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Balanced simplicial complex associated with $1 \times p$ polyomino

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Abstract

Balanced simplicial complexes are important objects in combinatorics and commutative algebra. A d-dimensional simplicial complex is balanced if its vertices can be coloured into $d+1$ colors, so there is no monochromatic edge. In this article, we establish two results concerning balanced simplicial complexes assigned to tilings of $m \times n$ board in a plane and a torus by I p-omino tile.

1 Introduction

Simplicial complexes are abstract objects that acquire most topological, geometrical, logical, combinatorial and algebraic features of spaces. It permits us to study complex spaces, and their application in topological data analysis, neural networks and modern physics is irreplaceable. A particular class of simplicial complexes, balanced complexes, is significant in combinatorics, commutative algebra and computer science. They are intensively studied since their introduction by Richard Stanley in [10]. Among the researchers who made significant progress in understanding these complexes in the last twenty years are Izmestiev, Klee, Juhnke-Kubitzke, Novik, Juhnke-Kubitzke and Murai (see [5], [2], [6], [3] and [4]).
Recreational mathematics often pays the way for establishing new mathematical disciplines. One of the most exciting puzzles is those with polyominoes. A polyomino is a polyform whose cells are squares. Solomon Golomb, who wrote the first book on polyominoes [1], gave them this name in 1953. In combinatorics, challenges are often posed for tiling a prescribed region, or the entire plane, with polyominoes; see [8] for some problems investigated in mathematics and computer science.
Recently, Baralić and Liđan come out with the idea to assign a simplicial complex to a polyomino tiling problem. Their work is related to studying simplicial complexes of graph hereditary properties that were investigated in topological combinatorics.

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Some results were published in Liđan's PhD thesis [7] from 2022. Their approach open the way for applying topological and commutative algebra methods to study these questions. In this article, we present two exciting results regarding the tiling of the board $m \times n$ in a plane and on a torus by $1 \times p$ polyomino and the balanced property of the corresponding simplicial complexes.

2 Preliminaries

This section reviews basic facts about simplicial complexes associated with polyomino tilings.

Definition 1. *An abstract simplicial complex K on a vertex set $[m] = \{1, 2, \ldots, m\}$ is a collection of subsets of $[m]$ such that*

(i) for each $i \in [m]$, $\{i\} \in K$,

(ii) for every $\sigma \in K$, if $\tau \subset \sigma$ then $\tau \in K$.

We assume that $\varnothing \in K$.

The elements of K are called *faces*. The dimension of a face σ of a simplicial complex K is defined as $\dim \sigma \colon = |\sigma| - 1$, where $|\sigma|$ denotes the cardinality of σ. Faces of dimension 0 are called *vertices* while faces of dimension one are called edges. *The dimension of* K, denoted by $\dim K$, is the maximal dimension of a face of K.
A face of K is called *maximal* or *a facet* if it is not a face of any other face of K. A subset τ of $[m]$ is called a *missing face* of K if $\tau \notin K$. A missing face τ is called *minimal* if any proper set $\sigma \subset \tau$ is a face K. A simplicial complex is completely determined by the list of its facets or minimal missing faces. Softwares like SAGE and Macaulay 2 take advantage of both approaches.
Let K be a simplicial complex and d its dimension.

Definition 2. *K is called balanced if there exists a function $k \colon V \mapsto [d+1]$ such that, if for every edge $\{x, y\} \in K$ holds $k(x) \neq k(y)$.*

In other words, Definition 2 says that a d-dimensional simplicial complex if its vertices can be colored into $d+1$ colors so that there is no monochromatic edge. Balanced simplicial complexes were introduced by Stanley in [10]. Notable examples are Coxeter complexes, Tits buildings, and the order complex of a graded poset, with the vertex set partition given by the rank function.
Simplicial complexes arise in many exciting ways. They are essential for applications of topology, commutative algebra and combinatorics. Baralić and Liđan studied a class of simplicial complexes arising from polyomino tilings. Recall that a polyomino is a plane geometric figure formed by joining one or more equal squares edge to edge. Some of their results are available in Liđan's PhD thesis [7].
Let M be a given finite subset of a plane or a torus square grid and $\mathcal{T}$ be a given finite set of polyomino tiles. Polyomino tiling problem asks whether M can be tiled using the tiles from $\mathcal{T}$ without overlapping or placing a tile so that it covers a cell outside of M. The problem that originated from recreational mathematics is an NP problem; it motivated further research in combinatorics and computer science. We assign a simplicial complex to the situation in the following way.

Definition 3. *$K(M; \mathcal{T})$ is a simplicial complex whose i-faces correspond to a placement of $i+1$ polyomino tiles from $\mathcal{T}$ onto M without overlapping.*

Definition of $K(M; \mathcal{T})$ implies the next property.

Proposition 1. $\dim(K(M; \mathcal{T})) + 1$ *is maximal number of polyomino tiles from $\mathcal{T}$ that may be placed on M without overlapping.*

Geometrical features of simplicial complexes $K(M; \mathcal{T})$ can be visualized only in a handful of cases, such as the following example.

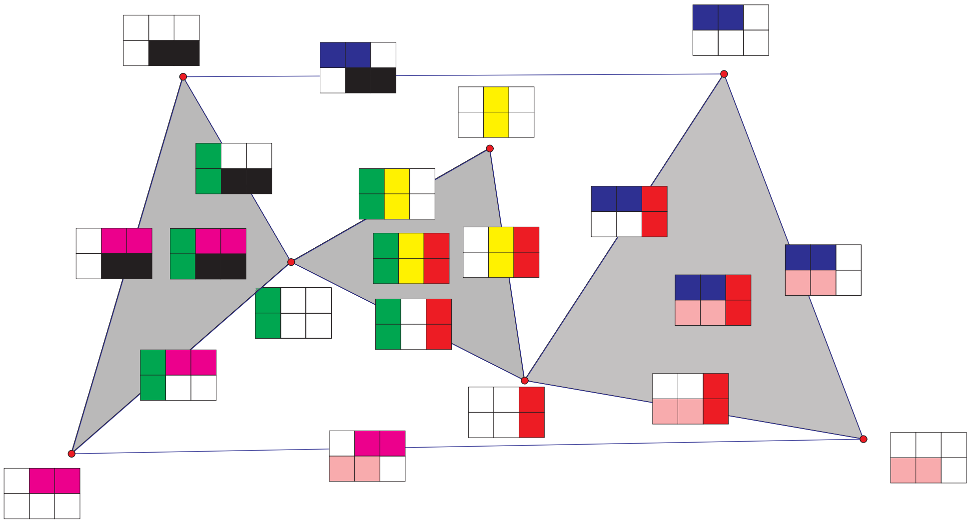

Figure 1: A geometrical realization of $K(M, \mathcal{T})$ when M is 2×3 plane board and $\mathcal{T}$ consists of dominoes.

Example 1. *If M is 2×3 plane board and $\mathcal{T}$ contains only dominoes, then a geometrical realization of the corresponding simplicial complex is presented in Figure 1.*

In the next section, we establish two results about balanced simplicial complexes associated with polyomino tilings.

3 Balanced simplicial complex associated with polyomino tilings

Let $\mathcal{T}$ consist of only I p-omino tile. Recall that I p-omino is $1 \times p$ tile. Let $K(p; m, n)$ and $T(p; m, n)$ be the simplicial complexes of polyomino tilings for planar $m \times n$ board and torus $m \times n$ board, respectively.
I-omino tiles are sensitive to diagonal colorings of square grids. Indeed, if we label the cells by the numbers $1, 2, \ldots, p$ as in Figure 2, I p-omino will always cover p different numbers, regardless of its placement. Therefore, we cannot put more than d_i tiles on the $m \times n$ board, where d_i is the number of cells labeled with i in Figure 2.

1	2	3	4	5	6	7	8	9	$\cdots$	p	1	2	3	4	5	$\cdots$
p	1	2	3	4	5	6	7	8	$\cdots$	$p-1$	p	1	2	3	4	$\cdots$
$p-1$	p	1	2	3	4	5	6	7	$\cdots$	$p-2$	$p-1$	p	1	2	3	$\cdots$
$p-2$	$p-1$	p	1	2	3	4	5	6	$\cdots$	$p-3$	$p-2$	$p-1$	p	1	2	$\cdots$
$p-3$	$p-2$	$p-1$	p	1	2	3	4	5	$\cdots$	$p-4$	$p-3$	$p-2$	$p-1$	p	1	$\cdots$
$p-4$	$p-3$	$p-2$	$p-1$	p	1	2	3	4	$\cdots$	$p-5$	$p-4$	$p-3$	$p-2$	$p-1$	p	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
1	2	3	4	5	6	7	8	9	$\cdots$	p	1	2	3	4	5	$\cdots$
p	1	2	3	4	5	6	7	8	$\cdots$	$p-1$	p	1	2	3	4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 2: Labeling of the cells in $m \times n$ board.

Lemma 1. $\dim K(p; m, n) = d_{a+1} - 1$ *for integers* $m = m_1 p + a$ *and* $n = n_1 p + b$ *such that* $m_1, n_1 \geq 1$ *and* $0 \leq a, b \leq p - 1$.

Proof: If $p \mid m$ or $p \mid n$, the corresponding horizontal or vertical side can be tiled by I p-omino in an obvious way. Therefore, the board can be tiled with $\frac{mn}{p}$ tiles, and each number in Figure 2 appears the same number of times. In this case, the statement is valid.
Let us analyze a challenging situation when $a, b > 0$. We will distinguish two cases: $a + b \leq p - 1$ and $a + b \geq p$.
If $a + b \leq p - 1$ we can conclude that $d_{a+1} = m_1 n_1 p + a n_1 + b m_1$, since there is no square labeled by $a + 1$ in the shaded region in Figure 3. The same figure depicts the positioning of $m_1 n_1 p + a n_1 + b m_1$ I p-ominoes in the board in this case.
Let us now consider the case $a + b \geq p$. We already know that $a + b \leq 2p - 2$.
In this case, from Figure 4, we can find that $d_{a+1} = m_1 n_1 p + a n_1 + b m_1 + a + b - p$ and that there is a tiling covering all squares labeled by $a + 1$. Similarly, as in the previous case, the critical argument is that in the shaded region in Figure 4, there is no $a + 1$ cell. It finishes our proof. ∎

Theorem 1. *Simplicial complex* $K(p; m, n)$, $m, n \geq p$ *is balanced.*

Proof: Consider the cells labeled by $a + 1$ in Figure 2 and denote them by x_1, x_2, $\ldots$, $x_{d_{a+1}}$ as in Figure 5. Each I p-omino tile has to cover exactly one of these cells. Indeed, this induces coloring of the vertices of $K(p; m, n)$ in d_{a+1} colors since two vertices are colored by the same color exactly when they cover the same square denoted by x_i for some i. By definition of $K(p; m, n)$, they do not span an edge. Because of Lemma 1, it follows that $K(p; m, n)$ is balanced.

∎

The following theorem establishes the analogues result for the simplicial complex $T(p; m, n)$ associated with the tiling of torus $m \times n$ board by I p-ominoes.

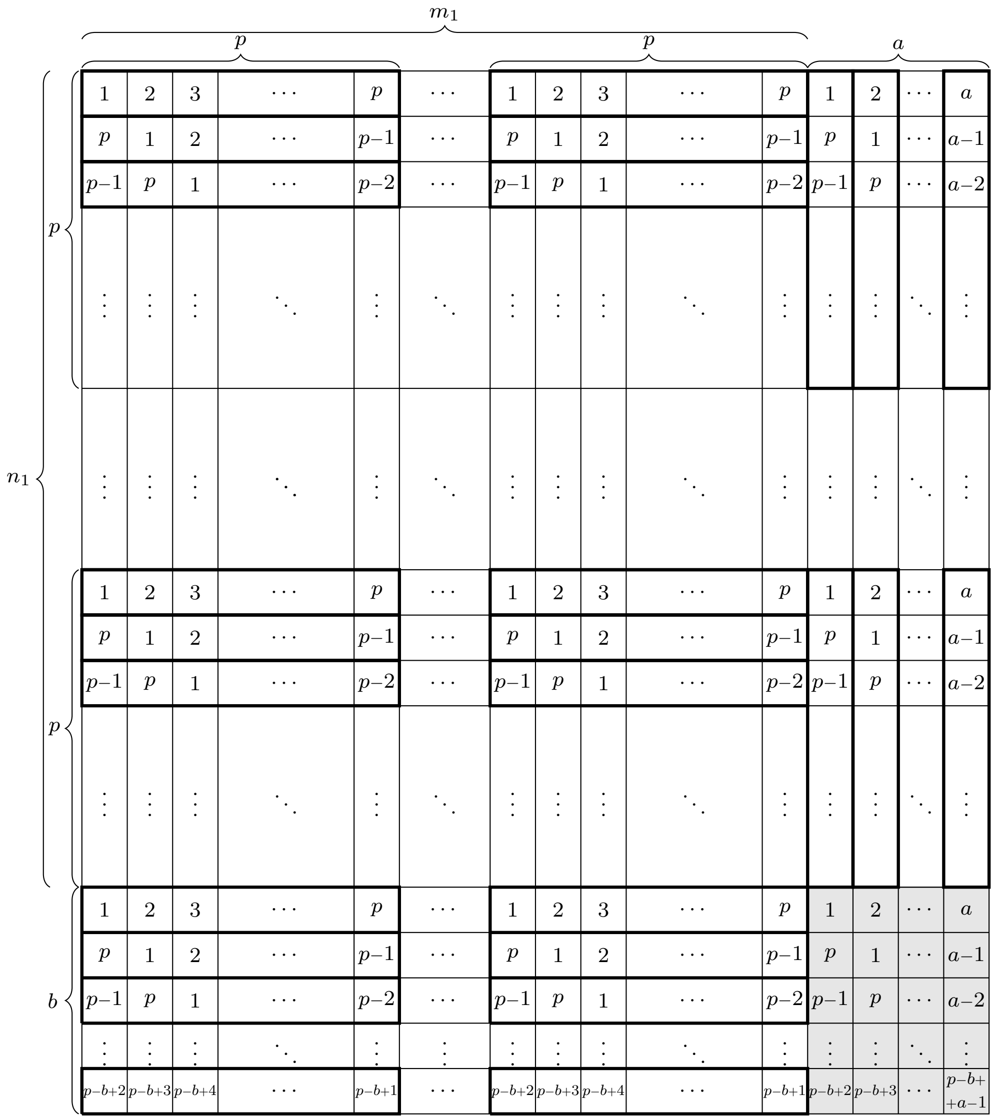

Figure 3: A placement of the maximal number of I p-ominoes in the board $m \times n$ in the case $a + b \leq p - 1$.

Theorem 2. *Simplicial complex $T(p; m, n)$ is a balanced complex if and only if $p \mid m$ and $p \mid n$.*

Proof: We can place at most $\lfloor \frac{mn}{p} \rfloor$ p-omino tiles on any $m \times n$ board so $\dim T(p; m, n) \leq \lfloor \frac{mn}{p} \rfloor - 1$. Observe that we can place mn vertical and mn horizontal I p-ominoes on torus $m \times n$ board, so $T(p; m, n)$ has $2mn$ vertices.
Assume that the vertices of $T(p; m, n)$ are properly colored. Two vertices can be colored in same way if the corresponding I p-omino placements overlap. Moreover, three vertices sharing the color correspond to the placements of tiles that cover a common square thanks to the tile of I-omino, which is convex. Celebrated Helly's theorem [9] implies that all vertices colored by the same color correspond to the tile placements having a joint cell. This implies that there are at most $2p$ vertices of $T(p; m, n)$ having same color.
We will examine the following cases separately.

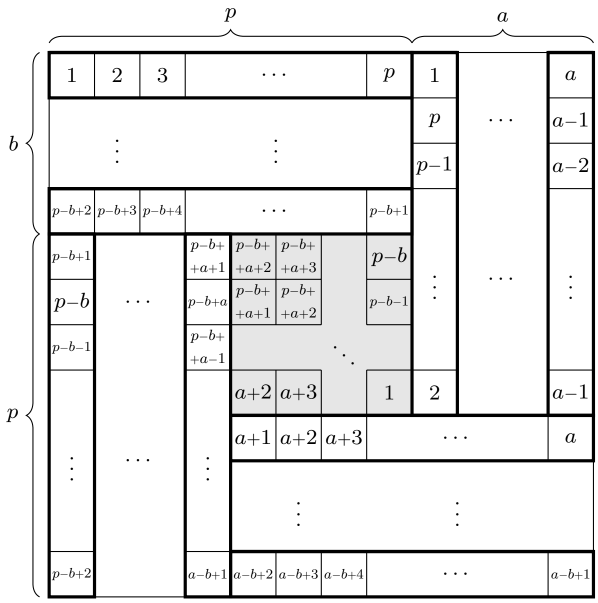

Figure 4: A placement of the maximal number I p-ominoes in the board $m \times n$ in the case $a + b \geq p$.

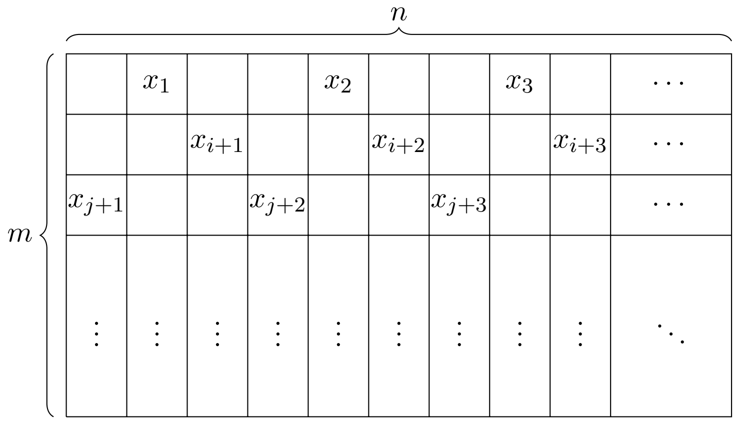

Figure 5: Coloring of vertices in simplicial complex $K(p; m, n)$.

1) Let $p|m$ and $p|n$. We can use the coloring used in the proof of Theorem 1 and verify that the complex is balanced.

2) Without loss of generality let assume that $p \nmid m$ and $p|n$. Obviously, $p|m \cdot n$. Then we used $\frac{m \cdot n}{p}$ colors, and there must be $2p$ vertices of $T(p; m, n)$ of each color. For each of $\frac{m \cdot n}{p}$ colors take a common cell for the corresponding placements of I p–omino tiles. Assume that these $\frac{m \cdot n}{p}$ cells are marked on the torus board.

   By placing one I p-omino tile on the board, we precisely cover one of the marked cells, so the coloring of each vertex is determined by the marked cell and the corresponding tile covers. On the other hand, in a row, we can have at most $\lfloor \frac{m}{p} \rfloor$ marked squares. We can show this in the following way. Between two consecutive marked cells in a row, we have to have precisely $p - 1$ empty cells to achieve that each vertex has its color and that I-omino cannot cover two marked squares. Since we have the torus board, we have at most $k + k(p-1)$ squares in each row, where k is the number of the marked cell in the row. The inequality is now proven.

However, it implies that the total number of marked cells in the board is at most $n \cdot \lfloor \frac{m}{p} \rfloor < n \cdot \frac{m}{p} = \frac{mn}{p}$ and that is the contradiction. Therefore, $T(p; m, n)$ is not balanced in this case.

3) Let $p \nmid m$ and $p \nmid n$. Then on the board we can put at most $\left\lfloor \frac{mn}{p} \right\rfloor - 1$ I p–ominoes, implying that

$$\left\lfloor \frac{mn}{p} \right\rfloor - 1 \geq \dim T(p; m, n).$$

If we used $\left\lfloor \frac{mn}{p} \right\rfloor$ colors for a proper coloring , then we would have at most $2p \cdot \left\lfloor \frac{mn}{p} \right\rfloor < 2p \cdot \frac{mn}{p} = 2mn$ vertices colored. But, this contradicts the fact that $T(p; m, n)$ has $2mn$ colored vertices.

Based on the considered cases, the statement has been proved. ∎

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