MATRIX PRODUCTS OF BINOMIAL COEFFICIENTS AND UNSIGNED STIRLING NUMBERS

MARIN KNEŽEVIĆ, VEDRAN KRČADINAC, AND LUCIJA RELIĆ

Dedicated to Professor Dragutin Svrtan on the occasion of his 70th birthday.

ABSTRACT. We study sums of the form $\sum_{k=m}^{n} a_{nk}b_{km}$, where a_{nk} and b_{km} are binomial coefficients or unsigned Stirling numbers. In a few cases they can be written in closed form. Failing that, the sums still share many common features: combinatorial interpretations, Pascal-like recurrences, inverse relations with their signed versions, and interpretations as coefficients of change between polynomial bases.

MSC: 05A10

1. INTRODUCTION

Let a_{nk} and b_{nk} be binomial coefficients $\binom{n}{k}$ or unsigned Stirling numbers of the first and second kind: $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ and $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ in Karamata-Knuth notation, respectively. We are interested in the sum $\sum_{k=m}^{n} a_{nk}b_{km}$. Denote by $A = [a_{nk}]$ and $B = [b_{nk}]$ the corresponding infinite lower-triangular matrices, indexed by the non-negative integers. The sum can be interpreted as the (n, m)-entry of the matrix product $A \cdot B$.

Our motivation are the following two sums, that can be written in closed form:

$$\sum_{k=m}^{n} \binom{n}{k} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} = \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\}, \tag{1}$$

$$\sum_{k=m}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] \binom{k}{m} = \left[\begin{matrix} n+1 \\ m+1 \end{matrix} \right]. \tag{2}$$

These are identities (6.15) and (6.16) in the book *Concrete Mathematics* [1]. Furthermore, the following sum are the Lah numbers [2, 3], denoted by $L(n, m)$:

$$\sum_{k=m}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] \left\{ \begin{matrix} k \\ m \end{matrix} \right\} = \frac{n!}{m!} \binom{n-1}{m-1}. \tag{3}$$

$\sum_k \binom{n}{k}\binom{k}{m}$ A038207	$\sum_k \binom{n}{k}\left[{k \atop m}\right]$ A094816	$\sum_k \binom{n}{k}\left\{{k \atop m}\right\}$ A008277
$\sum_k \left[{n \atop k}\right]\binom{k}{m}$ A130534	$\sum_k \left[{n \atop k}\right]\left[{k \atop m}\right]$ A325872*	$\sum_k \left[{n \atop k}\right]\left\{{k \atop m}\right\}$ A271703
$\sum_k \left\{{n \atop k}\right\}\binom{k}{m}$ A049020	$\sum_k \left\{{n \atop k}\right\}\left[{k \atop m}\right]$ A129062	$\sum_k \left\{{n \atop k}\right\}\left\{{k \atop m}\right\}$ A130191

TABLE 1. References for the sums in the On-line ency-
clopedia of integer sequences [7].

Closed-form expressions for two more sums are given in the next
section. In the remaining cases we do not know closed forms, but the
sums share many common features. Combinatorial interpretations are
outlined in Section 2. Row sums of the matrix $A \cdot B$ are discussed
in Section 3. Recurrences similar to Pascal's formula are proved in
Section 4. Inverse relations with signed versions of the sums are given
in Section 5, and in Section 6 the sums are interpreted as coefficients
of change between various polynomial bases.

Our main focus are sums with a_{nk} and b_{nk} equal to $\binom{n}{k}$, $\left[{n \atop k}\right]$ or $\left\{{n \atop k}\right\}$,
but we include some results on sums with $L(n,k)$ as well. Our nine
main sums are listed in the On-line encyclopedia of integer sequences [7]
as "triangles read by rows", see Table 1 (the entry marked by * is a
signed version). Many properties of the sums are reported in [7]. We
repeat some properties from [7], and reveal some new properties of the
sums. We mainly deal with properties that can be proved by counting
arguments.

2. COMBINATORIAL INTERPRETATIONS AND CLOSED FORMS

Identity (1) follows from the usual combinatorial interpretation of
Stirling numbers of the second kind. The right-hand side $\left\{{n+1 \atop m+1}\right\}$ counts
partitions of an $(n+1)$-element set S into $m+1$ blocks. The left-hand
side is obtained by distinguishing an element of S and assuming it is
covered by a block containing $n - k$ other elements of S.

The Stirling number of the first kind $\left[{n \atop m}\right]$ is usually interpreted as
the number of permutations of degree n with exactly m cycles. For the
proof of identity (2), however, another combinatorial interpretation is
more useful. Let Δ_n denote the board that remains after removing all
tiles below and on the side diagonal of a $n \times n$ square board. Consider
a new chess piece similar to a rook, but that only attacks tiles in its
row. Loehr [4] calls this piece a weak rook or wrook. He goes on to

show that the number of ways of placing k identical wrooks on Δ_n so that they don't attack each other is equal to $\left[\begin{smallmatrix} n \\ n-k \end{smallmatrix}\right]$. The right-hand side of (2) can now be interpreted as the number of ways of placing $n - m$ non-attacking wrooks on Δ_{n+1}. The left-hand side is obtained by partitioning the set of placements based on the number of wrooks not in the first column, but on the remaining Δ_n subboard. If $n - k$ wrooks are placed on the Δ_n subboard, the remaining $k - m$ wrooks must be placed in k rows of the first column that are not attacked. This can be done in $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \binom{k}{k-m} = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \binom{k}{m}$ ways.

In [6], a nice combinatorial interpretation of the Lah numbers is given: $L(n, m)$ counts partitions of an n-element set into m lists, i.e. non-empty totally ordered subsets. From this, identity (3) follows easily. Two more closed-form identities are

$$\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m} = 2^{n-m} \binom{n}{m} \tag{4}$$

and

$$\sum_{k=m}^{n} L(n, k) \, L(k, m) = 2^{n-m} L(n, m). \tag{5}$$

The left-hand side of (4) counts pairs (K, M), where $K \supseteq M$ are subsets of a fixed n-element set S, and $|M| = m$. The right-hand side is obtained by choosing M first. Identity (5) has an analogous proof using the combinatorial interpretation of Lah numbers from [6]. For two partitions Π_1, Π_2 of S into lists, we write $\Pi_1 \leq \Pi_2$ and say that Π_1 is a refinement of Π_2 if each list in Π_1 is a sublist of some list in Π_2. The left-hand side of (5) counts pairs (Π_1, Π_2) with $\Pi_1 \leq \Pi_2$ and $|\Pi_2| = m$. We first partition S into Π_1 with $|\Pi_1| = k \geq m$, then partition these k lists into m lists containing lists of Π_1 as elements, and finally concatenate the list of lists into elements of Π_2. We get the right side by partitioning S into m lists of Π_2 first and splitting them up into smaller lists of Π_1. This can be done in 2^{n-m} ways: breaks can be made in $n - m$ places, before every element except at the beginnings of the lists of Π_2.

Similar combinatorial interpretations can be given for the double Stirling sums $\sum_{k=m}^{n} \left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\} \left\{\begin{smallmatrix} k \\ m \end{smallmatrix}\right\}$ and $\sum_{k=m}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \left[\begin{smallmatrix} k \\ m \end{smallmatrix}\right]$. The former is the number of pairs (Π_1, Π_2) of ordinary partitions of an n-element set into blocks (non-empty subsets), with Π_1 a refinement of Π_2 and $|\Pi_2| = m$. The latter is the number of pairs of permutations (π_1, π_2), where π_1 is of degree n, π_2 is of degree equal to the number of cycles of π_1, and π_2 has exactly m cycles. However, since we do not see an easy way to count

the pairs if Π_2 or π_2 are chosen first, this does not lead to closed-form expressions for these two sums.

Combinatorial interpretations of the remaining sums in Table 1 are as follows. The sum $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \binom{k}{m}$ is the number of ordinary partitions of an n-element set with m blocks colored red, and the other blocks colored blue. The sum $\sum_{k=m}^{n} \binom{n}{k} \left[{k \atop m} \right]$ counts permutations with exactly m cycles of all subsets of an n-element set. Finally, $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left[{k \atop m} \right]$ is the number of pairs (Π, π), where Π is a partition of an n-element set into blocks, and π is a permutation with exactly m cycles of degree equal to the number of blocks of Π. Again, this does not lead to closed-form expressions, but the combinatorial interpretations of the sums will be used in the following sections to prove their properties.

3. Row sums

From the combinatorial interpretation of binomial coefficients and Stirling numbers, it is clear that $\sum_{m=0}^{n} \binom{n}{m} = 2^n$, $\sum_{m=0}^{n} \left[{n \atop m} \right] = n!$, and $\sum_{m=0}^{n} \left\{ {n \atop m} \right\} = B_n$. Here B_n is the n-th Bell number, i.e. the total number of partitions of an n-element set into blocks. In this section the goal is to determine

$$\sum_{m=0}^{n} \sum_{k=m}^{n} a_{nk} b_{km}.$$

This is the sum of the n-th row of the matrix $A \cdot B$.

From identities (1) and (2), we have

$$\sum_{m=0}^{n} \sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} = B_{n+1} \quad \text{and} \quad \sum_{m=0}^{n} \sum_{k=m}^{n} \left[{n \atop k} \right] \binom{k}{m} = (n+1)!.$$

Another row sum that can be written in closed form is

$$\sum_{m=0}^{n} \sum_{k=m}^{n} \binom{n}{k} \binom{k}{m} = 3^n.$$

The left-hand side counts the total number of pairs (K, M) of subsets of an n-set S with $K \supseteq M$. An alternative way of counting is to decide for each element $x \in S$ whether it is in M, in $K \setminus M$, or in $S \setminus K$, leading to the right-hand side.

The row sum of Lah numbers

$$\sum_{m=0}^{n} \sum_{k=m}^{n} \left[{n \atop k} \right] \left\{ {k \atop m} \right\} = \sum_{m=0}^{n} L(n, m)$$

can be interpreted as the total number of partitions of an n-set into lists. This is the "Lah equivalent" of the Bell number B_n and we will denote it by L_n. In [7], this is sequence number A000262.

Row sums of the remaining matrix products are given in the sequel. They have nice combinatorial interpretations and can be simplified to single sums.

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\left\{{n \atop k}\right\}\left[{k \atop m}\right]=\sum_{m=0}^{n}\left\{{n \atop m}\right\}m!.$$

These are the Fubini numbers, sequence A000670 in [7]. They count ordered partitions of an n-set, or weak orders on n elements.

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\left[{n \atop k}\right]\left[{k \atop m}\right]=\sum_{m=0}^{n}\left[{n \atop m}\right]m!.$$

This is the number of ordered factorizations of permutations of degree n into cycles, sequence A007840 in [7].

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\left\{{n \atop k}\right\}\binom{k}{m}=\sum_{m=0}^{n}\left\{{n \atop m}\right\}2^m.$$

The total number of partitions of an n-set with blocks colored red or blue. This is sequence A001861 in [7].

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\binom{n}{k}\left[{k \atop m}\right]=\sum_{m=0}^{n}\frac{n!}{m!}.$$

The total number of lists with elements from an n-set. Sequence number A000522 in [7].

$$\sum_{m=0}^{n}\sum_{k=m}^{n}\left\{{n \atop k}\right\}\left\{{k \atop m}\right\}=\sum_{m=0}^{n}\left\{{n \atop m}\right\}B_m.$$

The total number of pairs (Π_1,Π_2) of partitions of an n-set with $\Pi_1 \leq \Pi_2$. In [7], this is sequence number A000258.

4. Pascal-like recurrences

The binomial coefficients can be computed by Pascal's formula

$$\binom{n}{m}=\binom{n-1}{m-1}+\binom{n-1}{m}, \quad \binom{n}{0}=\binom{n}{n}=1.$$

Analogous recurrences for Stirling numbers are

$$\left[{n \atop m}\right]=\left[{n-1 \atop m-1}\right]+(n-1)\left[{n-1 \atop m}\right], \quad \left[{n \atop 0}\right]=\delta_{n0}, \quad \left[{n \atop n}\right]=1,$$

$$\left\{ {n \atop m} \right\} = \left\{ {n-1 \atop m-1} \right\} + m \left\{ {n-1 \atop m} \right\}, \quad \left\{ {n \atop 0} \right\} = \delta_{n0}, \quad \left\{ {n \atop n} \right\} = 1,$$

and for Lah numbers

$$L(n,m) = L(n-1, m-1) + (n+m-1)L(n-1, m)$$

with boundary values $L(n,0) = \delta_{n0}$ (the Kronecker delta), $L(n,n) = 1$. See [6] for proofs of the formulae by distinguishing an element of the underlying n-set S and counting. Our sums satisfy similar recurrences that can also be established by counting arguments.

To make the formulae more readable, we denote the sum $\sum_{k=m}^{n} a_{nk} b_{km}$ by $\left| {n \atop m} \right|$. For example, the double binomial sum $\left| {n \atop m} \right| = \sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}$ can be computed from

$$\left| {n \atop m} \right| = \left| {n-1 \atop m-1} \right| + 2 \left| {n-1 \atop m} \right|, \quad \left| {n \atop 0} \right| = 2^n, \quad \left| {n \atop n} \right| = 1.$$

This sum also satisfies the absorption identity $\left| {n \atop m} \right| = \frac{n}{m} \left| {n-1 \atop m-1} \right|$, just like the binomial coefficients.

By (1), the sum $\left| {n \atop m} \right| = \sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\}$ are shifted Stirling numbers of the second kind. Therefore,

$$\left| {n \atop m} \right| = \left| {n-1 \atop m-1} \right| + (m+1) \left| {n-1 \atop m} \right|, \quad \left| {n \atop 0} \right| = \left| {n \atop n} \right| = 1.$$

Similarly, by (2), the sum $\left| {n \atop m} \right| = \sum_{k=m}^{n} \left[{n \atop k} \right] \binom{k}{m}$ satisfies

$$\left| {n \atop m} \right| = \left| {n-1 \atop m-1} \right| + n \left| {n-1 \atop m} \right|, \quad \left| {n \atop 0} \right| = n!, \quad \left| {n \atop n} \right| = 1.$$

The sum $\left| {n \atop m} \right| = \sum_{k=m}^{n} \binom{n}{k} L(k,m)$ is sequence A271705 in [7], where the following recurrence is given:

$$\left| {n \atop m} \right| = \frac{n}{m} \left| {n-1 \atop m-1} \right| + n \left| {n-1 \atop m} \right|, \quad \left| {n \atop 0} \right| = \left| {n \atop n} \right| = 1.$$

The sum $\left| {n \atop m} \right| = \sum_{k=m}^{n} L(n,k) \binom{k}{m}$ is sequence A059110. It satisfies the same recurrence, but with different boundary values $\left| {n \atop 0} \right| = L_n$.

The double Lah sum $\left| {n \atop m} \right| = \sum_{k=m}^{n} L(n,k) L(k,m)$ satisfies

$$\left| {n \atop m} \right| = \frac{n}{m} \left| {n-1 \atop m-1} \right| + 2n \left| {n-1 \atop m} \right|, \quad \left| {n \atop 0} \right| = \delta_{n0}, \quad \left| {n \atop n} \right| = 1.$$

This sum also satisfies the absorption-like identity $\left| {n \atop m} \right| = \frac{n-m+1}{2m(m-1)} \left| {n \atop m-1} \right|$, while the Lah numbers satisfy $L(n,m) = \frac{n-m+1}{m(m-1)} L(n, m-1)$.

Now let $\left|{n \atop m}\right| = \sum_{k=m}^{n} \binom{n}{k} \left[{k \atop m}\right]$. This sum cannot be computed from the two values $\left|{n-1 \atop m-1}\right|$ and $\left|{n-1 \atop m}\right|$ alone, but we can give a Pascal-like recurrence involving more previous values:

$$\left|{n \atop m}\right| = \sum_{k=0}^{n-m} (n-1)^{\underline{k}} \left|{n-1-k \atop m-1}\right| + \left|{n-1 \atop m}\right|, \quad \left|{n \atop 0}\right| = \left|{n \atop n}\right| = 1.$$

Here $(n-1)^{\underline{k}} = (n-1) \cdot (n-2) \cdots (n-k)$ is the falling factorial. For the proof, recall that $\left|{n \atop m}\right|$ counts permutations with m cycles of subsets $T \subseteq S$, where S is a set of n elements. Fix an element $x \in S$ and divide the permutations according to whether they contain x or do not contain x. In the latter case there are clearly $\left|{n-1 \atop m}\right|$ permutations. In the former case, assume x is in a cycle with k other elements of S. This cycle can be chosen in $(n-1)^{\underline{k}}$ ways, and $m-1$ more cycles have to be chosen from the remaining $n-1-k$ elements. Thus, there are $\sum_{k=0}^{n-m} (n-1)^{\underline{k}} \left|{n-1-k \atop m-1}\right|$ permutations containing x.

The sum $\left|{n \atop m}\right| = \sum_{k=m}^{n} \left\{{n \atop k}\right\} \binom{k}{m}$ counts partitions of S with m blocks colored red, and the other blocks colored blue. Again, fix an element $x \in S$. If x is in a red block alone, there are $\left|{n-1 \atop m-1}\right|$ partitions. If x is in a red block with some other elements of S, there are $m \left|{n-1 \atop m}\right|$ partitions. Finally, if x is in a blue block with k other elements of S, there are $\binom{n-1}{k} \left|{n-1-k \atop m}\right|$ such partitions. Therefore, the following recursion holds:

$$\left|{n \atop m}\right| = \left|{n-1 \atop m-1}\right| + m \left|{n-1 \atop m}\right| + \sum_{k=0}^{n-m-1} \binom{n-1}{k} \left|{n-1-k \atop m}\right|.$$

The boundary values are $\left|{n \atop 0}\right| = B_n$ and $\left|{n \atop n}\right| = 1$.

The sum $\left|{n \atop m}\right| = \sum_{k=m}^{n} \left\{{n \atop k}\right\} \left\{{k \atop m}\right\}$ counts pairs (Π_1, Π_2) of partitions of S, where Π_2 has m blocks and Π_1 is a refinement of Π_2. Now let the fixed element $x \in S$ be contained in a block of Π_2 of size k. We can choose this block in $\binom{n-1}{k-1}$ ways and partition it into blocks of Π_1 in B_k ways. The remaining blocks of Π_2 and Π_1 can be chosen in $\left|{n-k \atop m-1}\right|$ ways. Therefore,

$$\left|{n \atop m}\right| = \sum_{k=1}^{n-m+1} \binom{n-1}{k-1} B_k \left|{n-k \atop m-1}\right|, \quad \left|{n \atop 0}\right| = \delta_{n0}, \quad \left|{n \atop n}\right| = 1.$$

For the sum $\left|{n \atop m}\right| = \sum_{k=m}^{n} \left[{n \atop k}\right] \left[{k \atop m}\right]$ a similar argument leads to the recurrence

$$\left|{n \atop m}\right| = \sum_{k=1}^{n-m+1} \binom{n-1}{k-1} \sum_{i=1}^{k} \left[{k \atop i}\right] (i-1)! \left|{n-k \atop m-1}\right|, \quad \left|{n \atop 0}\right| = \delta_{n0}, \quad \left|{n \atop n}\right| = 1,$$

41

and for the sum $\left| {n \atop m} \right| = \sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left[{k \atop m} \right]$ to the recurrence

$$\left| {n \atop m} \right| = \sum_{k=1}^{n-m+1} \binom{n-1}{k-1} \sum_{i=1}^{k} \left\{ {k \atop i} \right\} (i-1)! \left| {n-k \atop m-1} \right|, \quad \left| {n \atop 0} \right| = \delta_{n0}, \quad \left| {n \atop n} \right| = 1.$$

However, these increasingly complex recurrences become less useful as the coefficients are more difficult to evaluate than the sum $\left| {n \atop m} \right|$ directly.

5. Inverses

We denote signed versions of the Stirling numbers and their relatives by an exponent $^{\sigma}$, e.g. $\left[{n \atop m} \right]^{\sigma} = (-1)^{n-m} \left[{n \atop m} \right]$. For the matrix $A = [a_{nm}]$, we denote $A^{\sigma} = [a_{nm}^{\sigma}] = [(-1)^{n-m} a_{nm}]$. To avoid excessive bracketing, we write $\left[{n \atop m} \right]^{-1}$ for the inverse matrix A^{-1}.

It is well-known that $\binom{n}{m}^{-1} = \binom{n}{m}^{\sigma}$, $\left[{n \atop m} \right]^{-1} = \left\{ {n \atop m} \right\}^{\sigma}$, and $\left\{ {n \atop m} \right\}^{-1} = \left[{n \atop m} \right]^{\sigma}$. From this and the properties $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$ and $(A \cdot B)^{\sigma} = A^{\sigma} \cdot B^{\sigma}$, we can determine inverses of our sums. For example, let $a_{mn} = \left[{n \atop m} \right]$ and $b_{mn} = \left\{ {n \atop m} \right\}$. Then $A \cdot B$ are the Lah numbers $L(n,m)$ and we have

$$L(n,m)^{-1} = (A \cdot B)^{-1} = B^{-1} \cdot A^{-1} = A^{\sigma} \cdot B^{\sigma} = (A \cdot B)^{\sigma} = L(n,m)^{\sigma}.$$

Similarly it follows

$$\left(\sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} \right)^{-1} = \left(\sum_{k=m}^{n} \left[{n \atop k} \right] \binom{k}{m} \right)^{\sigma},$$

$$\left(\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \binom{k}{m} \right)^{-1} = \left(\sum_{k=m}^{n} \binom{n}{k} \left[{k \atop m} \right] \right)^{\sigma},$$

$$\left(\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left\{ {k \atop m} \right\} \right)^{-1} = \left(\sum_{k=m}^{n} \left[{n \atop k} \right] \left[{k \atop m} \right] \right)^{\sigma},$$

and

$$\left(\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left[{k \atop m} \right] \right)^{-1} = \left(\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left[{k \atop m} \right] \right)^{\sigma}.$$

The exponents $^{-1}$ and $^{\sigma}$ can be exchanged in the formulae above.

6. Polynomial bases

We denote the falling factorials by $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ and the rising factorials by $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$, following [1]. It is known that Stirling numbers of the second kind are coefficients of change from the standard polynomial basis of powers $(x^n \mid n \in \mathbb{N}_0)$ to the basis of falling factorials $(x^{\underline{m}} \mid m \in \mathbb{N}_0)$:

$$x^n = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} x^{\underline{m}}. \tag{6}$$

Stirling numbers of the first kind are coefficients of change from $(x^{\overline{n}})$ to (x^m):

$$x^{\overline{n}} = \sum_{m=0}^{n} \left[{n \atop m} \right] x^m. \tag{7}$$

From (6) and (7) it follows that coefficients of change from $(x^{\overline{n}})$ to $(x^{\underline{m}})$ are the Lah numbers $L(n,m) = \sum_{k=m}^{n} \left[{n \atop k} \right] \left\{ {k \atop m} \right\}$:

$$x^{\overline{n}} = \sum_{m=0}^{n} L(n,m) x^{\underline{m}}.$$

Ivo Lah's original definition of his numbers [2, 3] was a signed version of this relation. We concentrate on changes between polynomial bases with non-negative coefficients. The opposite changes have inverse coefficients, with alternating signs as shown in the previous section.

From the binomial theorem $(1+x)^n = \sum_{m=0}^{n} \binom{n}{m} x^m$ and (6) we see that the sums $\sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\}$ can be interpreted as coefficients of change from the polynomial basis $((1+x)^n)$ to the basis of falling factorials $(x^{\underline{m}})$:

$$(1+x)^n = \sum_{m=0}^{n} \left(\sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} \right) x^{\underline{m}}.$$

Similar interpretations can be given to other sums from Table 1. The double binomial sums $\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m}$ are coefficients of change from the basis $((2+x)^n)$ to the standard basis (x^m). The sums $\sum_{k=m}^{n} \binom{n}{k} \left[{k \atop m} \right]$ are coefficients of change from the basis $(\sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}})$ to the standard basis. The former basis contains a special case of Charlier polynomials [7], a family of orthogonal polynomials that can be written in terms of the generalized hypergeometric function (see [5]). Double Stirling sums of the second kind $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left\{ {k \atop m} \right\}$ are coefficients of change from the basis of Bell polynomials $(B_n(x))$, $B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$ to the basis of falling factorials $(x^{\underline{m}})$.

The two families of sums $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \left[{k \atop m} \right]$ and $\sum_{k=m}^{n} \left[{n \atop k} \right] \left[{k \atop m} \right]$ can be seen as coefficients of the polynomials $\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\overline{k}}$ and $\sum_{k=0}^{n} \left[{n \atop k} \right] x^{\overline{k}}$, i.e. coefficients of change to the standard basis (x^m). Similarly, the two families $\sum_{k=m}^{n} \left\{ {n \atop k} \right\} \binom{k}{m}$ and $\sum_{k=m}^{n} \left[{n \atop k} \right] \binom{k}{m}$ are coefficients of Bell polynomials $B_n(1+x)$ and polynomials $\sum_{k=0}^{n} \left[{n \atop k} \right] (1+x)^k$, respectively.

REFERENCES

[1] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics, 2nd edition*, Addison-Wesley Publishing Company, 1994.

[2] I. Lah, *A new kind of numbers and its application in the actuarial mathematics*, Boletim do Instituto dos Actuários Portugueses **9** (1954), 7–15.

[3] I. Lah, *Eine neue Art von Zahlen, ihre Eigenschaften und Anwendung in der mathematischen Statistik*, Mitteilungsbl. Math. Statist. **7** (1955), 203–212.

[4] N. A. Loehr, *Bijective Combinatorics*, CRC Press, 2011.

[5] N. Özmen, E. Erkuş-Duman, *On the Poisson-Charlier polynomials*, Serdica Math. J. **41** (2015), no. 4, 457–470.

[6] M. Petkovšek, T. Pisanski, *Combinatorial interpretation of unsigned Stirling and Lah numbers*, The Pi Mu Epsilon Journal **12** (2007), No. 7, 417–424.

[7] N. J. A. Sloane (editor), *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org (accessed December, 2020).

Email address: marin.k59@gmail.com

Email address: vedran.krcadinac@math.hr

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF ZAGREB, BIJENIČKA 30, HR-10000 ZAGREB, CROATIA

Email address: lucijarelic7@gmail.com