T_1(n,n) - T_1(n-1,n-1) - T_1(n+1,n) = 0$.

In the other case, if $\lceil\varphi(N)\rceil$ is <u>not</u> a triangular number, then by Theorem 4.2 there are $n \in \mathbb{N}$ and $k \in [n-1]$ such that $N = T_1(n,k)$, $\lceil\varphi(N)\rceil = T_1(n-1,k)$. Hence, $\lceil\varphi(N)\rceil - 1 = T_1(n-1,k) - 1 = T_1(n-1,k-1)$ and $\lceil\varphi\big(\lceil\varphi(N)\rceil - 1\big)\rceil = \lceil\varphi\big(T_1(n-1,k-1)\big)\rceil = T_1(n-2,k-1)$. Therefore, by Eq.(2.5) for $n-2$ instead of n and $k-1$ instead of k and $j = 1$ we have $\delta(N) = 2\cdot T_1(n-1,k) - T_1(n-2,k-1) - T_1(n,k) = 0$.

Hence, for all $N \in \mathbb{N}$ we have obtained $\delta(N) = 0$ and this proves the proposition. $\square$

Note that Theorem 4.2 and Corollary 4.4 together with Proposition 5.3 tell us how many *subtowers*, that is regular states with $\underbrace{ss\ldots s}_{k}\underbrace{tt\ldots t}_{N-k}$, $s,t \in \{0,1,2,3\}$, $s \neq t$, $k \in \mathbb{N}$ and how many discs (this is the *height* of the subtower) have to be built in the iterative solution of the Tower of Hanoi with four pegs and N discs by Hinz, namely $n = n_1(N) = \left\lfloor \frac{1}{2}(1+\sqrt{8N-7}) \right\rfloor$ subtowers of heights $1 = \psi^{n-1}(N)$, $\psi^{n-2}(N), \ldots, \psi^1(N)$ and $\psi^0(N) = N$, respectively. Note that the subtower of height N is the ordinary tower of N discs. (For more details on the Tower of Hanoi with more than three pegs we refer to [9, Chapter 5].) For example, let $N = 25$, then by the above remark and looking at the red path in Figure 1 we obtain the sequence $25 \to 19 \to 14 \to 10 \to 6 \to 3 \to 1$, which tells us that in order to perform the iterative solution of the Reve's puzzle we have to build in all 7 subtowers oh heights $1, 3, 6, 10, 19$ and 25, respectively.

However, this is only one of all possible ways to solve the Reve's puzzle, since for instance the sequence $25 \to 18 \to 13 \to 9 \to 5 \to 3 \to 1$ (see the blue path in Figure 1) is also a solution by Eq.(5.2). To conclude, we want to count the number of paths in Table 1 from N to 1, where the only allowed directions are north and northwest.

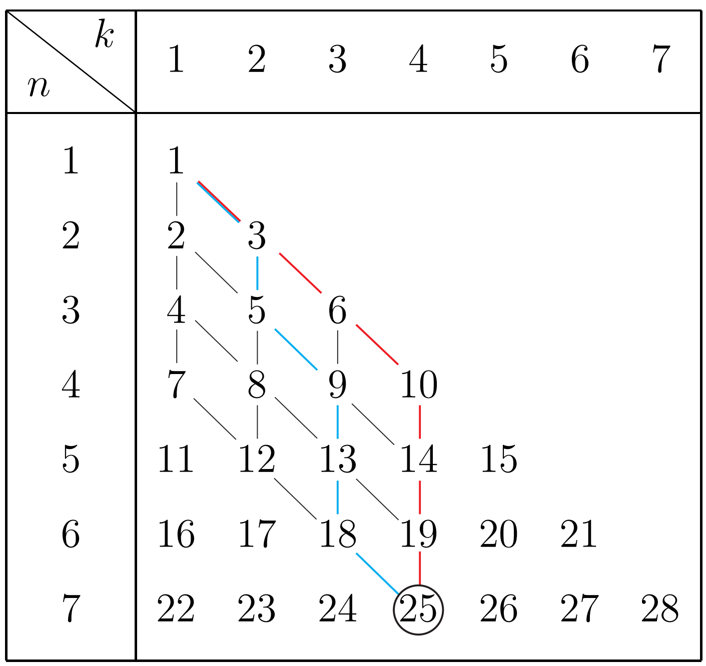

Figure 1: Two different optimal paths for the Reve's puzzle with 25 discs

Corollary 5.4 *Let $N \in \mathbb{N}$, then there are in Table 1 exactly*

$$\binom{n_1(N)-1}{\binom{n_1(N)+1}{2}-N} \tag{5.10}$$

paths in north and northwest direction of length $\ell(N) = n_1(N) - 1$ from $N \in \mathbb{N}$ to 1, where $n_1(N) = \left\lfloor \frac{1}{2}(1+\sqrt{8N-7}) \right\rfloor$. The solution is unique only for all N and $N+1$, where N is a triangular number.

Proof. Let $N \in \mathbb{N}$, then by Theorem 2.4 N is in the nth row and kth column of $T_1(n,k)$, where $n = n_1(N) = \left\lfloor \frac{1}{2}(1+\sqrt{8N-7}) \right\rfloor$ and $k = k_1(N) = N - \frac{1}{2}n_1(N)\cdot\big(n_1(N)-1\big) = N - \binom{n_1(N)}{2}$. The number of steps in north direction is given by $n_1(N) - k_1(N) = n_1(N) - \Big(N - \frac{1}{2}n_1(N)\cdot\big(n_1(N)-1\big)\Big) = \binom{n_1(N)+1}{2} - N$. Since the total length of each path is $n_1(N)-1$, the number of steps in northwest direction is $n_1(N)-1-\big(n_1(N)-k_1(N)\big) = k_1(N)-1$. A path can be described by a sequence of bits 0 and 1 of length $n_1(N)-1$, where the bit 0 indicates the north direction and 1 the northwest direction. Therefore, the total number of paths from N to 1 is equal to the number of sequences of $n_1(N)-1$ bits of which $n_1(N) - k_1(N) = \binom{n_1(N)+1}{2} - N$ are 0 and $k_1(N)-1$ are 1. This number is exactly the binomial coefficient given in Eq.(5.10).

If N is a triangular number, then $k_1(N) = n_1(N)$ and hence $\binom{n_1(N)-1}{0} = 1$. For $N+1$, N a triangular number, we have $k_1(N) = 1$, hence $\binom{n_1(N)-1}{n_1(N)-1} = 1$ and this proves the corollary. $\square$

Remark 5.5 *Theorem 4.2, Corollary 4.4, Proposition 5.3 and Corollary 5.4 give a complete answer to the remark by H. E. Dudeney in [6, p. 132] (at least for the case with 4 pegs):* " If the number of cheeses in the case of four stools is not triangular, ..., then there will be more than one way of making the piles and subsidiary tables will be required. This is the case with the Reve's 8 cheeses. But I will leave the reader to work out for himself the extension of the problem." *Obviously, the words cheeses, stools and piles stand for discs, pegs and subtowers, respectively. Indeed, for $N = 8$ we have $n = n_1(8) = \left\lfloor \frac{1}{2}(1+\sqrt{8\cdot 8-7}) \right\rfloor = 4$ and by Eq.(5.10) $\binom{4-1}{\binom{4+1}{2}-8} = \binom{3}{2} = 3$ different sequences of subtowers have to be built, namely $8 \to 4 \to 2 \to 1$, $8 \to 5 \to 2 \to 1$ and $8 \to 5 \to 3 \to 1$ (cf. also [9, pp. 240–245], private communication by Hinz, 2019).*

We remark that constructing all paths from a number N up to the number 1 as defined in the above corollary one obtains the two-dimensional (complete) *grid graph* $P_m \square P_n$, that is the Cartesian product of the *path graphs* P_m on $m = k_1(N) = N - \binom{n_1(N)}{2}$ vertices and P_n on $n = n_1(N) - k_1(N) + 1 = \binom{n_1(N)+1}{2} - N + 1$ vertices (see Figure 2). Therefore $P_m \square P_n$ has $|P_m \square P_n| = |P_m| \cdot |P_n| = k_1(N)\cdot\big(n_1(N) - k_1(N) + 1\big)$ vertices and $||P_m \square P_n|| = \big(|P_m|-1\big)\cdot|P_n| + \big(|P_n|-1\big)\cdot|P_m| = 2|P_m|\cdot|P_n| - \big(|P_m|+|P_n|\big) = 2k_1(N)\cdot\big(n_1(N)-k_1(N)+1\big) - \big(n_1(N)+1\big)$ edges.

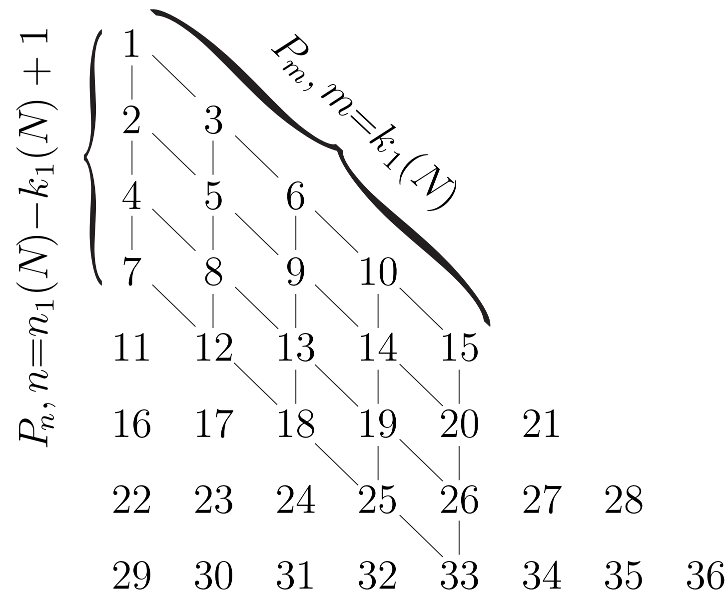

Figure 2: Grid graph $P_m \Box P_n$, $m = 5$, $n = 4$

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Introduction to and Strengthening of the Atiyah–Sutcliffe Conjectures for Several Types of four points Configurations

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Abstract

In 2001 Sir M. F. Atiyah formulated a conjecture C1 and later with P. Sutcliffe two stronger conjectures C2 and C3. These conjectures, inspired by physics (spin-statistics theorem of quantum mechanics), are geometrically defined for any configuration of points in the Euclidean three space. The conjecture C1 is proved for $n = 3, 4$ and for general n only for some special configurations (M. F. Atiyah, M. Eastwood and P. Norbury, D.Đoković). Interestingly the conjecture C2 (and also stronger C3) is not yet proven even for arbitrary four points in a plane. So far we have verified the conjectures C2 and C3 for parallelograms, cyclic quadrilaterals and some infinite families of tetrahedra.

We have also proposed a strengthening of conjecture C3 for configurations of four points (Four Points Conjectures).

For almost collinear configurations (with all but one point on a line) we propose several new conjectures (some for symmetric functions) which imply C2 and C3. By using computations with multi-Schur functions we can do verifications up to $n = 9$ of our conjectures. We can also verify stronger conjecture of Đoković which imply C2 for his nonplanar configurations with dihedral symmetry.

Finally we mention that by minimizing a geometrically defined energy, figuring in these conjectures, one gets a connection to some complicated physical theories, such as Skyrmions and Fullerenes.

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1 Introduction on Geometric Energies

In this Section we describe some geometric energies, introduced by Atiyah. To construct first geometric energy consider n distinct ordered points, $\mathbf{x}_i \in \mathbb{R}^3$ for $i = 1, ..., n$. For each pair $i \neq j$ define the unit vector

$$\mathbf{v}_{ij} = \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \tag{1.1}$$

giving the direction of the line joining $\mathbf{x}_i$ to $\mathbf{x}_j$. Now let $t_{ij} \in \mathbb{CP}^1$ be the point on the Riemann sphere associated with the unit vector $\mathbf{v}_{ij}$, via the identification $\mathbb{CP}^1 \cong S^2$, realized as stereographic projection. Next, set p_i to be the polynomial in t with roots t_{ij} $(j \neq i)$, that is

$$p_i = \alpha_i \prod_{j \neq i} (t - t_{ij}) \tag{1.2}$$

where α_i is a certain normalization coefficient. In this way we have constructed n polynomials which all have degree $n-1$, and so we may write

$$p_i = \sum_{j=1}^{n} m_{ij} t^{j-1}.$$

Finally, let M_n be the $n \times n$ matrix with entries m_{ij}, and let $\mathcal{D}_n$ be its determinant

$$\mathcal{D}_n = \mathcal{D}_n(\mathbf{x}_1, ..., \mathbf{x}_n) = \det M_n. \tag{1.3}$$

This geometrical construction is relevant to the Berry-Robbins problem, which is concerned with specifying how a spin basis varies as n point particles move in space, and supplies a solution provided it can be shown that $\mathcal{D}_n$ is always non-zero. For $n = 2, 3, 4$ it can be proved that $\mathcal{D}_n \neq 0$ (Atiyah $n = 3$, Eastwood and Norbury $n = 4$) and numerical computations suggest that $|\mathcal{D}_n| \geq 1$ for all n, with the minimal value $|\mathcal{D}_n| = 1$ being attained by n collinear points.

The geometric energy is the n-point energy defined by

$$E_n = -\log |\mathcal{D}_n|, \tag{1.4}$$

so minimal energy configurations maximize the modulus of the determinant.

This energy is geometrical in the sense that it only depends on the directions of the lines joining the points, so it is translation, rotation and scale invariant. Remarkably, the minimal energy configurations, studied numerically for all $n \leq 32$, are essentially the same as those for the Thomson problem.

2 Eastwood–Norbury formulas for Atiyah determinants

In this section we first recall Eastwood–Norbury formula for Atiyah determinant for three or four points in Euclidean three–space. In the case $n = 3$ the (non normalized) Atiyah determinant reads as

$$D_3 = d_3(r_{12}, r_{13}, r_{23}) + 8r_{12}r_{13}r_{23}$$

where

$$d_3(a, b, c) = (a + b - c)(b + c - a)(c + a - b)$$

and r_{ij} $(1 \leq i < j \leq 3)$ is the distance between the i^{th} and j^{th} point.

The normalized Atiyah determinant for 3 points is

$$\mathcal{D}_3 = \frac{D_3}{8r_{12}r_{13}r_{23}}$$

and it is evident that $|\mathcal{D}_3| = \mathcal{D}_3 \geq 1$.

In the case $n = 4$ the (non normalized) Atiyah determinant D_4 has real part given by a polynomial (with 248 terms) as follows:

$$\Re(D_4) = 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + A_4 + 288V^2 \quad (2.5)$$

where

$$A_4 = \sum_{l=1}^{4} \left(\sum_{(l \neq) i=1}^{4} r_{li}((r_{lj} + r_{lk})^2 - r_{jk}^2) \right) d_3(r_{ij}, r_{ik}, r_{jk})$$

(here $\{j, k\} = \{1, 2, 3, 4\} \setminus \{l, i\}$) and V denotes the volume of the tetrahedron with vertices our four points:

$$\begin{aligned} 144V^2 = \; & r_{12}^2 r_{34}^2 (r_{13}^2 + r_{14}^2 + r_{23}^2 + r_{24}^2 - r_{12}^2 - r_{34}^2) + \text{ two similar terms} \\ & -(r_{12}^2 r_{13}^2 r_{23}^2 + \text{ three similar terms}) \end{aligned} \quad (2.6)$$

We now state two formulas which will be used later:

1. Alternative form of A_4:

$$\begin{aligned} A_4 = \; & \sum_{l=1}^{4} \big((d_3(r_{il}, r_{jl}, r_{kl}) + 8r_{il}r_{jl}r_{kl} + r_{il}(r_{il}^2 - r_{jk}^2) + \\ & r_{jl}(r_{jl}^2 - r_{ik}^2) + r_{kl}(r_{kl}^2 - r_{ij}^2)\big) d_3(r_{ij}, r_{ik}, r_{jk}), \end{aligned} \quad (2.7)$$

where for each l we write $\{1, 2, 3, 4\} \setminus \{l\} = \{i < j < k\}$.

2. The sum of the second and the fourth term of (2.5) can be rewritten as

$$\begin{aligned} & 144V^2 - 2d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = \\ & = (r_{12} - r_{34})^2 (r_{13}^2 r_{24}^2 + r_{14}^2 r_{23}^2 - r_{12}^2 r_{34}^2) + \text{ two such terms } + \\ & + 4r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} - \\ & - r_{12}^2 r_{13}^2 r_{23}^2 - r_{12}^2 r_{14}^2 r_{24}^2 - r_{13}^2 r_{14}^2 r_{34}^2 - r_{23}^2 r_{24}^2 r_{34}^2. \end{aligned} \quad (2.8)$$

It is well known that this quantity is always nonpositive.

The imaginary part $\Im(D_4))$ of Atiyah determinant can be written as a product of $144V^2$ with a polynomial (with integer coefficients) having 369 terms.

The normalized Atiyah determinant for 4 points is

$$\mathcal{D}_4 = \frac{D_4}{2^{\binom{4}{2}} \prod\limits_{1\le i<j\le 4} r_{ij}}.$$

The original Atiyah conjecture in our cases is equivalent to nonvanishing of the determinants D_3 and D_4.

A stronger conjecture of Atiyah and Sutcliffe ([4],Conjecture 2) states in our cases that $|D_3| \ge 8r_{12}r_{13}r_{23}$ ($\Leftrightarrow |\mathcal{D}_3| \ge 1$) and $|D_4| \ge 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34}$ ($\Leftrightarrow |\mathcal{D}_4| \ge 1$).

From the formula (2.5) above, with the help of the simple inequality $d_3(a,b,c) \le abc$ (for $a,b,c \ge 0$), Eastwood and Norbury got "almost" the proof of the stronger conjecture by exhibiting the inequality

$$\Re(D_4) \ge 60r_{12}r_{13}r_{23}r_{14}r_{24}r_{34}.$$

To remove the word "almost" seems to be not so easy (at present not yet done even for planar configuration of four points).

A third conjecture (stronger than the second) of Atiyah and Sutcliffe ([4], Conjecture 3) can be expressed, in the four point case, in terms of polynomials in the edge lengths as

$$|D_4|^2 \ge \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk}) \tag{2.9}$$

where the product runs over the four faces of the tetrahedron.
(cf. `ftp://ftp.maths.adelaide.edu.au/pure/meastwood/atiyah.ps`)

In the first part of this paper we study some infinite families of quadrilaterals and tetrahedra and verify both Atiyah and Sutcliffe conjectures for several such infinite families. In this version of the paper we propose a somewhat stronger conjecture than (2.9) which reads as follows:

Conjecture 2.1 $\boxed{\textit{(Four Points Conjectures)}}$

$$\begin{aligned} &\Re(D_4) - (4+\tfrac{3}{4})\cdot 288V^2 \ge \\ &\ge 64 \prod_{1\le i,j\le 4} r_{ij} + \sum_{\{i<j<k\}\subset\{1,2,3,4\}} (4+\tfrac{1}{4}\delta) r_{il}r_{jl}r_{kl} d_3(r_{ij}, r_{ik}, r_{jk}) \end{aligned} \tag{2.10}$$

where

$$\delta = \begin{cases} \dfrac{d_3(r_{ij}, r_{ik}, r_{jk})}{r_{ij}r_{ik}r_{jk}} & , \textbf{\textit{weak version}} \\ 1 & , \textbf{\textit{strong version}} \end{cases}$$

Proposition 2.2 *Any of the Four Points Conjectures (2.10) imply conjecture (2.9).*

Proof .
By using the inequality $1 \geq d_3(a,b,c)/(abc), (a,b,c > 0)$ (see Appendix 2, Proposition 2.13) we see that the strong version implies the weak version of conjecture. We then rewrite the rhs of the weak version of (2.10) as follows:

$$\prod_{1\leq i,j\leq 4} r_{ij} \left(\frac{1}{4} \sum_{l=1}^{4} \left(\frac{d_3(r_{ij}, r_{ik}, r_{jk})}{r_{ij} r_{ik} r_{jk}} + 8 \right)^2 \right)$$

Finally, by the quadratic–geometric (QG) inequality we obtain

$$\geq \prod_{1\leq i,j\leq 4} r_{ij} \left(\prod_{l=1}^{4} \left(\frac{d_3(r_{ij}, r_{ik}, r_{jk})}{r_{ij} r_{ik} r_{jk}} + 8 \right) \right)^{\frac{2}{4}} = \left(\prod_{l=1}^{4} \left(d_3(r_{ij}, r_{ik}, r_{jk}) + 8 r_{ij} r_{ik} r_{jk} \right) \right)^{\frac{1}{2}}$$

Thus we obtain:

$$|D_4|^2 \geq |\Re(D_4)|^2 \geq |\Re(D_4) - (4+\tfrac{3}{4}) \cdot 288 V^2|^2 \geq \prod_{l=1}^{4} (d_3(r_{ij} r_{ik} r_{jk}) + 8 r_{ij} r_{ik} r_{jk})$$

i.e. the inequality (2.9). ■

Remark 2.3 *In terms of trigonometry (see subsection "Atiyah determinant for triangles and quadrilaterals via trigonometry" on page 114), the weak Four Points Conjecture can be written simply as*

$$\Re(D_4) - (4+\tfrac{3}{4}) \cdot 288 V^2 \geq \left(\prod_{1\leq i,j\leq 4} r_{ij} \right) \left(4 \sum_{l=1}^{4} c_l^2 \right)$$

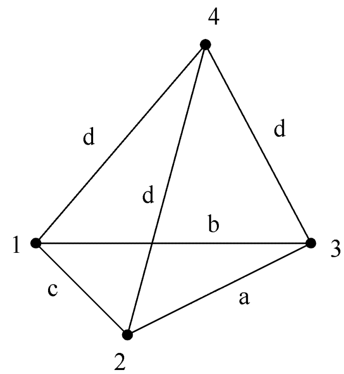
where

$$c_l := \cos^2 \frac{X^{(l)}}{2} + \cos^2 \frac{Y^{(l)}}{2} + \cos^2 \frac{Z^{(l)}}{2}, \quad l = 1,2,3,4.$$

and $X^{(l)}$, $Y^{(l)}$, $Z^{(l)}$ are the angles of the triangle opposite to the vertex l.

2.1 Atiyah–Sutcliffe conjecture for (vertically) upright tetrahedra (or pyramids)

We call a tetrahedron upright if some of its vertices (say 4) is equidistant from all the remaining vertices (1, 2 and 3, which we can think as lying in a horizontal plane.)



$r_{23} = a$
$r_{13} = b$
$r_{12} = c$
$r_{14} = r_{24} = r_{34} = d$

Note that then $d \geq R =$ the circumradius of the base triangle 123, then by Heron's formula we have: $R = abc/\sqrt{(a+b+c)d_3(a,b,c)}$.

Here, as before, $d_3(a,b,c) = (a+b-c)(a-b+c)(-a+b+c)$, $(a,b,c > 0)$.

The left hand side of the strong Four Points Conjecture 2.10 (but without $\frac{3}{4}$ term!) can be evaluated as follows, by using Eastwood-Norbury formula (2.5)

$$\begin{aligned} LHS &= \Re(D_4) - 4 \cdot 288V^2 = \\ &= 64 \prod_{1 \leq i < j \leq 4} r_{ij} - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + A_4 - 3 \cdot 288V^2 \end{aligned}$$

where

$$-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = -4d_3(a,b,c)d^3$$

$$\begin{aligned} A_4 = & \sum_{l=1}^{4} \left(\sum_{(l\neq)i=1}^{4} r_{li}((r_{lj}+r_{lk})^2 - r_{jk}^2) d_3(r_{ij}, r_{ik}, r_{jk}) \right) = \\ = & \sum_{cyc(a,b,c)} \left[c((b+d)^2 - d^2) + b((c+d)^2 - d^2) + d((b+c)^2 - a^2) \right] d_3(a,d,d) + \\ & + \left[d((d+d)^2 - a^2) + d((d+d)^2 - b^2) + d((d+d)^2 - c^2) \right] d_3(a,b,c) = \\ = & \quad [4bcd + ((b+c)^2 - a^2)d + b^2c + bc^2](2a^2d - a^3)] + \cdots + \\ & + [12d^3 - (a^2+b^2+c^2)d] d_3(a,b,c) \qquad (by \;\; 2.7) \end{aligned}$$

$$\begin{aligned} & -3 \cdot 288V^2 = \\ & = -6[(b^2+c^2-a^2)a^2d^2 + (c^2+a^2-b^2)b^2d^2 + (a^2+b^2-c^2)c^2d^2 - a^2b^2c^2] (by \; 2.6) \\ & = -6[(2a^2b^2 + 2a^2c^2 + 2b^2c^2)d^2 - a^2b^2c^2] - a^4 - b^4 - c^4 \\ & = -6[(a+b+c)d_3(a,b,c)d^2 - a^2b^2c^2] \end{aligned}$$

Similarly the right hand side of the Conjecture 2.10

$$
\begin{aligned}
RHS = &\quad 64 \prod_{1\le i<j\le 4} r_{ij} + \sum_{l=1}^{4} \left(4+\frac{1}{4}\right) r_{il}r_{jl}r_{kl} d_3(r_{ij}, r_{ik}, r_{jk}) \\
= &\quad 64abcd^3 + \left(4+\frac{1}{4}\right) bcd(2a^2d - a^3) + \text{ two such terms}+ \\
&\quad + \left(4+\frac{1}{4}\right) d^3 d_3(a,b,c)
\end{aligned}
$$

Now we can rewrite the difference

$$LHS - RHS = I + II$$

where

$$I = \sum_{cyc}(b^2c+bc^2)(2a^2d-a^3)-(a^2+b^2+c^2)d_3(a,b,c)d+6a^2b^2c^2-24\frac{a^2b^2c^2}{a+b+c}d$$

and

$$
\begin{aligned}
II = &\quad \left(4-\frac{1}{4}\right) d_3(a,b,c)d^3 + \sum_{cyc}((b+c)^2-a^2)d(2a^2d-a^3)- \\
&\quad -6\sum_{cyc}(b^2+c^2-a^2)a^2d^2 - \frac{1}{4}\sum_{cyc} bcd(2a^2d-a^3) + 24\frac{a^2b^2c^2}{a+b+c}d
\end{aligned}
$$

Then we can further simplify

$$
\begin{aligned}
I = &\quad \left[4abc(ab+ac+bc) - (a^2+b^2+c^2)d_3(a,b,c) - \frac{24a^2b^2c^2}{a+b+c}\right] d+ \\
&+6a^2b^2c^2 - \sum_{sym} a^3b^2c
\end{aligned}
$$

and

$$
\begin{aligned}
II = d &\left[\frac{15}{4}d_3(a,b,c)d^2 + (a+b+c)\left(\frac{7}{2}abc - 4d_3(a,b,c)\right)d + 24\frac{a^2b^2c^2}{a+b+c}+\right. \\
&\left. +\frac{1}{4}abc(a^2+b^2+c^2) - (a+b+c)\left(\sum_{sym} a^3b - a^4 - b^4 - c^4\right)\right]
\end{aligned}
$$

Lemma 2.4 *We have the following strengthening of the basic inequality for our function $d_3(a,b,c) = (a+b-c)(a-b+c)(-a+b+c)$:*

$$d_3(a,b,c) \le \frac{9a^2b^2c^2}{(a+b+c)(a^2+b^2+c^2)} \quad \left(\le \frac{27a^2b^2c^2}{(a+b+c)^3} \le abc\right)$$

Proof .
We have

$$
\begin{aligned}
&9a^2b^2c^2-(a^2+b^2+c^2)(a+b+c)d_3(a,b,c)=\\
&=9a^2b^2c^2-(a^2+b^2+c^2)(2a^2b^2+2a^2c^2+2b^2c^2-a^4+b^4+c^4)=\\
&=3a^2b^2c^2-a^4b^2-a^2b^4-a^4c^2-a^2c^4-b^4c^2-b^2c^4+a^6+b^6+c^6=\\
&(a^2-b^2)[a^2(a^2-c^2)-b^2(b^2-c^2)]+c^2(a^2-c^2)(b^2-c^2)\geq 0\\
&(\text{if we assume } a\geq b\geq c\geq 0)
\end{aligned}
$$

(a special instance of a Schur inequality) ■
(Note that this result follows from the formula $OG^2 = R^2-(a^2+b^2+c^2)/9$ for the distance of the circumcenter and the centroid of a triangle.)

Now we have

Lemma 2.5 *The quantity I is increasing w.r.t. d and it is positive for $d \geq R$.*

Proof .
We prove that the coefficient of d in I is positive by using that $(ab+ac+bc)(a+b+c) \geq 9abc$ and Lemma 2.4 .

The proof of positivity of I reduces to the positivity of the following quantity:

$$
\begin{aligned}
&\{[4abc(ab+ac+bc)-(a^2+b^2+c^2)d_3(a,b,c)](a+b+c)-24a^2b^2c^2\}^2-\\
&-(a+b+c)^3(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2-6abc)^2d_3(a,b,c)
\end{aligned}
$$

which by substituting $a = b+h$ and $b = c+k$ and then expanding has all coefficients positive (and ranging from 1 to 32151). ■

Lemma 2.6 *The quantity II is increasing w.r.t. d and it is positive for $d \geq R$.*

Proof .
Let $II = d\cdot III$. Then

$$\frac{\partial III}{\partial d} = \left(\tfrac{15}{2}d_3(a,b,c)d-\tfrac{a+b+c}{2}d_3(a,b,c)\right)+\tfrac{7(a+b+c)}{2}(abc-d_3(a,b,c))$$

The second term is positive by Proposition 2.13. For the first term we have:

$$
\begin{aligned}
&\tfrac{15}{2}d_3(a,b,c)d-\tfrac{a+b+c}{2}d_3(a,b,c)\geq \tfrac{15}{2}d_3(a,b,c)R-\tfrac{a+b+c}{2}d_3(a,b,c)\geq\\
&\left(\tfrac{15abc}{(a+b+c)^{3/2}}-\sqrt{d_3(a,b,c)}\right)\tfrac{a+b+c}{2}\sqrt{d_3(a,b,c)}\geq 0
\end{aligned}
$$

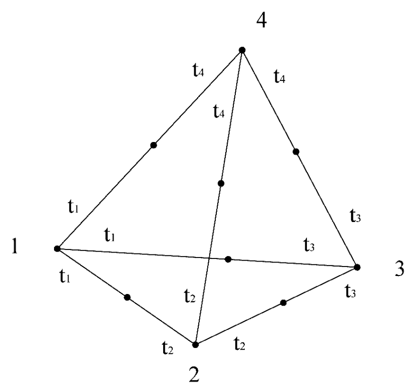
by Lemma (2.4).
The proof of positivity of II reduces to the positivity of the following quantity:

$$
\begin{aligned}
&\frac{15}{4}d_3(a,b,c)R^2+(a+b+c)\left(\frac{7}{2}abc-4d_3(a,b,c)\right)R+24\frac{a^2b^2c^2}{a+b+c}+\\
&+\tfrac{1}{4}abc(a^2+b^2+c^2)-(a+b+c)\left(\textstyle\sum_{sym}a^3b-a^4-b^4-c^4\right)
\end{aligned}
$$

which can be nicely visualized by `Maple` using tangential coordinates ($a = v+w$, $b = u+w$, $c = u+v$). ■

2.2 Atiyah–Sutcliffe conjectures for edge–tangential tetrahedra

By edge–tangential tetrahedron we shall mean any tetrahedron for which there exists a sphere touching all its edges (i.e. its 1–skeleton has an inscribed sphere.) For each i from 1 to 4 we denote by t_i the length of the segment (lying on the tangent line) with one endpoint the vertex and the other the point of contact of the tangent line with a sphere.



$$r_{ij} = t_i + t_j, \ (1 \le i < j \le 4)$$

Now we shall compute all the ingredients appearing in the Eastwood–Norbury formula for D_4 in terms of elementary symmetric functions of the (tangential) variables t_1, t_2, t_3, t_4 (recall $e_1 = t_1 + t_2 + t_3 + t_4$, $e_2 = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4$, $e_3 = t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4$, $e_4 = t_1t_2t_3t_4$).

$$64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} = 64 \prod_{1 \le i < j \le 4} (t_i + t_j) = 64 s_{3,2,1} =$$

$$= 64 \begin{vmatrix} e_3 & e_4 & 0 \\ e_1 & e_2 & e_3 \\ 0 & 1 & e_1 \end{vmatrix} = 64e_3e_2e_1 - 64e_4e_1^2 - 64e_3^2$$

Here we have used Jacobi–Trudi formula for the triangular Schur function $s_{3,2,1}$ (see [10], (3.5)). Furthermore we have

$$-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = 128e_4e_2 - 32e_4e_1^2 - 32e_3^2$$

$$288V^2 = 128e_4e_2 - 32e_3^2$$

In order to compute A_4 we first compute, for fixed l the following quantities

$$d_3(r_{ij}, r_{ik}, r_{jk}) = 8t_it_jt_k$$

$$\sum_{(l \neq) i=1}^{4} r_{li}((r_{lj} + r_{lk})^2 - r_{jk}^2) = 4(3t_l(t_1 + t_2 + t_3 + t_4) + 2(t_it_j + t_it_k + t_jt_k))t_l.$$

Thus we get:

$$A_4 = 32(3e_1^2 + 4e_2)e_4 = 96e_4e_1^2 + 128e_4e_2.$$

Now we adjust terms in D_4, in order to get shorter expression, as follows

$$\begin{aligned} D_4 = \; & (64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} - 2\cdot 288V^2)+ \\ & +(-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) - 288V^2) + A_4 + 4\cdot 288V^2 \\ = \; & (64e_3e_2e_1 - 64e_4e_1^2 - 256e_4e_2) + (-32e_4e_1^2)+ \\ & +(96e_4e_1^2 + 128e_4e_2) + 4\cdot 288V^2 \\ = \; & 64e_3e_2e_1 - 128e_4e_2 + 1152V^2 \\ = \; & 64e_2(e_3e_1 - 2e_4) + 1152V^2 \\ = \; & 64e_2(2e_4 + m_{211}) + 1152V^2, \end{aligned}$$

where $m_{211} = t_1^2t_2t_3 + \cdots$ denotes the monomial symmetric function associated to the partition $(2,1,1)$.

In order to verify the third conjecture of Atiyah and Sutcliffe

$$|D_4|^2 \geq \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk})$$

we note first that

$$\begin{aligned} d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk} &= (8t_it_jt_k + 8(t_i+t_j)(t_i+t_k)(t_j+t_k)) \\ &= 8(t_i+t_j+t_k)(t_it_j+t_it_k+t_jt_k) \end{aligned}$$

and state the following:

Lemma 2.7 *For any nonnegative real numbers $t_1, t_2, t_3, t_4 \geq 0$ the following inequality*

$$\begin{aligned} &(t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4)^2(2t_1t_2t_3t_4 + m_{211}(t_1,t_2,t_3,t_4))^2 \geq \\ &\geq \textstyle\prod_{\{i<j<k\}\subset\{1,2,3,4\}}(t_i+t_j+t_k)(t_it_j+t_it_k+t_jt_k) \end{aligned} \tag{2.11}$$

holds true.

Proof of Lemma 2.7.
The difference between the left hand side and the right hand side of the above inequality (2.11), written in terms of monomial symmetric functions is equal to

$$\begin{aligned} LHS - RHS = \; & m_{6321} + 3m_{6222} + m_{543} + 2m_{5421} + 7m_{5322} + 5m_{5331}+ \\ & +3m_{444} + 7m_{4431} + 8m_{4422} + 8m_{4332} + 3m_{3333} \geq 0 \end{aligned}$$

■

Remark 2.8 *One may think that the inequality in Lemma 2.7 can be obtained as a product of two simpler inequalities. This is not the case, because the following inequalities hold true:*

$$\begin{aligned} (t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4)^2 \;&\leq \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (t_i+t_j+t_k) \\ (2t_1t_2t_3t_4 + m_{211}(t_1,t_2,t_3,t_4))^2 \;&\geq \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (t_it_j+t_it_k+t_jt_k) \end{aligned}$$

Now we continue with verification of the third conjecture of Atiyah and Sutcliffe for edge tangential tetrahedron:

$$\begin{aligned}
|D_4|^2 \geq \quad & (D_4)^2 \geq [64e_2(2e_4 + m_{211})]^2 \\
\geq \; & 8^4 \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (t_i + t_j + t_k)(t_i t_j + t_i t_k + t_j t_k) \quad \text{(by Lemma 2.7)} \\
= \; & \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (d_3(r_{ij}, r_{ik}, r_{jk}) + 8 r_{ij} r_{ik} r_{jk})
\end{aligned}$$

so the strongest Atiyah–Sutcliffe conjecture is verified for edge–tangential tetrahedra.

2.3 Verification of the strong Four Points Conjecture for edge–tangential tetrahedra

The strong Four Points Conjecture 2.10 for edge tangential tetrahedra is equivalent to positivity of the following quantity:

$$\begin{aligned}
& \Re(D_4) - 64\prod r_{ij} - (4+\tfrac{3}{4})288V^2 - \sum_{l=1}^{4}(4+\tfrac{1}{4}) r_{il} r_{jl} r_{kl}\, d_3(r_{ij}, r_{ik}, r_{jk}) \\
& = (-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + A_4) + 288V^2 - (4+\tfrac{3}{4})288V^2 \\
& \quad - \sum_{l=1}^{4}(4+\tfrac{1}{4}) r_{il} r_{jl} r_{kl}\, d_3(r_{ij}, r_{ik}, r_{jk}) \\
& = (-32m_{3111} - 32m_{222} + 96m_{3111} + 320m_{2211}) - 240m_{2211} + 120m_{222} \\
& \quad - (34m_{3111} + 136m_{2211} + 34m_{222}) \\
& = 30m_{3111} + 54m_{222} - 56m_{2211}
\end{aligned}$$

In terms of augmented monomial symmetric functions

$$\widetilde{m}_\lambda(t_1, t_2, t_3, t_4) = \sum_{\sigma\in S_4} t^{\sigma.\lambda}$$

the last quantity is equal to

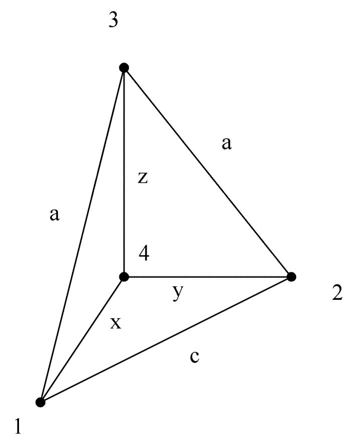
$$= 5\widetilde{m}_{3111} + 9\widetilde{m}_{222} - 14\widetilde{m}_{2211} \ (\geq 0 \text{ by Muirheads's inequality})$$

Thus, the strong Four Points Conjecture is verified for the edge–tangential tetrahedra.

Note that the verification of this conjecture which is stronger than A–S conjecture C3 is somewhat simpler (at least for edge–tangential tetrahedra).

2.4 Trirectangular tetrahedra

A tetrahedron is called trirectangular if it has a vertex at which all the face angles are right angles. The opposite face to such a vertex we call a base. We label the edge lengths as follows



$$
\begin{aligned}
r_{12} &= c \\
r_{13} &= b \\
r_{23} &= a \\
r_{14} &= x \\
r_{24} &= y \\
r_{34} &= z
\end{aligned}
$$

We have following obvious relations: $a^2 = y^2+z^2$, $b^2 = x^2+z^2$, $c^2 = x^2+y^2$. By using them we can get

$$
\begin{array}{ll}
d_3(ax, by, cz) & = 2xyz(ayz + bxz + cxy - abc) \\
d_3(a, b, c) & = 2(ax^2 + by^2 + cz^2 - abc), \\
d_3(x, y, c) & = 2xy(x + y - c), \\
d_3(x, b, z) & = 2xz(x + z - b), \\
d_3(a, y, z) & = 2yz(y + z - a)
\end{array}
\tag{2.12}
$$

and

$$
\begin{aligned}
&\Re(D_4) - 64abcxyz - 288V^2 = \\
&= 4xyz \sum_{cyc} 2ax^2 + \sum_{cyc} (2ab + cz + z^2)(x + y) - 10abc
\end{aligned}
\tag{2.13}
$$

where $\sum_{cyc}$ has three terms [1] corresponding to a cycle $((a, x) \to (b, y) \to (c, z))$. By writing $x + y = x + y - c + c$ and using the identity

$$
\sum_{cyc} c^2 z = \sum_{cyc} (x^2 + y^2)z = \sum_{cyc} (x + y)z^2 = \sum_{cyc} z^2(x + y - c) + \sum_{cyc} ax^2
$$

we get that the second cyclic sum is equal to

$$
\sum_{cyc} (2ab+cz+z^2)(x+y) = 6ab+\sum_{cyc} (2ab+cz+2z^2)(x+y-c)+2\sum_{cyc} ax^2 \tag{2.14}
$$

By inserting this into (2.13) we get

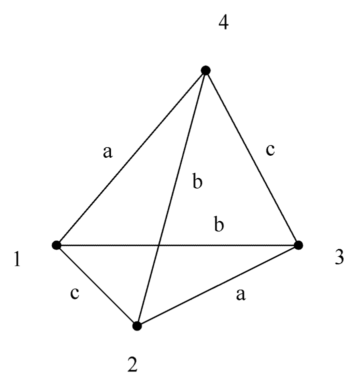
$$
\Re(D_4)-64abcxyz-288V^2 = 4xyz(2d_3(a, b, c)+\sum_{cyc} (2ab+cz+2z^2)(x+y-c))
$$

Hence $\Re(D_4) \geq 64abcxyz$ so the verification of the C2 of Atiyah–Sutcliffe for trirectangular tetrahedra is finished.

[1] $\sum_{cyc} f(a, b, c, x, y, z) = f(a, b, c, x, y, z) + f(b, c, a, y, z, x) + f(c, a, b, z, x, y)$

2.5 Atiyah–Sutcliffe conjectures for regular and semi–regular tetrahedra

Semiregular (SR) tetrahedra are one of the simplest configurations of tetrahedra. These tetrahedra have opposite edges equal and hence all faces are congruent. Sometimes semi–regular tetrahedra are called isosceles tetrahedra.



$$r_{23} = r_{14} = a$$
$$r_{13} = r_{24} = b$$
$$r_{12} = r_{34} = c$$

By (2.8) we get

$$288V^2 - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = 0 \; (\Rightarrow 288V^2 = 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}))$$

By (2.7) we get

$$\begin{aligned} A_4 &= \sum_{l=1}^{4} (d_3(r_{il}, r_{jl}, r_{kl}) + 8r_{il}r_{jl}r_{kl})d_3(r_{ij}, r_{ik}, r_{jk}) \\ &= 4d_3(a,b,c)^2 + 32abc\, d_3(a,b,c) \end{aligned}$$

The quantity in the weak Four Points Conjecture is

$$\begin{aligned} & l.h.s - r.h.s = \\ &= A_4 - \left(4 + \frac{3}{4}\right) 288V^2 - \sum_{l=1}^{4} \left(4 + \frac{1}{4}\frac{d_3(r_{ij}, r_{ik}, r_{jk})}{r_{ij}r_{ik}r_{jk}}\right) r_{il}r_{jl}r_{kl}d_3(r_{ij}, r_{ik}, r_{jk}) \\ &= 4d_3(a,b,c)^2 + 32abc\, d_3(a,b,c) - (16+3)d_3(a^2,b^2,c^2) - [16abc\, d_3(a,b,c) + d_3(a,b,c)^2] \\ &= 3(d_3(a,b,c)^2 - d_3(a^2,b^2,c^2)) + 16(abc\, d_3(a,b,c) - d_3(a^2,b^2,c^2)) \geq 0 \end{aligned}$$

by using the inequalities $abc \geq d_3(a,b,c)$ and $d_3(a,b,c)^2 \geq d_3(a^2,b^2,c^2)$ (see Appendix 2, Proposition 2.13; also see [19] or [20]).
This proves the weak Four Points Conjecture for semiregular tetrahedra. ■

We can also prove strong Four Points Conjecture by noting that

$$\frac{\Re(D_4)}{64a^2b^2c^2} \geq 1 + \frac{1}{8}\sum_{l}\left(4 + \frac{1}{4}\right)(\delta_l - 1)$$

(where $\delta_l = \dfrac{d_3(r_{ij}, r_{ik}, r_{jk})}{r_{ij}r_{ik}r_{jk}}$) is equivalent to

$$\sum \delta_l(\delta_l - 1) - \left(\frac{1}{2} + \frac{1}{32}\right)(\delta_l - 1) = \sum (\delta_l - 1)\left(\delta_l - \frac{1}{2} - \frac{1}{32}\right) \geq 0\,.$$

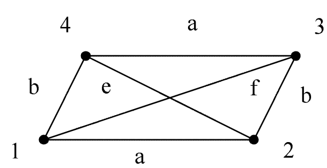
The proof of the strong Four Points Conjecture for semiregular tetrahedra reduces to the positivity of the following expression

$$4(d_3(a,b,c)^2 - d_3(a^2,b^2,c^2)) + 15(abc\, d_3(a,b,c) - d_3(a^2,b^2,c^2)) \geq 0$$

which is also true by the same argument.

2.6 Atiyah–Sutcliffe conjectures for parallelograms

Given a parallelogram with vertices 1, 2, 3 and 4 denote by a, b its side lengths and by e, f its diagonals.



$r_{12} = r_{34} = a$
$r_{23} = r_{14} = b$
$r_{24} = e$
$r_{13} = f$

For the numbers a, b, e, f we have the basic relation (”a parallelogram law”)

$$e^2 + f^2 = 2(a^2 + b^2) \tag{2.15}$$

By using this relation we can rewrite various quantities in the Eastwood-Norbury formula.

Proposition 2.9 *We have the following identities*

1. $d_3(a,b,e) = (a+b-e)(a-b+e)(-a+b+e) = (a+b-e)(a+b-f)(a+b+f)$
2. $\Delta := (a+b+e)d_3(a,b,e) = (a+b+f)d_3(a,b,f) =$
   $= (a+b+e)(a+b+f)(a+b-e)(a+b-f) =$
   $= 2a^2b^2 + 2a^2e^2 + 2b^2e^2 - a^4 - b^4 - e^4 = 2a^2b^2 + 2a^2f^2 + 2b^2f^2 - a^4 - b^4 - f^4$
3. $4ab + e^2 - f^2 = 2(a+b+f)(a+b-f)$, $4ab + f^2 - e^2 = 2(a+b+e)(a+b-e)$
4. $d_3(a^2, b^2, ef) = (a^2 + b^2 - ef)\Delta$
5. $d_3(a,b,e)d_3(a,b,f) - d_3(a^2,b^2,ef) = (2ab - 2ef - (a+b)(e+f))\Delta$
6. $ed_3(a,b,f) + fd_3(a,b,e) = (a+b-e)(a+b-f)(e^2 + f^2 + (a+b)(e+f))$
7. $(4ab + e^2 - f^2)ed_3(a,b,f) + (4ab + f^2 - e^2)fd_3(a,b,e) = 2((a+b)(e+f) - 2ef)\Delta$

Proof .
For *1.* we write $(a-b+e)(-a+b+e) = e^2-(a-b)^2 = 2a^2+2b^2-f^2-(a-b)^2 = (a+b-f)(a+b+f)$. Identity *2.* follows from *1.* directly. For *3.* we substitute $e^2 = 2a^2+2b^2-f^2$ and simplify. For *4.* we compute and use *2.*:

$$(a^2-b^2+ef)(-a^2+b^2+ef) = e^2f^2-(a^2-b^2)^2 = e^2(2a^2+2b^2-e^2)+2a^2b^2-a^4-b^4 = \Delta$$

For *5.* we first use *1.* and then *4.*: $d_3(a,b,e)d_3(a,b,f)-d_3(a^2,b^2,ef) = (a+b+f)(a+b-e)(a+b-f)d_3(a,b,f) - (a^2+b^2-ef)\Delta = [(a+b)^2-(a+b)(e+f)+ef]\Delta - (a^2+b^2-ef)\Delta = [2ab+2ef-(a+b)(e+f)]\Delta$

For *6.* we use *1.* twice.

For *7.* we first use *3.* and then *2.*:

$$\begin{aligned} l.h.s. &= 2(a+b+f)(a+b-f)ed_3(a,b,f)+2(a+b+e)(a+b-e)fd_(a,b,e) \\ &= 2[(a+b-f)e+(a+b-e)f]\Delta \end{aligned}$$

■

Now we apply Eastwood-Norbury formula (note that $288V^2 = 0$, $D_4 = real$)

$$D_4 - 64\prod r_{ij} = -4d_3(a^2,b^2,c^2)+A_4$$

where

$$\begin{aligned} A_4 &= 2[d_3(a,b,e)+8abe+e(e^2-f^2)]d_3(a,b,f)+2[d_3(a,b,f)+8abf+f(f^2-e^2)]d_3(a,b,e) \\ &= I_0+I_1+I_2 \end{aligned}$$

where

$$\begin{aligned} I_0 &= 4d_3(a,b,e)d_3(a,b,f) \\ I_1 &= 2[4abe+e(e^2-f^2)]d_3(a,b,f)+2[4abf+f(f^2-e^2)]d_3(a,b,e) \\ &= 4((a+b)(e+f)-2ef)\Delta \quad \text{(by 7.)} \\ I_2 &= 2[4abe\, d_3(a,b,f)+4abf\, d_3(a,b,e)] \\ &= 8ab(a+b-e)(a+b-f)(e^2+f^2+(a+b)(e+f)) \quad \text{(by } \mathit{6.}\text{)} \end{aligned}$$

By using *5.* we have

$$\begin{aligned} D_4 - 64\prod r_{ij} &= 4(d_3(a,b,e)d_3(a,b,f)-d_3(a^2,b^2,ef))+I_1+I_2 \\ &= 4((2ab+2ef-(a+b)(e+f))\Delta+((a+b)(e+f)-2ef)\Delta)+I_2 \\ &= 8ab\Delta+I_2 \geq 0 \end{aligned}$$

This proves the Atiyah–Sutcliffe conjecture (C2) for parallelograms. The Atiyah–Sutcliffe conjecture (C3) for parallelograms

$$D_4^2 \geq (d_3(a,b,e)+8abe)^2(d_3(a,b,f)+8abf)^2$$

is equivalent to the positivity of

$$D_4 - d_3(a,b,e)d_3(a,b,f) - 8[abf\, d_3(a,b,e)+abe\, d_3(a,b,f)] - 64a^2b^2ef \geq 0$$

but we can prove a stronger statement

$$\begin{aligned}&D_4 - 2d_3(a,b,e)d_3(a,b,f) - 8[abf\, d_3(a,b,e) + abe\, d_3(a,b,f)] - 64a^2b^2ef = \\ &= 8ab\Delta - 2d_3(a,b,e)d_3(a,b,f) = 2[4ab - (a+b-e)(a+b-f)]\Delta \geq 0\end{aligned}$$

because the triangle inequalities $b < e + a$ and $a < f + b$ imply

$$(a+b-e)(a+b-f) < 2a \cdot 2b = 4ab.$$

Thus we have verified also C3 for parallelograms.

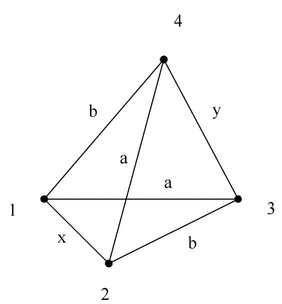
Finally we verify our strong Four Point Conjecture for parallelograms as follows

$$\begin{aligned}&D_4 - 64\prod r_{ij} - \sum(4+\tfrac{1}{4})r_{il}r_{jl}r_{kl}d_3(r_{ij}, r_{jl}, r_{ik}) \\ &= 8ab\Delta - \tfrac{1}{4}(I_2/4) \\ &= 8ab(a+b-e)(a+b-f)[(a+b+e)(a+b+f) - \tfrac{1}{16}(e^2+f^2+(a+b)(e+f))] \\ &= \tfrac{1}{2}ab(a+b-e)(a+b-f)[16((a+b)^2+(a+b)(e+f)+ef) - (2a^2+2b^2+(a+b)(e+f))] \\ &= \tfrac{1}{2}ab(a+b-e)(a+b-f)[14(a^2+b^2)+32ab+15(a+b)(e+f)+16ef] \geq 0\end{aligned}$$

■

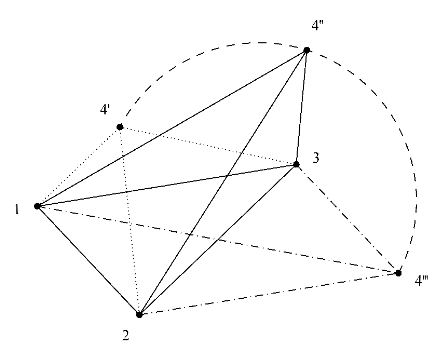
2.7 Atiyah–Sutcliffe conjectures for "wedge" tetrahedra

A tetrahedron with two pairs of opposite edges having the same length we simply call a "wedge" tetrahedron.



$$r_{12} = x$$
$$r_{34} = y$$
$$r_{13} = r_{24} = a$$
$$r_{23} = r_{14} = b$$

If $x = y = c$ we get a semiregular tetrahedron and if all points lie in a plane then we get either a parallelogram or an isosceles trapezium.



Again we compute the data appearing in the Eastwood–Norbury formula

$$\begin{aligned}&-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) =\\&= -4d_3(xy, a^2, b^2)\\&= -4(xy - a^2 + b^2)(xy + a^2 - b^2)(a^2 + b^2 - xy)\end{aligned} \tag{2.16}$$

and we have the basic inequalities

$$xy + b^2 \geq a^2,\ xy + a^2 \geq b^2,\ a^2 + b^2 \geq xy \tag{2.17}$$

The positivity of the volume

$$144V^2 = (xy-a^2+b^2)(xy+a^2-b^2)(2a^2+2b^2-x^2-y^2) \quad (\leq 2d_3(a^2, b^2, xy)) \tag{2.18}$$

gives us one more basic inequality

$$2a^2 + 2b^2 \geq x^2 + y^2 \tag{2.19}$$

We have

$$\begin{aligned}A_4 = &\ 2[a((b+x)^2 - a^2) + b((a+x)^2 - b^2) + x((a+b)^2 - y^2)]d_3(a,b,y)\\&+2[a((b+y)^2 - a^2) + b((a+y)^2 - b^2) + y((a+b)^2 - x^2)]d_3(a,b,x)\end{aligned} \tag{2.20}$$

By using identity

$$d_3(a,b,c) = a(b^2 + c^2 - a^2) + b(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2) - 2abc \tag{2.21}$$

we can rewrite A_4 as follows

$$\begin{aligned}A_4 &= 2[4abx + d_3(a,b,x) - x(a^2 + b^2 - x^2) + 2abx + x((a+b)^2 - y^2)]d_3(a,b,y)\\&\quad +2[4aby + d_3(a,b,y) - y(a^2 + b^2 - y^2) + 2aby + y((a+b)^2 - x^2)]d_3(a,b,x)\\&= 2[4abx + d_3(a,b,x) - x((a-b)^2 - x^2) + x((a+b)^2 - y^2)]d_3(a,b,y)\\&\quad +2[4aby + d_3(a,b,y) - y((a-b)^2 - y^2) + y((a+b)^2 - x^2)]d_3(a,b,x)\\&= \{8abx + d_3(a,b,x) + [d_3(a,b,x) - 2x((a-b)^2 - x^2)] + 2x((a+b)^2 - y^2)\}d_3(a,b,y)\\&\quad +\{8aby + d_3(a,b,y) + [d_3(a,b,y) - 2y((a-b)^2 - y^2)] + 2y((a+b)^2 - x^2)\}d_3(a,b,x)\end{aligned} \tag{2.22}$$

Now we compute

$$\begin{aligned}&d_3(a,b,x) - 2x((a-b)^2 - x^2) =\\&= (a+b-x)(a-b+x)(-a+b+x) + 2x(a-b+x)(-a+b+x)\\&= (a+b+x)(a-b+x)(-a+b+x)\end{aligned}$$

The contribution $A_4^{[\,]}$ of both square brackets in A_4 is equal to

$$\begin{aligned}A_4^{[\,]} &:= [d_3(a,b,x) - 2x((a-b)^2 - x^2)]d_3(a,b,y) +\\&\quad + [d_3(a,b,y) - 2y((a-b)^2 - y^2)]d_3(a,b,x)\\&= (x^2 - (a-b)^2)(y^2 - (a-b)^2)[(a+b+x)(a+b-y) + (a+b+y)(a+b-x)]\\&= (x^2 - (a-b)^2)(y^2 - (a-b)^2)(2(a+b)^2 - 2xy)\\&= 4ab(x^2 - (a-b)^2)(y^2 - (a-b)^2) + 2(x^2 - (a-b)^2)(y^2 - (a-b)^2)(a^2 + b^2 - xy)\end{aligned}$$

$$(2.23)$$

At this point we have discovered the following beautiful identity

$$\begin{aligned}&[x^2-(a-b)^2][y^2-(a-b)^2]=\\&=(xy-a^2+b^2)(xy+a^2-b^2)+(a-b)^2(2a^2+2b^2-x^2-y^2)\end{aligned}\qquad(2.24)$$

By this identity we can write

$$\begin{aligned}A_4^{[\ ]}&=4ab(x^2-(a-b)^2)(y^2-(a-b)^2)\\&+2(a-b)^2(2a^2+2b^2-x^2-y^2)(a^2+b^2-xy)\\&+2d_3(a^2,b^2,xy)\end{aligned}$$

Lemma 2.10 *We have the following inequality for "wedge" tetrahedra*

$$d_3(a^2,b^2,xy)\le 2ab(x^2-(a-b)^2)(y^2-(a-b)^2)$$

Proof .
Recall that

$$d_3(a^2,b^2,xy)=(a^2+b^2-xy)(a^2-b^2+xy)(-a^2+b^2+xy)$$

Let $a\ge b$. Then the triangle inequalities $a\le b+x$ and $a\le b+y$ imply $(a-b)^2\le xy$ i.e. $a^2+b^2-xy\le 2ab$. Since $2a^2+2b^2-x^2-y^2\ge 0$ (inequality (2.19)) then from our inequality (2.24) it follows that

$$(a^2-b^2+xy)(-a^2+b^2+xy)\le (x^2-(a-b)^2)(y^2-(a-b)^2)$$

By multiplying the last two inequalities Lemma follows. ∎

As a consequence of Lemma we get immediately that

$$A_4\ge A_4^{[\ ]}\ge 4d_3(a^2,b^2,xy)$$

because the remaining terms in A_4 are all nonnegative. This verifies the A–S conjecture C2 for "wedge" tetrahedra.

Remark 2.11 *Instead of splitting* $2(a+b)^2-2xy=4ab+2(a^2+b^2-xy)$ *(used above), we can use the identity*

$$2(a+b)^2-2xy=4(a^2+b^2-xy)+2(xy-(a-b)^2)$$

to obtain explicit formula for $A_4^{[\ ]}$*:*

$$\begin{aligned}&A_4^{[\ ]}=\\&[4(a^2+b^2-xy)+2(xy-(a-b)^2)][(xy-a^2+b^2)(xy+a^2-b^2)+\\&+(a-b)^2(2a^2+2b^2-x^2-y^2)]=\\&=4d_3(a^2,b^2,xy)+4(a^2+b^2-xy)(2a^2+2b^2-x^2-y^2)(a-b)^2+\\&+2(xy-(a-b)^2)(x^2-(a-b)^2)(y^2-(a-b)^2)\end{aligned}$$

which, without using Lemma 2.10, implies inequality

$$A_4^{[\ ]}\ge 4d_3(a^2,b^2,xy)$$

needed for the verification of A–S conjecture C2 for "wedge" tetrahedra.

Now we state a final formula for "wedge" tetrahedra:

$$\boxed{\begin{array}{ll} & \underline{\text{First explicit formula for wedge tetrahedra}}: \\ \Re(D_4) = & (d_3(a,b,x)+8abx)(d_3(a,b,y)+8aby)+d_3(a,b,x)d_3(a,b,y)+ \\ & +2x((a+b)^2-y^2)d_3(a,b,y)+2y((a+b)^2-x^2)d_3(a,b,x)+ \\ & +4(a^2+b^2-xy)(2a^2+2b^2-x^2-y^2)(a-b)^2+ \\ & +2(xy-(a-b)^2)(x^2-(a-b)^2)(y^2-(a-b)^2) \\ & +288V^2 \end{array}}$$

which implies a strengthened A–S conjecture C3 for wedge tetrahedra

$$\begin{aligned} \Re(D_4) \geq &\ (d_3(a,b,x)+8abx)(d_3(a,b,y)+8aby)+d_3(a,b,x)d_3(a,b,y)+288V^2 \\ \geq &\ (d_3(a,b,x)+8abx)(d_3(a,b,y)+8aby) \end{aligned}$$

In the sequel we obtain an alternative formula for the real part of the Atiyah determinant for a wedge tetrahedra.

We group terms in A_4 differently as follows:

$$\begin{aligned} A_4 = &\ 2[4abx+d_3(a,b,x)+x(4ab+x^2-y^2)]d_3(a,b,y)+ \\ &+2[4aby+d_3(a,b,y)+x(4ab+y^2-x^2)]d_3(a,b,x) \end{aligned}$$

By letting

$$2a^2+2b^2-x^2-y^2 =: 2h \quad (\geq 0)$$

we can rewrite

$$4ab+x^2-y^2 = 4ab+x^2+(2h+x^2-2a^2-2b^2) = 2(h+x^2-(a-b)^2)$$

and similarly for

$$4ab+y^2-x^2 = 2(h+y^2-(a-b)^2)$$

Thus

$$\begin{aligned} A_4 = &\ 4d_3(a,b,x)d_3(a,b,y)+8abx\,d_3(a,b,y)+8aby\,d_3(a,b,x)+ \\ &+4h(x\,d_3(a,b,y)+y\,d_3(a,b,x))+4A_4' \end{aligned}$$

where

$$\begin{aligned} A_4' &= x(x^2-(a-b)^2)d_3(a,b,y)+y(y^2-(a-b)^2)d_3(a,b,x) \\ &= (x^2-(a-b)^2)(y^2-(a-b)^2)[x(a+b-y)+y(a+b-x)] \\ &= (x^2-(a-b)^2)(y^2-(a-b)^2)[(x-y)^2+x(a+b-x)+y(a+b-y)] \\ &= [(xy-a^2+b^2)(xy+a^2-b^2)+2(a-b)^2h][(x-y)^2+x(a+b-x)+y(a+b-y)] \end{aligned} \tag{2.25}$$

by our identity (2.24).

Note that

$$-144V^2+2d_3(a^2,b^2,xy) = (xy-a^2+b^2)(xy+a^2-b^2)(x-y)^2$$

So

$$\begin{aligned}A_4' &= (2d_3(a^2,b^2,xy) - 144V^2) + 2(a-b)^2h[x(a+b-y) + y(a+b-x)]\\ &+ (xy - a^2 + b^2)(xy + a^2 - b^2)(x(a+b-x) + y(a+b-y))\end{aligned}$$

By writing

$$\begin{aligned}&4A_4' = 2A_4' + 2A_4' =\\ &= 2(x^2-(a-b)^2)(y^2-(a-b)^2)[x(a+b-y)+y(a+b-x)]+\\ &+\{4d_3(a^2,b^2,xy) - 288V^2 + 4(a-b)^2h[x(a+b-y)+y(a+b-x)]+\\ &+2(xy-a^2+b^2)(xy+a^2-b^2)(x(a+b-x)+y(a+b-y))\} =\\ &= 4d_3(a^2,b^2,xy) - 288V^2 + [2(x^2-(a-b)^2)(y^2-(a-b)^2) + 4(a-b)^2h]\cdot\\ &\cdot(x(a+b-y)+y(a+b-x))+\\ &+2(xy-a^2+b^2)(xy+a^2-b^2)[x(a+b-x)+y(a+b-y)]\end{aligned}$$

we obtain the following explicit formula for the real part of Atiyah determinant for "wedge" tetrahedron:

<u>Second explicit formula for wedge tetrahedra</u>:

$$\begin{aligned}\Re(D_4) = &(d_3(a,b,x) + 8abx)(d_3(a,b,y) + 8aby) + 3d_3(a,b,x)d_3(a,b,y)+\\ &+2x((a+b)^2 - y^2)d_3(a,b,y) + 2y((a+b)^2 - x^2)d_3(a,b,x)+\\ &2(x^2y^2 - (a^2-b^2))[x(a+b-x) + y(a+b-y)]+\\ &+2[(a-b)^2(x(a+b-y) + y(a+b-x))](2a^2 + 2b^2 - x^2 - y^2)\end{aligned}$$

which implies another strengthening of the Atiyah–Sutcliffe conjecture C3 for "wedge" tetrahedra

$$\begin{aligned}\Re(D_4) &\geq (d_3(a,b,x) + 8abx)(d_3(a,b,y) + 8aby) + 3d_3(a,b,x)d_3(a,b,y)\\ &\geq (d_3(a,b,x) + 8abx)(d_3(a,b,y) + 8aby)\end{aligned}$$

2.8 Atiyah determinant for triangles and quadrilaterals via trigonometry

Denote the three points x_1, x_2, x_3 simply by symbols $1,2,3$ and let X, Y and Z denote the angles of the triangle at vertices 1, 2 and 3 respectively. Then we can express the Atiyah determinant $D_3 = d_3(r_{12}, r_{13}, r_{23}) + 8r_{12}r_{13}r_{23}$ as follows

$$D_3 = 4r_{12}r_{13}r_{23}\left(\cos^2\frac{X}{2} + \cos^2\frac{Y}{2} + \cos^2\frac{Z}{2}\right).$$

This follows, by using cosine law and sum to product formula for cosine, from the following identity

$$\begin{aligned}d_3(a,b,c) + 8abc &= (a+b-c)(a-b+c)(-a+b+c) + 8abc\\ &= a((b+c)^2 - a^2) + b((c+a)^2 - b^2) + c((a+b)^2 - c^2).\end{aligned}$$

Now we shall translate the Eastwood–Norbury formula for (planar quadrilaterals) into a trigonometric form. Denote the four points x_1, x_2, x_3, x_4 simply by symbols $1,2,3,4$ and denote by

$$(X^{(1)}, Y^{(1)}, Z^{(1)}),\ \ (X^{(2)}, Y^{(2)}, Z^{(2)}),\ \ (X^{(3)}, Y^{(3)}, Z^{(3)}),\ \ (X^{(4)}, Y^{(4)}, Z^{(4)})$$

the angles of the triangles 234, 341, 412, 123 in this cyclic order (i.e. the angle of a triangle 412 at vertex 2 is $Z^{(3)}$ etc.).

Next we denote by c_l, $(1 \leq l \leq 4)$, the sums of cosines squared of half-angles of the l–th triangle i.e.:

$$c_l := \cos^2 \frac{X^{(l)}}{2} + \cos^2 \frac{Y^{(l)}}{2} + \cos^2 \frac{Z^{(l)}}{2}, \quad l = 1, 2, 3, 4.$$

Similarly, we denote by $\widehat{c}_l$, $(1 \leq l \leq 4)$, the sum of cosines squared of half-angles at the l–th vertex of our quadrilateral thus

$$\begin{aligned}
\widehat{c}_1 &= \cos^2 \frac{Z^{(2)}}{2} + \cos^2 \frac{Y^{(3)}}{2} + \cos^2 \frac{X^{(4)}}{2} \\
\widehat{c}_2 &= \cos^2 \frac{Z^{(3)}}{2} + \cos^2 \frac{Y^{(4)}}{2} + \cos^2 \frac{X^{(1)}}{2} \\
\widehat{c}_3 &= \cos^2 \frac{Z^{(4)}}{2} + \cos^2 \frac{Y^{(1)}}{2} + \cos^2 \frac{X^{(2)}}{2} \\
\widehat{c}_4 &= \cos^2 \frac{Z^{(1)}}{2} + \cos^2 \frac{Y^{(2)}}{2} + \cos^2 \frac{X^{(3)}}{2}
\end{aligned}$$

Then the term A_4 in the Eastwood–Norbury formula can be rewritten as

$$\begin{aligned}
A_4 &= \sum_{l=1}^{4} (4 r_{li} r_{lj} r_{lk} \widehat{c}_l) \cdot 4 r_{ij} r_{ik} r_{jk} (c_l - 2) \\
&= 16 r_{12} r_{13} r_{23} r_{14} r_{24} r_{34} \sum_{l=1}^{4} \widehat{c}_l (c_l - 2).
\end{aligned}$$

where for each l we write $\{1, 2, 3, 4\} \setminus \{l\} = \{i < j < k\}$.

In order to rewrite the term $-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23})$ into a trigonometric form we recall a theorem of Möbius ([12]) which claims that for any quadrilateral 1234 in a plane the products $r_{12}r_{34}$, $r_{13}r_{24}$ and $r_{14}r_{23}$ are proportional to the sides of a triangle whose angles are the differences of angles in the quadrilateral 1234:

$$\begin{aligned}
X &= \sphericalangle 134 - \sphericalangle 124 \\
Y &= \sphericalangle 214 - \sphericalangle 234 \\
Z &= \sphericalangle 413 - \sphericalangle 423
\end{aligned}$$

Thus

$$-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = -16 r_{12} r_{13} r_{23} r_{14} r_{24} r_{34} (c - 2)$$

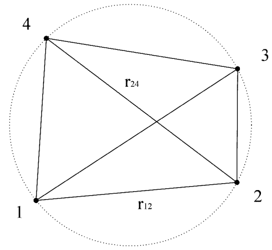
where

$$c = \cos^2 \frac{X}{2} + \cos^2 \frac{Y}{2} + \cos^2 \frac{Z}{2}.$$

Thus we have obtained a trigonometric formula for Atiyah determinant of quadrilaterals

$$\begin{aligned}\Re(D_4) &= \prod_{1\le i<j\le 4} r_{ij}\left(64-16(c-2)+16\sum_{l=1}^{4}\widehat{c}_l(c_l-2)\right)\\ &= 16\prod_{1\le i<j\le 4} r_{ij}\left(6-c+\sum_{l=1}^{4}\widehat{c}_l(c_l-2)\right)\end{aligned}$$

Now we shall verify Atiyah–Sutcliffe conjecture for cyclic quadrilaterals.



Ptolemy's theorem

$r_{12}r_{34}+r_{23}r_{14}=r_{13}r_{24}$

In this case, by a well known Ptolemy's theorem, we see that

$$-4d_3(r_{12}r_{34},r_{13}r_{24},r_{14}r_{23})=0 \quad (\Leftrightarrow c=2)$$

By using the equality of angles $Z^{(2)}=X^{(1)}$, $Z^{(3)}=X^{(2)}$, $Z^{(4)}=X^{(3)}$, $Z^{(1)}=X^{(4)}$ and $Y^{(1)}+Y^{(3)}=\pi=Y^{(2)}+Y^{(4)}$ (angles with vertex on a circle's circumference with the same endpoints are equal or supplement of each other)we obtain

$$\begin{aligned}\widehat{c}_1 &= \cos^2\frac{X^{(1)}}{2}+\sin^2\frac{Y^{(1)}}{2}+\cos^2\frac{Z^{(1)}}{2}=c_1-\cos Y^{(1)},\\ \widehat{c}_2 &= \cos^2\frac{X^{(2)}}{2}+\sin^2\frac{Y^{(2)}}{2}+\cos^2\frac{Z^{(2)}}{2}=c_2-\cos Y^{(2)},\\ \widehat{c}_3 &= \cos^2\frac{X^{(3)}}{2}+\sin^2\frac{Y^{(3)}}{2}+\cos^2\frac{Z^{(3)}}{2}=c_3-\cos Y^{(3)},\\ \widehat{c}_4 &= \cos^2\frac{X^{(4)}}{2}+\sin^2\frac{Y^{(4)}}{2}+\cos^2\frac{Z^{(4)}}{2}=c_4-\cos Y^{(4)}.\end{aligned}$$

Now we have

$$\begin{aligned}\Re(D_4) = &\left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(64+16\sum_{l=1}^{4}(c_l-\cos Y^{(l)})(c_l-2)\right)\\ \ge &\left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(64+16\sum_{l=1}^{4}(c_l-1)(c_l-2)\right)\end{aligned}$$

(here we have used that $2 \leq c_l (\leq \frac{9}{4})$ for each $l = 1, 2, 3, 4$)

$$\begin{aligned}
&\geq \left(\prod_{1\leq i<j\leq 4} r_{ij}\right)\left(64 + 16\sum_{l=1}^{4}(c_l - 2) + 16\sum_{l=1}^{4}(c_l - 2)^2\right) \\
&\geq \left(\prod_{1\leq i<j\leq 4} r_{ij}\right)\left(64 + 16\sum_{l=1}^{4}(c_l - 2) + 4\left(\sum_{l=1}^{4}(c_l - 2)\right)^2\right)
\end{aligned}$$

(by quadratic–arithmetic inequality)

$$\begin{aligned}
&= \left(\prod_{1\leq i<j\leq 4} r_{ij}\right)\left(\left(8 + \sum_{l=1}^{4}(c_l - 2)\right)^2 + 3\left(\sum_{l=1}^{4}(c_l - 2)\right)^2\right) \\
&= \left(\prod_{1\leq i<j\leq 4} r_{ij}\right)\left(\left(\sum_{l=1}^{4}c_l\right)^2 + 3\left(\sum_{l=1}^{4}(c_l - 2)\right)^2\right) \\
&\geq \left(\prod_{1\leq i<j\leq 4} r_{ij}\right)\left(\sum_{l=1}^{4}c_l\right)^2 \geq 16\sqrt{c_1c_2c_3c_4}\prod_{1\leq i<j\leq 4} r_{ij}
\end{aligned}$$

by A–G inequality.

Finally,

$$\begin{aligned}
|D_4|^2 = {} & |\Re(D_4)|^2 \geq 4^4 c_1c_2c_3c_4 \prod_{1\leq i<j\leq 4} r_{ij}^2 \\
= {} & \prod_{l=1}^{4}(4r_{ij}r_{ik}r_{jk}c_l) = \prod_{l=1}^{4}(d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk})
\end{aligned}$$

where for each l we write $\{1, 2, 3, 4\}\backslash\{l\} = \{i < j < k\}$. This finishes verification of Atiyah–Sutcliffe conjectures for cyclic quadrilaterals.

Proposition 2.12 *The weak Four Points Conjecture for cyclic quadrilaterals holds true.*

Proof .
From the formula obtained above we proceed along a different path

$$
\begin{aligned}
\Re(D_4) = &\left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(64+16\sum_{l=1}^{4}(c_l-\cos Y^{(l)})(c_l-2)\right)\\
= &\left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(4\sum_{l=1}^{4}\Big[4+4(c_l-\cos Y^{(l)})(c_l-2)\Big]\right)\\
= &\left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(4\sum_{l=1}^{4}\Big[c_l^2+(c_l-2)[3(c_l-2)+4(1-\cos Y^{(l)})]\Big]\right)\\
\ge &\left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(4\sum_{l=1}^{4}c_l^2\right)\text{(because } 2\le c_l \text{ for each } l=1,2,3,4)\\
= &\prod r_{ij}\left(\frac{1}{4}\sum_{l=1}^{4}\left(\frac{d_3(r_{ij},r_{ik},r_{jk})+8r_{ij}r_{ik}r_{jk}}{r_{ij}r_{ik}r_{jk}}\right)^2\right)
\end{aligned}
$$

and this verifies the weak Four Points Conjecture for cyclic quadrilaterals. ■

2.9 Verification of the Strong four points conjecture for a triangle with orthocenter

Let $1,2,3,4$ be the points so that $4=$ orthocenter of the triangle $\triangle 123$ and let X,Y,Z denote the angles of the triangle $\triangle 123$ at vertices $1,2,3$ respectively. Note that $\angle 243=\pi-X$, $\angle 341=\pi-Y$, $\angle 142=\pi-Z$. Then, using the notations of (2.8), we have the following formula for the normalized Atiyah determinant

$$
\mathcal{D}_4=\frac{D_4}{64\prod\limits_{1\le i<j\le 4} r_{ij}}=1-\frac{1}{4}(c-2)+\frac{1}{4}\sum_{l=1}^{4}\widehat{c}_l(c_l-2) \tag{2.26}
$$

where

$$
\begin{aligned}
c &= \cos^2\tfrac{\angle 243-\angle 213}{2}+\cos^2\tfrac{\angle 341-\angle 321}{2}+\cos^2\tfrac{\angle 142-\angle 132}{2}\\
&= \sin^2 X+\sin^2 Y+\sin^2 Z\\
c_1 &= \cos^2\tfrac{\frac{\pi}{2}-Z}{2}+\cos^2\tfrac{\frac{\pi}{2}-Y}{2}+\cos^2\tfrac{\pi-X}{2}=\tfrac{3-\cos X+\sin Y+\sin Z}{2}\\
c_2 &= \tfrac{3+\sin X-\cos Y+\sin Z}{2},\quad c_3=\tfrac{3+\sin X+\sin Y-\cos Z}{2}\\
c_4 &= \cos^2\tfrac{X}{2}+\cos^2\tfrac{Y}{2}+\cos^2\tfrac{Z}{2}=\tfrac{3+\cos X+\cos Y+\cos Z}{2}\\
\widehat{c}_1 &= \cos^2\tfrac{X}{2}+\cos^2\tfrac{\frac{\pi}{2}-Y}{2}+\cos^2\tfrac{\frac{\pi}{2}-Z}{2}=\tfrac{3+\cos X+\sin Y+\sin Z}{2}\\
\widehat{c}_2 &= \tfrac{3+\sin X+\cos Y+\sin Z}{2},\quad \widehat{c}_3=\tfrac{3+\sin X+\sin Y+\cos Z}{2}\\
\widehat{c}_4 &= \cos^2\tfrac{\pi-X}{2}+\cos^2\tfrac{\pi-Y}{2}+\cos^2\tfrac{\pi-Z}{2}=\tfrac{3-\cos X-\cos Y-\cos Z}{2}
\end{aligned}
$$

Now we compute

$$
\begin{aligned}
(\widehat{c}_1-1)(c_1-2)-\frac{c-2}{4} &= \frac{1+\cos X+\sin Y+\sin Z}{2}\cdot\frac{-1-\cos X+\sin Y+\sin Z}{2}- \\
&\quad -\frac{1}{4}(\sin^2 X+\sin^2 Y+\sin^2 Z-2)= \\
&= -\frac{\cos X}{2}+\frac{\sin Y\sin Z}{2}=\frac{\cos Y\cos Z}{2} \\
(\widehat{c}_2-1)(c_2-2)-\frac{c-2}{4} &= -\frac{\cos Y}{2}+\frac{\sin X\sin Z}{2}=\frac{\cos X\cos Z}{2} \\
(\widehat{c}_3-1)(c_3-2)-\frac{c-2}{4} &= -\frac{\cos Z}{2}+\frac{\sin X\sin Y}{2}=\frac{\cos X\cos Y}{2} \\
(\widehat{c}_4-1)(c_4-2)-\frac{c-2}{4} &= \frac{1-\cos X-\cos Y-\cos Z}{2}\cdot\frac{\cos X+\cos Y+\cos Z-1}{2}- \\
&\quad -\frac{1}{4}(\sin^2 X+\sin^2 Y+\sin^2 Z-2) \\
&= -\frac{1}{2}+\frac{1}{2}(\cos X+\cos Y+\cos Z)- \\
&\quad -\frac{1}{2}(\cos X\cos Y+\cos X\cos Z+\cos Y\cos Z)
\end{aligned}
$$

By adding last four equations we get

$$
\begin{aligned}
&\sum_{l=1}^{4}(\widehat{c}_l-1)(c_l-2)-(c-2)= \\
&= -\tfrac{1}{2}+\tfrac{1}{2}(\cos X+\cos Y+\cos Z) \\
&= c_4-2
\end{aligned}
\tag{2.27}
$$

Claim:

$$
c_4 \geq \frac{1}{4}(c_1+c_2+c_3+c_4) \tag{2.28}
$$

Proof of claim: It is equivalent to the inequality (for angles of any triangle)

$$
2(\cos X+\cos Y+\cos Z)\geq \sin X+\sin Y+\sin Z \tag{*}
$$

We can suppose that $0\leq X,Y\leq \pi/2$. Then $Z=\pi-X-Y$. The identity

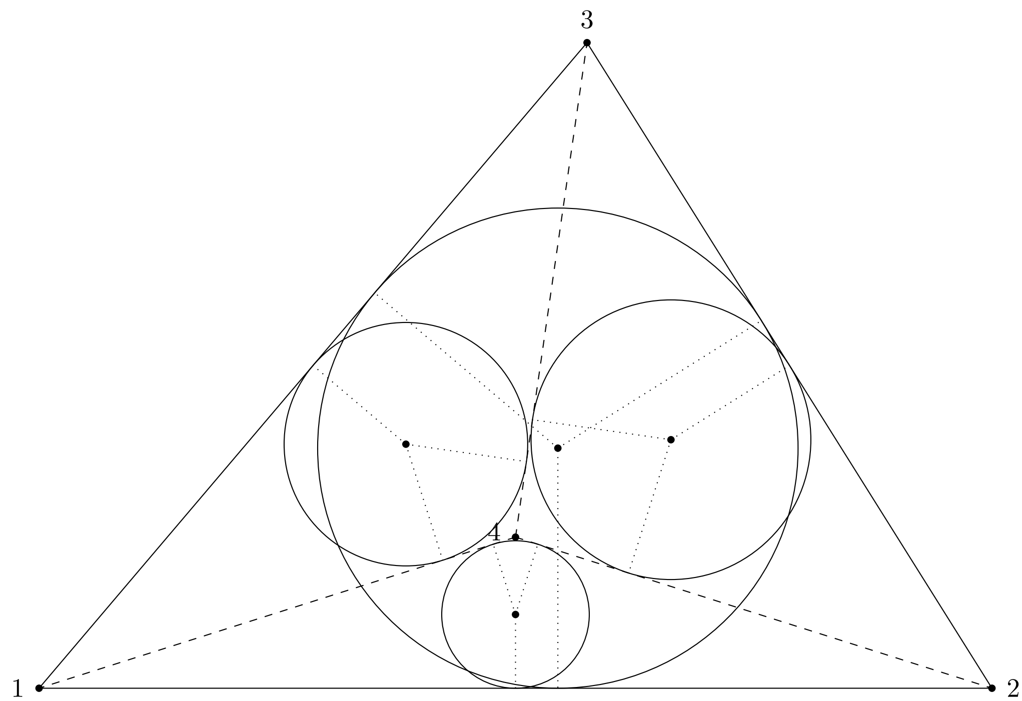
$$
\begin{aligned}
&2(\cos X+\cos Y-\cos(X+Y))-\sin X-\sin Y-\sin(X+Y)= \\
&= 2(1-\sin X)(1-\sin Y)+(1-\cos X)(\sin Y+\cos Y-1)+ \\
&\quad +(1-\cos Y)(\sin X+\cos X-1) \\
&\geq 0
\end{aligned}
$$

proves the Claim directly (The inequality (*) can be deduced also from a Blundon's fundamental inequality).

By combining (2.26), (2.27) and (2.28) we obtain

$$
\mathcal{D}_4 \geq 1+\frac{1}{4}\sum_{l=1}^{4}\left(1+\frac{1}{4}\right)(c_l-2)
$$

what verifies the strong four points Conjecture for a four point configuration of vertices and the orthocenter of any triangle.


3
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2

Appendix A - New proof of the Eastwood–Norbury formula

The four points in the Euclidean space E^3 viewed as $C \times R$:
$P_i : x_i = (z_i, r_i),\ z_i \in \mathbb{C},\ r_i \in \mathbb{R}$

$R_{ij} := r_{ij} + r_i - r_j,\ z_{ij} := z_i - z_j$

$R_{ij}R_{ji} = r_{ij}^2 - (r_i - r_j)^2 = |z_{ij}|^2 = -z_{ij}z_{ji}$

$p_1 = \left(z + \frac{\bar{z}_{12}}{R_{12}}\right)\left(z + \frac{\bar{z}_{13}}{R_{13}}\right)\left(z + \frac{\bar{z}_{14}}{R_{14}}\right)$

$p_2 = \left(z + \frac{\bar{z}_{21}}{R_{21}}\right)\left(z + \frac{\bar{z}_{23}}{R_{23}}\right)\left(z + \frac{\bar{z}_{24}}{R_{24}}\right)$

$p_3 = \left(z + \frac{\bar{z}_{31}}{R_{31}}\right)\left(z + \frac{\bar{z}_{32}}{R_{32}}\right)\left(z + \frac{\bar{z}_{34}}{R_{34}}\right)$

$p_4 = \left(z + \frac{\bar{z}_{41}}{R_{41}}\right)\left(z + \frac{\bar{z}_{42}}{R_{42}}\right)\left(z + \frac{\bar{z}_{43}}{R_{43}}\right)$

Matrix of coefficients of $\{p_1, p_2, p_3, p_4\}$

$$M_4 = \left|\begin{array}{cccc} \boxed{1} & \cdot & \cdot & \frac{\bar{z}_{12}}{R_{12}}\frac{\bar{z}_{13}}{R_{13}}\frac{\bar{z}_{14}}{R_{14}} \\ 1 & \boxed{\frac{\bar{z}_{21}}{R_{21}} + \frac{\bar{z}_{23}}{R_{23}} + \frac{\bar{z}_{24}}{R_{24}}} & \cdot & \cdot \\ 1 & \cdot & \boxed{\frac{\bar{z}_{31}}{R_{31}}\frac{\bar{z}_{32}}{R_{32}} + \frac{\bar{z}_{31}}{R_{31}}\frac{\bar{z}_{34}}{R_{34}} + \frac{\bar{z}_{32}}{R_{32}}\frac{\bar{z}_{34}}{R_{34}}} & \cdot \\ 1 & \cdot & \cdot & \boxed{\frac{\bar{z}_{41}}{R_{41}}\frac{\bar{z}_{42}}{R_{42}}\frac{\bar{z}_{43}}{R_{43}}} \end{array}\right| \begin{array}{l} \\ \cdot A \\ \cdot B \\ \cdot C \end{array}$$

$A = z_{21},\ B = z_{31}z_{32},\ C = z_{41}z_{42}z_{43}$

Normalized Atiyah determinant

$$D_4 = \underbrace{\det(M_4)}_{antisym.} \cdot \underbrace{z_{21} \cdot z_{31}z_{32} \cdot z_{41}z_{42}z_{43}}_{antisym.} = \sum 1 \cdot \left(\frac{z_{21}\bar{z}_{21}}{R_{21}} + \frac{z_{21}\bar{z}_{23}}{R_{23}} + \frac{z_{21}\bar{z}_{24}}{R_{24}}\right) \cdot$$
$$\cdot \left(\frac{z_{31}\bar{z}_{31}z_{32}\bar{z}_{32}}{R_{31}R_{32}} + \frac{z_{31}\bar{z}_{31}z_{32}\bar{z}_{34}}{R_{31}R_{34}} + \frac{z_{32}\bar{z}_{32}z_{32}\bar{z}_{34}}{R_{32}R_{34}}\right) R_{14}R_{24}R_{34} =$$
$$= \sum (R_{12}R_{24} + \underbrace{z_{21}z_{24}})(R_{13}R_{23}R_{34} + R_{13}\underbrace{z_{32}\bar{z}_{34}} + R_{23}\underbrace{z_{31}\bar{z}_{34}})R_{14} +$$
$$+ (R_{13}R_{24}R_{34}\underbrace{z_{21}\bar{z}_{23}} + R_{13}R_{24}R_{32}\underbrace{z_{21}\bar{z}_{34}} + R_{24}\underbrace{z_{21}\bar{z}_{23}z_{31}\bar{z}_{34}})R_{14}$$

(where summations are over all permutations of indices).

By writing $z_{ij}\bar{z}_{kl} = C[i,j,k,l] + \sqrt{-1}\,S[i,j,k,l]$ and using a Lagrange identity (involving the dot product of two cross products; a fact mentioned by N. Wildberger to the author) we have $S[i,j,k,l]S[p,q,r,s] = C[i,j,p,q]C[k,l,r,s] - C[i,j,r,s]C[k,l,p,q]$ (we have discovered this identity independently) and using the formula

$$C[i,j,k,l] = \mathrm{Re}(z_{ij}\bar{z}_{kl}) = \tfrac{1}{2}[|z_{il}|^2 + |z_{jk}|^2 - |z_{ik}|^2 - |z_{jl}|^2] =$$
$$= \tfrac{1}{2}[r_{il}^2 + r_{jk}^2 - r_{ik}^2 - r_{jl}^2] - (r_i - r_j)(r_k - r_l)$$

we obtain our derivation of the Eastwood–Norbury formula.

By this new method we obtained a polynomial formula for the planar configurations of 5 points (by S_5–symmetrization of a "one page" expression) and a rational formula for the spatial 5 point configuration (this last formula has almost 100000 terms).

This settles one of the Eastwood–Norbury conjectures. We do not yet have definite geometric interpretations for the "nonplanar" part of the formula involving heights r_i, $i = 1, \ldots, 5$.

Our trigonometric (euclidean) Eastwood–Norbury formula
(where $c_{i\_jk} := \cos(ij, ik)$ and $c_{ij\_kl} := \cos(ij, kl)$):

$$\begin{aligned}16Re(D_4) = \quad & (1 + c_{3\_12} + c_{2\_34})(1 + c_{1\_24} + c_{4\_13})+\\ & (1 + c_{2\_13} + c_{3\_24})(1 + c_{4\_12} + c_{1\_34})+\\ & (1 + c_{3\_12} + c_{1\_34})(1 + c_{2\_14} + c_{4\_23})+\\ & (1 + c_{1\_23} + c_{3\_14})(1 + c_{2\_34} + c_{4\_12})+\\ & (1 + c_{2\_13} + c_{1\_24})(1 + c_{3\_14} + c_{4\_23})+\\ & (1 + c_{1\_23} + c_{2\_14})(1 + c_{3\_24} + c_{4\_13})+\\ & 2(c_{14\_23}c_{13\_24} - c_{14\_23}c_{12\_34} + c_{13\_24}c_{12\_34})+\\ & 72(\text{normalized volume})^2.\end{aligned}$$

Open problems: Hyperbolic (euclidean) version for $n \geq 4$ ($n \geq 5$) points in terms of distances, or in terms of angles.

Appendix B

Here we first recall a remarkable inequality of I. Schur (c.f. J. Michael Steele: The Cauchy–Schwarz Master Class, Cambridge University Press, 2004.)

For all values $x, y, z \geq 0$ and all $\alpha \geq 0$ we have

$$\begin{aligned} I_\alpha(x,y,z) &:= \sum x^\alpha (x-y)(x-z) = \\ &= x^\alpha(x-y)(x-z) + y^\alpha(y-x)(y-z) + z^\alpha(z-x)(z-y) \geq 0 \end{aligned}$$

with equality iff either $x = y = z$ or two of the variables are equal and the third is zero. Note that I_α is a symmetric function. For a proof we can assume $0 \leq x \leq y \leq z$. Then clearly $x^\alpha(x-y)(x-z) \geq 0$ and by grouping the other two terms we get $(z-y)[z^\alpha(z-x) - y^\alpha(y-x)] \geq 0$ by observing that $z \geq y$ and $z - x \geq y - x$.

Now we state and prove several properties of a function

$$d_3(x,y,z) := (x+y-z)(x-y+z)(-x+y+z) \quad (x,y,z \geq 0)$$

which frequently appears in the main part of this paper.

We note that the area $A = A(a,b,c)$ of a triangle with sides lengths a, b, c is given, according to the Heron-s formula:

$$\begin{aligned} (4A)^2 &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\ &= (a+b+c)d_3(a,b,c) \\ &= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 \end{aligned}$$

Properties of the function d_3:

Proposition 2.13 *We have the following identities and inequalities:*

1. $xyz - d_3(x,y,z) = \sum x(x-y)(x-z) \geq 0$
2. $(x^2+y^2+z^2)d_3(x,y,z) \leq xyz(xy+xz+yz)$
3. $d_3(x,y,z)^2 - d_3(x^2,y^2,z^2) =$
$= \sum x^2(y^2 - yz + z^2 - x^2)^2 + \left(\sum x(y^2 - yz + z^2 - x^2)\right)^2 \geq 0$
4. $d_3(x,y,z)^2 - d_3(x^2,y^2,z^2) =$
$= 8x^2y^2z^2 - 2(xyz + x^3 + y^3 + z^3)d_3(x,y,z) \geq 0$
5. $(x+y+z)^2 d_3(x,y,z)^2 - 3(x^2+y^2+z^2)d_3(x^2,y^2,z^2) =$
$= 4\sum x^4(x^2-y^2)(x^2-z^2) \geq 0$
6. $(x+y+z)(X+Y+Z)d_3(x,y,z)d_3(X,Y,Z) - 3(xX+yY+zZ)d_3(xX,yY,zZ) =$
$2\sum(x^2(x^2-y^2)X^2(X^2-Z^2) + X^2(X^2-Y^2)x^2(x^2-z^2)) + (x^2(Y^2-Z^2) +$
$y^2(Z^2-X^2) + z^2(X^2-Y^2))^2 \geq 0$

Proof .

All identities *1.–6.* can be easily checked by expansion. The inequality in *1.* follows from Schur's inequality ($\alpha = 1$), in *3.* it is evident since the rhs is the sum of four squares (see [5]). Case *4.* follows from *3.* Case *5.* follows from Schur's inequality ($\alpha = 2$). Case *6.* follows from a generalization of the case $\alpha = 2$ of Schur's inequality:

$$\begin{aligned} &II_2(x,y,z,X,Y,Z) = \sum x(x-y)X(X-Z) = \\ &= x(x-y)X(X-Z) + y(y-x)Y(Y-Z) + z(z-x)Z(Z-Y) \geq 0 \end{aligned}$$

(by letting $y = x + h$, $z = y + k$, $Y = X + H$, $Z = Y + K$, $h, k, H, K \geq 0$). ■

Corollary 2.14 *From the Proposition we get the following inequalities:*

$$d_3(x,y,z) \leq xyz \quad (from\ 1.\ )$$

and a stronger inequality $d_3(x,y,z) \leq 4x^2y^2z^2/(xyz + x^3 + y^3 + z^3)$ *(from 4.)*

From 3. we have the inequality

$$d_3(x,y,z)^2 \geq d_3(x^2,y^2,z^2)$$

which can also be obtained from 5. (which implies famous Finsler–Hadwiger inequality) by using the inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$.

The inequality 6. , with the help of Chebyshev inequality

$$(x+y+z)(X+Y+Z) \leq 3(xX + yY + zZ) \quad (x \leq y \leq z, \quad X \leq Y \leq Z)$$

gives us the following inequality (which seems to be new):

$$d_3(x,y,z)d_3(X,Y,Z) \geq d_3(xX, yY, zZ)$$

(when $0 \leq x \leq y \leq z$, $0 \leq X \leq Y \leq Z$*).*

Remark 2.15 *If* a, b, c *are side lengths of a triangle then inequality* $d_3(a,b,c) \leq abc$ *follows also directly from the following identity*

$$abc - d_3(a,b,c) = \frac{1}{2}[(-a+b+c)(b-c)^2 + (a-b+c)(a-c)^2 + (a+b-c)(a-b)^2]$$

from which we also have the following inequality

$$8(abc - d_3(a,b,c))^3 \geq 27d_3(a,b,c)(a-b)^2(a-c)^2(b-c)^2$$

Appendix C - Equations for mixed Atiyah 3pt energies

$$\begin{aligned}
\Delta_3 &= D_3 - 1 = e^{-p}\,\frac{\sinh(p-a)\sinh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}, \quad \left(p = \frac{a+b+c}{2}\right)\\
\Delta_3^+ &= D_3^+ - 1 = e^{p}\,\frac{\sinh(p-a)\sinh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}\\
P_1 &= (X-\Delta_3)(X-\Delta_3^+) =\\
&= X^2 - 2\frac{\cosh(p)\sinh(p-a)\sinh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}X + \frac{\sinh^2(p-a)\sinh^2(p-b)\sinh^2(p-c)}{\sinh^2(a)\sinh^2(b)\sinh^2(c)}\\
\Delta_3^{(1)} &= D_3^{001} - 1 = -e^{-(p-c)}\,\frac{\sinh(p)\sinh(p-a)\sinh(p-b)}{\sinh(a)\sinh(b)\sinh(c)} = -e^{-p+c}\frac{\sinh(p)}{\sinh(c)}\sin^2(A)\\
\Delta_3^{(6)} &= D_3^{110} - 1 = -e^{p-c}\,\frac{\sinh(p)\sinh(p-a)\sinh(p-b)}{\sinh(a)\sinh(b)\sinh(c)} = -e^{+p-c}\frac{\sinh(p)}{\sinh(c)}\sin^2(A)\\
P_2 &= (X-\Delta_3^{(1)})(X-\Delta_3^{(6)}) =\\
&= X^2 - 2\frac{\sinh(p)\sinh(p-a)\sinh(p-b)\cosh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}X + \frac{\sinh^2(p)\sinh^2(p-a)\sinh^2(p-b)}{\sinh^2(a)\sinh^2(b)\sinh^2(c)}\\
P_3 &= (X-\Delta_3^{(2)})(X-\Delta_3^{(5)}) =\\
&= X^2 - 2\frac{\sinh(p)\sinh(p-a)\cosh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}X + \frac{\sinh^2(p)\sinh^2(p-a)\sinh^2(p-c)}{\sinh^2(a)\sinh^2(b)\sinh^2(c)}\\
P_4 &= (X-\Delta_3^{(3)})(X-\Delta_3^{(4)}) =\\
&= X^2 - 2\frac{\sinh(p)\cosh(p-a)\sinh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}X + \frac{\sinh^2(p)\sinh^2(p-b)\sinh^2(p-c)}{\sinh^2(a)\sinh^2(b)\sinh^2(c)}\\
P(X) &= P_1P_2P_3P_4 = \prod_{i=0}^{7}\left(X-\Delta^{(i)}\right) = X^8 + \cdots + e_7X - e_8\\
e_8 &= \prod_{i=0}^{7}\Delta^{(i)} = \frac{\sinh^6(p)\sinh^6(p-a)\sinh^6(p-b)\sinh^6(p-c)}{\sinh^6(a)\sinh^6(b)\sinh^6(c)}\\
e_8 &= \frac{(1-c_1^2)^2(1-c_2^2)^2(1-c_3^2)^2}{4096} = \frac{\sin^4(A)\sin^4(B)\sin^4(C)}{4096}\\
e_7 &= \frac{1}{256}(1-c_1^2)(1-c_2^2)(1-c_3^2)(-1+2c_1c_2c_3+c_1^2+c_2^2+c_3^2) = \frac{1}{256}\sin^2(A)\sin^2(B)\sin^2(C)\Phi^2\\
P(X) &= X^8 + 2X^7 + \frac{1}{4}(4+\sigma_1-\Phi^2)X^6 + \frac{1}{4}(\sigma_1-2\Phi^2)X^5 - \frac{1}{32}(3\sigma_3 - 2\sigma_2 + (2\sigma_1-8)\Phi^2)X^4 -\\
&\quad - \tfrac{1}{32}(\sigma_3+2\sigma_1\Phi^2)X^3 - \tfrac{1}{256}(\sigma_1\sigma_3 + (4\sigma_2-\sigma_3)\Phi^2)X^2 - \tfrac{1}{256}\sigma_3\Phi^2X + \tfrac{1}{4096}\sigma_3^2
\end{aligned}$$

where $\sigma_1 = \sin^2(A)+\sin^2(B)+\sin^2(C), \sigma_2 = \sin^2(A)\sin^2(B)+\sin^2(A)\sin^2(C)+\sin^2(B)\sin^2(C)$, $\sigma_3 = \sin^2(A)\sin^2(B)\sin^2(C), \Phi^2 = 4\cos(\sigma)\cos(\sigma-A)\cos(\sigma-B)\cos(\sigma-C), \sigma = \frac{A+B+C}{2}$

$$\prod_{i=0}^{7}D_3^{(i)} = \frac{1}{128}(\sigma_3-2\sigma_2)\Phi^2 + \frac{1}{4096}(\sigma_3^2 - 16\sigma_1\sigma_3 + 256(\sigma_2-\sigma_3))$$

$$
\begin{aligned}
P_-(X) &= X^4 + (1+\Phi)X^3 + \frac{1}{4}(c_1^2+c_2^2+c_3^2+3c_1c_2c_3+3\Phi)X^2 + \frac{1}{8}(-1+2c_1c_2c_3+c_1^2+c_2^2+c_3^2+\\
&\quad + (1+c_1c_2c_3)\Phi)X - \frac{1}{64}(1-c_1^2)(1-c_2^2)(1-c_3^2)\\
&= \frac{1}{8}X(2X+\Phi)\left[(2X+1)(2X+1+\Phi)+c_1c_2c_3\right] - \frac{1}{64}(1-c_1^2)(1-c_2^2)(1-c_3^2)\\
&= \frac{1}{8}X(2X+\Phi)\left[(2X+1)(2X+1+\Phi)+\frac{1+\Phi^2-(c_1^2+c_2^2+c_3^2)}{2}\right] - \frac{1}{64}(1-c_1^2)(1-c_2^2)(1-c_3^2)\\
&= \frac{1}{16}X(2X+\Phi)\left[2(2X+1)(2X+1+\Phi)-2+s_1^2+s_2^2+s_3^2\right] - \frac{1}{64}s_1^2s_2^2s_3^2\\
P(X) &= P_-(X)P_+(X), \text{ where } P_+(X) = P_-(X)\Big|_{\Phi\to-\Phi}\\
P_-(X)\Big|_{\Phi=0} &= \frac{1}{16}X\cdot 2X\left[2(2X+1)^2-2+s_1^2+s_2^2+s_3^2\right] - \frac{1}{64}s_1^2s_2^2s_3^2\\
&= X^3(X+1) + \frac{1}{8}(s_1^2+s_2^2+s_3^2)X^2 - \frac{1}{64}s_1^2s_2^2s_3^2\\
P(0) &= \frac{1}{4096}s_1^4s_2^4s_3^4\\
\prod_{i=0}^{7} D_3^{(i)} &= P(-1) = \left[\frac{1}{16}(-1)(-2+\Phi)[-2(-1+\Phi)-2+s_1^2+s_2^2+s_3^2] - \frac{1}{64}s_1^2s_2^2s_3^2\right]\cdot\\
&\qquad \cdot\left[\frac{1}{16}(-1)(-2-\Phi)[-2(-1-\Phi)-2+s_1^2+s_2^2+s_3^2] - \frac{1}{64}s_1^2s_2^2s_3^2\right]\\
&= \left[\frac{1}{16}(2-\Phi)[-2\Phi+\sigma_1] - \frac{1}{64}\sigma_3\right]\left[\frac{1}{16}(2+\Phi)[2\Phi+\sigma_1] - \frac{1}{64}\sigma_3\right]\\
&= \left[\frac{1}{16}(2\Phi^2-4\Phi+2\sigma_1-\Phi\sigma_1) - \frac{1}{16}\sigma_3\right]\left[\frac{1}{16}(2\Phi^2+4\Phi+2\sigma_1+\Phi\sigma_1) - \frac{1}{16}\sigma_3\right]\\
&= \left[\frac{1}{16}(2\Phi^2+2\sigma_1) - \frac{1}{64}\sigma_3 - \frac{1}{16}(4+\sigma_1)\Phi\right]\left[\frac{1}{16}(2\Phi^2+2\sigma_1) - \frac{1}{64}\sigma_3 + \frac{1}{16}(4+\sigma_1)\Phi\right]\\
&= \frac{1}{64^2}\left\{\left[4\left(\Phi^2+\sigma_1\right)-\sigma_3\right]^2 - 4^2\left(4+\sigma_1\right)^2\Phi^2\right\}.
\end{aligned}
$$

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Refinements of Euler's inequalities in plane, space and n-space

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Abstract

We improve Euler's inequality $R \geq 2r$, where R and r are triangle's circumradius and inradius, respectively, and prove some consequences of it. We also show non-Euclidean version of this result. Next, we improve $3D$ analogue of Euler's inequality for tetrahedra $R \geq 3r$ and discuss recursive way to improve analogues of Euler's inequality for simplices. We end with some open problems, including possible CEEG (classical Euclidean elementary geometry) proof of Grace-Danielsson's inequality $d^2 \leq (R-3r)(R+r)$, where d is the distance between the centers of the insphere and the circumsphere of a tetrahedron.

Keywords: Triangle inequalities, tetrahedron and volume inequalities, Euler's inequality in $2D$, $3D$, and n-dimensional space, non-Euclidean geometry.

MSC 2010 : 51M04, 51M09, 51M16

1 Introduction

In this paper we shall discuss some results about refinements of the famous *Euler's inequality* about the ratio of circumradius R and inradius r of an n-dimensional simplex T:

$$R \geq nr. \tag{E_n}$$

The equality is attained if and only if T is a regular simplex.

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Proof. Consider the simplex T' whose vertices are centroids of faces of T. Then T' is clearly similar to T with the similarity coefficient equal to n. Therefore, the circumradius R' of T' is $1/n$ of R, that is, $R = nR'$. But, $R' \geq r$, because the smallest ball that just touch all the faces of T is exactly the inscribed ball of T, so $R \geq nr$. It is clear that the equality is attained if and only if T is a regular simplex. This completes the proof of (E_n). $\square$

Let us recall that Euler proved the inequality $R \geq 2r$ for triangles back in 1765.

2 Triangle

The following theorem refines (E_2), i.e. the Euler inequality $R \geq 2r$ for triangles.

Theorem 1. *For any triangle with side lengths a, b, c and with circumradius R and inradius r we have*

$$2abcR \geq (abc + a^3 + b^3 + c^3)r \geq 4abcr, \tag{1}$$

with equality if and only if the triangle is equilateral.

Before we prove Theorem 1, let us make some remarks. Let

$$d_3(x, y, z) := (x + y - z)(x - y + z)(-x + y + z).$$

This is a symmetric function in x, y, z and hence can be expressed in terms of elementary symmetric functions $e_1 = x + y + z$, $e_2 = xy + yz + zx$, $e_3 = xyz$. So, we have $d_3(x, y, z) = 4e_1e_2 - e_1^3 - 8e_3$. Schur's inequality (from 1915) tells us that $d_3 \leq e_3$ (whose variables are positive reals). For an (a, b, c)-triangle with area S and half-perimeter s, since $R/r = abc/4S/S/s = abc/4(s-a)(s-b)(s-c) = 2abc/d_3(a, b, c) \geq 2$, it follows that $d_3(a, b, c) \leq abc$, and this is a geometric proof of Schur's inequality. So, Euler's inequality (E_2) implies Schur's inequality.

The right inequality in (1) is just the $AM - GM$ inequality for numbers a^3, b^3 and c^3. The left inequality in (1) is equivalent, using Heron's formula $16S^2 = (a+b+c)d_3(a, b, c)$ and formulas $R = abc/4S$ and $r = 2S/(a+b+c)$ to the following inequality:

$$(2abc)^2 \geq (abc + a^3 + b^3 + c^3)d_3(a, b, c). \tag{2}$$

Proof of (2) is given in [1]; one algebraic and one geometric proof; see also [5]. We provide here a short algebraic proof. (A quick proof of Heron's formula is to square the cosine law and the sine law $\cos C = (a^2+b^2-c^2)/2ab$, $\sin C = 2S/ab$ and use Pythagoras' rule $cos^2 C + sin^2 C = 1$. Pythagoras' rule is equivalent to the axiom on parallels. And in fact, the cosine and the sine law and consequently Heron's formula are all equivalent to the axiom on parallels.)

Proof. Recall the cyclic sum notation: $\sum f(a,b,c) := f(a,b,c) + f(b,c,a) + f(c,a,b)$, as well as the cyclic product $\prod f(a,b,c) := f(a,b,c)f(b,c,a)f(c,a,b)$. In terms of cyclic sums and products (2) can be written as

$$4\prod a^2 \geq \left(\prod a + \sum a^3\right) d_3(a,b,c). \tag{3}$$

It is easy to check by expansion that in terms of cyclic sums we have the following sum of squares:

$$d_3(a,b,c)^2 - d_3(a^2,b^2,c^2) = \sum (a(b^2-bc+c^2-a^2))^2 + \left(\sum a(b^2-bc+c^2-a^2)\right)^2 \geq 0.$$

On the other hand (as it is again easy to check):

$$d_3(a,b,c)^2 - d_3(a^2,b^2,c^2) = 2(2abc)^2 - 2(abc + a^3 + b^3 + c^3)d_3(a,b,c).$$

From the last two identities and inequalities we obtain (3), and hence (2). □

If we express everything in (2) in terms of elementary symmetric functions e_1, e_2, e_3 in variables a, b, c, we obtain the following symmetric inequality in side lengths for any triangle (note, $\prod a + \sum a^3 = e_1^3 - 3e_1e_2 + 4e_3$):

$$e_1^6 + 12e_1^3e_3 + 12e_1^2e_2^2 + 36e_3^2 \geq 7e_1^4e_2 + 40e_1e_2e_3. \tag{4}$$

Again, this inequality is tight in the sense that it becomes equality if and only if the triangle is equilateral. We shall apply our refinement of Euler's inequality to upper bound the area of a triangle.

Theorem 2. *The area S of a triangle with side lengths a, b, c can tightly be bounded from above as*

$$4S^2 \leq \frac{(a+b+c)a^2b^2c^2}{abc + a^3 + b^3 + c^3} \leq \frac{(a+b+c)^4}{2^2 3^3}. \tag{5}$$

Proof. The left hand side of (5) is equivalent to (1). Namely, (1) can be stated as

$$\frac{R}{r} = \frac{(a+b+c)abc}{8S^2} \geq \frac{abc + a^3 + b^3 + c^3}{2abc},$$

and this inequality is obviously equivalent to the left inequality in (5). The right inequality in (5) follows from the AM-GM inequality $abc \leq (\frac{a+b+c}{3})^3$ and the inequality $\frac{abc}{abc+a^3+b^3+c^3} \leq 1/4$, and this is equivalent to the AM-GM inequality $abc \leq \frac{a^3+b^3+c^3}{3}$. □

Note that (5) is an improvement of the *standard isoperimetric inequality* for triangles

$$S \le \frac{\sqrt{3}(a+b+c)^2}{36}.$$

This is also explained in [1] and [5]. The left inequality in (5) improves the inequality $S \le \frac{3(abc)^{2/3}}{4}$, which, in turn, improves the standard isoperimetric inequality for triangles, as proved by Theorem 4.1. in [11]. So, we have the following chain of bounds for the area S of an (a, b, c)-triangle:

$$4S^2 \le \frac{(a+b+c)a^2b^2c^2}{abc+a^3+b^3+c^3} \le \frac{3(abc)^{4/3}}{4} \le \frac{(a+b+c)^4}{2^2 3^3}. \tag{6}$$

3 Noneuclidian triangles

The following Lemma from [2] allows us to convert inequalities for Euclidean into spherical and hyperbolic triangles.

Lemma 3. *If $f(a, b, c) \ge 0$ is an inequality which holds for all Euclidean triangles with side lengths a, b, c, then $f(s(a), s(b), s(c)) \ge 0$ for all spherical or hyperbolic triangles with side lengths a, b, c, where $s(x) = x/2$ in Euclidean geometry, $s(x) = \sin(x/2)$ in spherical geometry and $s(x) = \sinh(x/2)$ in hyperbolic geometry.*

Using this Lemma, the inequality (2) or (4) can easily be converted into corresponding spherical or hyperbolic analogues. This is elaborated in more details in [1] and [5]. For instance, in hyperbolic geometry, the inequality (4) holds with symmetric functions in variables $\sinh(a/2)$ etc.

4 Tetrahedron

Now we go to the 3-dimensional ($3D$) space. Let $T = ABCD$ be a tetrahedron (3-simplex) with edge lengths a, b, c, a', b', c', (a, b, c forming the triangle ABC, a against a' etc.), volume $V = vol(ABCD)$, surface area $S (= S_A + \ldots + S_D)$ and R and r the circumradius and inradius of T, respectively. It is well known that the numbers aa', bb' and cc' are side lengths of the triangle called *Crelle's triangle* of T. Recall, Crelle's triangle $A^*B^*C^*$ of T against D is the image of the triangle ABC under the inversion I with the center D and radius ρ whose square is equal $\rho^2 = DA \cdot DB \cdot DC = a'b'c'$. $A^* = I(A)$, etc. (Recall, the inversion I maps a point $X \ne D$ to the point X^* on the ray DX, such that the product of distances is $DX \cdot DX^* = \rho^2$; point D is mapped to the infinity.) There are three more Crelle's triangles.

The area Crelle of the Crelle triangle is given by the *Crelle formula*: Crelle $= 6RV$ (see e.g. [3]).

Proof of Crelle's formula. The distance h of D to the plane $A^*B^*C^*$ is equal to $\rho^2/2R$, because the circumsphere of T is mapped by I to the plane $A^*B^*C^*$. The ratio of volumes $vol(DA^*B^*C^*) : vol(DABC)$ is equal to the ratio of products of lengths of lateral edges, that is

$$vol(DA^*B^*C^*) : vol(DABC) = \frac{DA^* \cdot DB^* \cdot DC^*}{DA \cdot DB \cdot DC}.$$

Namely, if h_C and h_C^* are distances from C and C^* to the plane DAB (or DA^*B^*) and S_C, S_C^* the areas of triangles DAB and DA^*B^*, respectively, then $h_C : h_C^* = DC : DC^*$ and $S_C : S_C^* = (DA \cdot DB) : (DA^* \cdot DB^*)$ and since $V = vol(DABC) = S_C h_C/3$, $vol(DA^*B^*C^*) = S_C^* h_C^*/3$, it follows that

$$\frac{vol(DA^*B^*C^*)}{vol(DABC)} = \frac{h_C^* S_C^*}{h_C S_C} = \frac{DA^* \cdot DB^* \cdot DC^*}{DA \cdot DB \cdot DC}.$$

So,

$$vol(DA^*B^*C^*) = V \cdot \frac{DA^*}{DA} \cdot \frac{DB^*}{DB} \cdot \frac{DC^*}{DC} = V \cdot \Big(\frac{\rho}{DA}\Big)^2 \cdot \Big(\frac{\rho}{DB}\Big)^2 \cdot \Big(\frac{\rho}{DC}\Big)^2 = V\rho^2.$$

On the other hand, $vol(DA^*B^*C^*) = Crelle \cdot \frac{h}{3}$, so $V\rho^2 = Crelle \cdot \frac{h}{3} = \left(\frac{Crelle}{3}\right)\rho^2$, and hence $Crelle = 6RV$. By symmetry, the other three Crelle's triangles have the same area. $\square$.

By using Crelle's formula, the standard isoperimetric inequality for tetrahedra (see [3]) can be derived, as we shall prove later in the text

$$2^3 3^4 V^2 \leq \sqrt{3} S^3. \tag{7}$$

The following inequality (see [4]) is also crucial for the rest of our argument:

$$2^3 3^2 V^2 \leq d_3(aa', bb', cc'). \tag{8}$$

Proof of (8). By Heron's formula for Crelle's triangle and by Crelle's formula we have $16 Crelle^2 = (aa' + bb' + cc') d_3(aa', bb', cc') = 16(6RV)^2$. So, (8) is equivalent to

$$aa' + bb' + cc' \leq 8R^2 \tag{9}$$

(with equality iff T is equifacial).

To prove (9), recall Lagrange's center mass theorem, which says that if $A_1, \ldots, A_n$ is any set of points in space and G their centroid (center of mass), then $\sum \overrightarrow{GA_i} = \mathbf{0}$. For any point P, the moment of inertia I_P is defined as $I_P := \sum PA_i^2$. Then

$$I_P = \sum (\overrightarrow{PG} + \overrightarrow{GA_i})^2 = \sum GA_i^2 + nPG^2 + 2\overrightarrow{PG} \cdot \sum \overrightarrow{GA_i} = I_G + nPG^2.$$

By adding these equalities for $P = A_1, \ldots, A_n$, we get

$$2\sum_{i<j} A_iA_j^2 = \sum I_{A_i} = nI_G + n\sum GA_i^2 = 2nI_G.$$

From the last two equalities we obtain

$$nI_P = \sum_{i<j} A_iA_j^2 + n^2PG^2.$$

For $n = 4$ and $A_1, \ldots, A_4 = A, B, C, D$ the vertices of T, this implies that for any point P in space we have

$$4\sum PA^2 = \sum a^2 + 16PG^2.$$

If $P = O$, the circumcenter of T, then the last equality implies

$$16R^2 = \sum a^2 + 16OG^2.$$

Since $a^2 + a'^2 \geq 2aa'$, etc., (9) follows since $OG^2 \geq 0$. This proves the inequality (8). Equality holds if and only if $a = a'$, $b = b'$ and $c = c'$, and $G = O$. On the other hand, if T is equifacial, then $G = O$, because if P is any of the vertices of T (recall, $a = a'$ etc.), from above equality we get by symmetry that G has the same distance from all four vertices of T, so $G = O$. $\square$.

We shall prove (see [1]) the following refinement, or rather improvement of Euler's inequality (E_3), i.e. $R \geq 3r$.

Theorem 4.

$$\left(\frac{R}{r}\right)^2 \geq \frac{3(aa' + bb' + cc')}{d_3^{1/3}} \geq 9. \tag{10}$$

Here, $d_3 = d_3(aa', bb', cc')$. Equality on the left hand side of (10) becomes equality if and only if the tetrahedron T is equifacial, i.e. $a = a'$, $b = b'$ and $c = c'$. The right inequality becomes equality if and only if $aa' = bb' = cc'$ (i.e. if and only if the Crelle triangle is equilateral).

Proof. Start with Crelle's formula and the equality $3V = rS$. We have (C is the area of Crelle's triangle)

$$\left(\frac{R}{r}\right)^2 = \frac{C^2S^2}{2^2 3^4 V^4} = \frac{16C^2}{2^6 3^4} \cdot \left(\frac{S^3}{V^2}\right)^{2/3} \cdot \frac{1}{V^{8/3}}. \tag{11}$$

Heron's formula for the Crelle triangle yields

$$16C^2 = (aa' + bb' + cc')d_3(aa', bb', cc').$$

The isoperimetric inequality (7) tells us that $\frac{S^3}{V^2} \geq \frac{2^3 3^4}{\sqrt{3}}$. The inequality (8) says

$$\frac{1}{V^{8/3}} \geq \frac{2^4 \cdot 3^{8/3}}{d_3(aa', bb', cc')^{4/3}}.$$

Inserting these estimates to (11) yields the left inequality in (10). The right inequality in (10) follows from the Schur inequality and the AM-GM inequality for the numbers aa', bb', cc'. Namely,

$$d_3(aa', bb', cc') \leq (aa')(bb')(cc') \leq \left(\frac{aa' + bb' + cc'}{3}\right)^3.$$

It is clear that the left inequality in (10) becomes an equality if and only if both (7) and (8) are equalities, i.e. if and only if the tetrahedron is equifacial. □

5 Simplex

Finally, we go to the n-dimensional space. With obvious notations we have the following standard isoperimetric inequality for an n-simplex $T = A_0 \ldots A_n$ with basically the same proof as for tetrahedra:

$$(n!S^n)^2 \geq n^{3n}(n+1)^{n+1}V^{2(n-1)}. \tag{12}$$

In fact, even a stronger upper bound for V holds in terms of the product of surface areas S_i of T (see [3], [6], [7]):

$$V^{2(n-1)} \leq (n!)^2 \frac{(n+1)^{n-1}}{n^{3n}} (S_0 \times S_1 \times \cdots \times S_n)^{2n/(n+1)}. \tag{13}$$

For reader's covenients, and since (12) and (13) are not very well known, we provide a proof of them bellow in the text (for $n = 3$).

Next, as we know the crux of the proof of (8) is the inequality $16R^2 \geq \sum a^2$, where $\sum a^2$ is the sum of squared edge lengths of T. The same proof works for an n-simplex T:

$$(n+1)^2 R^2 \geq \sum a^2. \tag{14}$$

Define *Crelle's (n-1)-simplex* $C_{n-1}^{(0)}$ of the simplex $T = A_0A_1 \dots A_n$ against the vertex A_0 of the simplex T as the image $A_1^*A_2^* \dots A_n^* = I_0(A_1 \dots A_n)$, $A_i^* = I_0(A_i)$, $i = 1, \dots, n$, where $I_0 = I_0(A_0, \rho_0)$, $\rho_0^2 = \prod_{i=1}^n A_0A_i$, is the inversion with the center A_0 and the radius of inversion ρ_0.

The ratio of volumes of pyramides $A_0A_1^* \dots A_n^*$ and $A_0A_1 \dots A_n$ is equal to the ratio of products of side lengths of lateral edges $A_0A_1^* \times \cdots \times A_0A_n^*$ and $A_0A_1 \times \cdots \times A_0A_n$. Little calculation yields that the volume is equal to:

$$vol_{n-1}(C_{n-1}^{(0)}) = 2nRV\rho_0^{2(n-3)}. \tag{15}$$

For $n = 3$, formula (15) reduces to Crelle's formula $C = 6RV$. In order to symmetrize everything, consider C_{n-1}, the product of all $vol_{n-1}(C_{n-1}^{(i)})$, $i = 0, 1, \dots, n$, where $C_{n-1}^{(i)}$ is Crelle's $(n-1)$-simplex of T against the vertex A_i. Note that for $n = 2$, $C_1 = abc = 4RS$, and (1) or (5) can be applied. Also, $C_2 = (6RV)^4$ and $C_3 = (8RV)^5 \prod a^2$.

Now we can recursively improve Euler's inequality (E_n), $n > 3$, as follows. Start with $n = 4$ and apply (8) to all five Crelle's tetrahedra in C_3. Then use (14). This gives an upper bound for the volume V of a 4-simplex T in terms of its edge lengths. Then use again (14) and $rS = 4V$, (13), (14) to estimate R/r. Then continue that way to higher dimensions.

The obtained intrinsic lower bound for R/r is certainly better than the ambiental Euler's bound n. We shall not go into details here, but the interested reader can check that the procedure works on a simple example with 4-simplex, say, origin and four unit vertices on coordinate axes.

6 Isoperimetric and some other inequalities

We shall now prove (12) and (13) for $n = 3$, but the same proof by "mutatis mutandis" works for all n. Let us prove (7), i.e.

$$\frac{S^3}{V^2} \geq 2^3 3^3 \sqrt{3}, \tag{16}$$

with equality if and only if T is equifacial.

Proof. Let $T = ABCD$ be a tetrahedron and $S_A, \ldots, S_D$ be face areas of T (S_A opposite to A etc.); in the sum they give S. Let $\mathbf{n}_A$ be the unit outward normal vector to the face with area S_A etc. Then the vector sum $\sum S_A\mathbf{n}_A = 0$ (because vector on the left is perpendicular to all vectors including itself, so it has to be zero). Let W_A be the volume of the tetrahedron T_A spanned by $\mathbf{n}_B$, $\mathbf{n}_C$, $\mathbf{n}_D$ and similarly W_B and so on. Then we have $W_A S_B S_C S_D = 3V^2/4$, as well as the other three such formulas (this is called the sine law for tetrahedra, see [6]). Indeed, by multiplying the above null-sum of vectors by $\times S_A\mathbf{n}_A$ (cross product) and then by $S_B\mathbf{n}_B$ (scalar product), and taking into account that $W_A = ((\mathbf{n}_B \times \mathbf{n}_C)\mathbf{n}_D)/6$ and so on, the claim above follows easily. Next, we need the inequality $27V \leq 8\sqrt{3} \cdot R^3$. To prove this, let d be the distance from the circumcenter of T to the plane BCD and R_A the circumradius of the triangle BCD. Then $R_A^2 = R^2 - d^2$ and $V \leq S_A(R+d)/3$, because the altitude to BCD is at most $R+d$. Recall that the area K of any triangle with side lengths a, b, c satisfies the inequality $K \leq \sqrt{3}(abc)^{2/3}/4$, or equivalently, $K \leq 3\sqrt{3} \cdot R^2/4$. Hence,

$$S_A \leq 3\sqrt{3} \cdot \frac{R_A^2}{4} = \frac{3\sqrt{3}}{4}(R^2 - d^2).$$

Therefore,

$$V \leq \frac{S_A}{3}(R+d) \leq \frac{\sqrt{3}}{4}(R^2 - d^2)(R+d).$$

The AM-GM inequality yields

$$2(R^2 - d^2)(R+d) = (2R - 2d)(R+d)(R+d) \leq \left(\frac{4R}{3}\right)^3.$$

Thus, finally, $V \leq 8\sqrt{3} \cdot R^3/27$. Now, all four tetrahedra $T_A, \ldots, T_D$ with volumes $W_A, \ldots, W_D$, respectively, do not intersect in their interiors and their union E is a tetrahedron inscribed in the unit sphere. By the above discussion, the volume W of E is at most $8\sqrt{3}/27$. Hence, by the AM-GM we have

$$\prod W_A \leq \left(\sum \frac{W_A}{4}\right)^4 = \left(\frac{W}{4}\right)^4 \leq \left(\frac{8\sqrt{3}}{27 \cdot 4}\right)^4 = \frac{2^4}{3^{10}}.$$

Finally, by multiplying all four sine-law formulas, by using above results and the AM-GM we get

$$V^4 = \frac{16}{9} \cdot \left(\prod S_A\right)^{3/2} \left(\prod W_A\right)^{1/2} \leq \frac{16}{9} \cdot \left(\sum \frac{S_A}{4}\right)^6 \cdot \frac{2^2}{3^5} = \frac{S^6}{2^6 3^7}.$$

By taking the square root (16) follows. Equality is clear from the proof. □

The same proof (with obvious changes) works for any simplex, so we have just proved (12) and (13). Some consequences of the above proof and (13) are the inequalities (see also [6], [7]):

$$(n!V)^2 n^n \leq (n+1)^{n+1} R^{2n}, \qquad (n!V)^2 2^n \leq (n+1) \left(\prod a^2\right)^{2/(n+1)}, \quad (17)$$

with equalities if and only if the simplex is regular.

The right inequality in (17) was first conjectured by this author back in 1969, when as an undergraduate student proposed it as an open problem in *Elemente der Mathematik*, and then proved three years later by G. Korchmáros, see [8] and [9]. In 1981, J. Zh. Zhang and L. Yang gave it a different aspect, [10]. Further development can be traced in [6] and [7].

7 Some open problems

We end with some open problems. In [11] we proved non- Euclidean versions of some classical triangle inequalities. The problems and proposals for new research projects are the following.

What are non-Euclidean versions of some of the 3-dimensional or n-dimensional inequalities described in this paper and some other ("classical") inequalities for tetrahedra and simplices?

For example, we can apply the inequalities (2) or (4) to Crelle's triangle of a (Euclidean) tetrahedron and thus obtain the corresponding inequality for tetrahedra, symmetric in products of opposite edges. Then by Lemma 3 it can be applied to hyperbolic (and spherical) tetrahedra. What is $3D$-analogue of Lemma 3? And in higher dimensions?

Next, what is the hyperbolic version of (8) and of our main inequality (10), when hyperbolic tetrahedron has circumsphere?

Note that for a hyperbolic tetrahedron we can form the corresponding Crelle's triangle with side lengths as products of sinh of (halfs of) opposite dihedral angles, and similarly in the spherical case. What is the hyperbolic $3D$-analogue of Crelle's formula? And more generally of formula (15) for hyperbolic simplices?

What are hyperbolic (and spherical) versions of (12), (13) and (17)?

It is well known that computing volumes is a hard task even in Euclidean geometry and needless to say in hyperbolic geometry. Even for a simplex, computing Cayley-Menger determinant is a hard task. Therefore, having on disposal various upper (and lower) bounds for volume of a tetrahedron could be important in many applications in physics, astrophysics, IT technology, crystallography and other disciplines when using triangulations in order to estimate the volume of a body. For instance, one could imagine that in future we could want to estimate the volume in deep space determined by four (or more) stars, knowing their mutual distances. Then this volume should be estimated by an inequality for a hyperbolic tetrahedron in the (supposedly hyperbolic) universe and sum all of them.

The last problem is about Grace-Danielsson's inequality (see (18)). It was proposed in 1949, but without clear proof by Danielsson and claimed earlier by Grace. If $d = OI$ is the distance between the circumcenter O and the incenter I of the tetrahedron T, then

$$d^2 \leq (R - 3r)(R + r). \tag{18}$$

If T is equifacial then $O = I$ and since $R \geq 3r$, (18) holds in this case. Can our results provide a proof in general case? Is there a short elegant proof of (18) in the sense of CEEG (classical Euclidean elementary geometry)? Recall, Euler proved in 1767 that for a triangle $d^2 = (R - 2r)R$. A short proof is by inversion w.r.t. the incircle of a triangle: the image of the circumcircle has radius $r^* = r/2 = Rr^2/(R^2 - d^2)$. At the moment it's not clear whether a similar method works for tetrahedra. The inequality (18) was proved via computer aided argument and by quantum information theory in 2015 and 2018 (see lit. in [5]). So, the interest for finding a proof of (18) even among computer scientists shows its importance. In higher dimensions there are not even any documented hypothesis analogous to Euler's formula or (18) inequality. Perhaps some arguments involving Crelle's triangle or Crelle's simplices could be helpful in higher dimensions and allow us to use recursive method as we did with Euler's inequality. At the moment it is not known how to approach geometrically to this elusive inequality. In any case, it is a good project for further research.

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Notes