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The Birkhoff polytope of the groups F_4 and H_4

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Abstract

We compute the set of facets of the polytope which is the convex hull of the Coxeter groups F_4 or H_4:

- For the group F_4 we found 2 orbits of facets which contradicts previous results published in [19].
- For the group H_4 we found 1063 orbits of facets which provides a counterexample to the conjecture of [19].

1 Introduction

Given a finite group G acting linearly on a real vector space $\mathbb{R}^n$, there is a strong interest in finding the orbits of facets of the convex hull Gx of a vector x.
For a coxeter group G with its natural action on $\mathbb{R}^n$, the structure is well known and given by the Wythoff construction (See [5]). Other representations were considered in [4] for the alternating group. Another very interesting case for a group having a n-dimensional representation is to consider the action of the group on itself. For the symmetric group S_n this gets us the Birkhoff polytope.
In [1, 19] the description was extended to other Coxeter groups by introducing the Birkhoff tensors B_G which are facets of conv G. In [18] the symmetry group of conv G are determined for all the finite Coxeter groups.
The authors proved the following result:

Theorem 1. *For $G = \mathsf{A}_n$, B_n, $\mathsf{I}_2(n)$ and H_3 all facets of* conv G *are Birkhoff tensors.*

They also proved that for D_4 the result does not hold and they claim in Theorem 8.1 that this implies that the result does not hold for D_n and E_n. Note that the authors also claimed that F_4 satisfy the theorem but we prove that this is not true. The authors conjectured [19, Problem 8.1] that for H_4 the result does hold. As it turns out, this is not true since while there is just one orbit of Birkhoff tensors, there is more than one orbit of facets:

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Theorem 2. *The polytope* conv H_4 *has* 188455824000 *facets in* 1063 *orbits.*

The proof of this result is computational with the algorithms presented in Section 3 and the results presented in Section 4.

2 Definitions

2.1 Convex geometry

A set $S \subset \mathbb{R}^n$ is called *convex* if for all $x, y \in S$ we have $[x, y] \subset S$. A convex set is called a *polytope* if there exist vectors $x_1 \dots, x_N$ such that

$$S = \left\{ x \in \mathbb{R}^n \text{ s.t. } x = \sum_{i=1}^N \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1 \right\}$$

and S is called the convex hull $Conv(\{x_1, \dots, x_N\})$ of $x_1, \dots, x_N$. A point $x \in S$ which cannot be expressed as the middle of two points $y_1, y_2 \in S$ is called a *vertex*. A polytope P has finitely many vertices and the vertices $v_1, \dots, v_N$ allow to write $P = Conv(\{v_1, \dots, v_N\})$.
For a polytope P there exist a number of linear forms f_i and constants C_i such that

$$P = \{x \in \mathbb{R}^n \text{ s.t. } f_i(x) \leq C_i \text{ for } 1 \leq i \leq M\}$$

The inequalities f_i which cannot be expressed as the sum of two other such inequalities are called facets. Passing from the description from vertices to the description by facets is called the *dual description problem*. See [21] for an introduction to the subject. This paper is just two special cases of dual descriptions problems being solved.

2.2 Coxeter group

A *finite Coxeter group* G is a finite group of isometries acting on a real space $\mathbb{R}^n$ which is generated by N reflections. A Coxeter group is called irreducible if there exist a non-trivial subspace preserved by G. Such groups were classified by Coxeter himself and we refer to [13] for related definitions and terminology. In dimension 4 there are the A_4, B_4, D_4, F_4, H_4.
Coxeter groups are tightly related to regular polytopes (see [3]). For example the isometry group of the 24-cell is the Coxeter group F_4. Also the isometry group of the 120-cell and 600-cell is the Coxeter group H_4.

3 Algorithms

The effective computation of dual description of polytopes is a classic problem in [11]. We designed software for using symmetries of polytope when computing their dual descriptions. The initial version of the code was written in `GAP` with some parts in `C/C++`. After several extensions, the code was completely ported into `C++`. As a side product of that, we are allowed to select the numeric type of the occurring matrix entries.

3.1 Dual description algorithms

In order to compute the facets of the polytope, we used the recursive adjacency decomposition technique. This method has been used for many different computations and is explained in [2, 6, 10, 20].
The idea of the adjacency decomposition technique is to compute one facet and from this facet to compute the adjacent facets. The obtained facets are then tested for equivalence. Computing the facets adjacent to a given facet is itself a dual description problem, therefore one may need to apply the method recursively hence the name recursive adjacency decomposition method.
Some early termination criterion are given in [6] and allow us to avoid having to compute the adjacencies of all the facets.
For a long time we used the code developed in [8] which is a package of [12]. For several reasons we have developed a new `C++` implementation (see [9]) that allow us to gain additional speed and functionality.

3.2 Fields

The commonly used numerical type is **mpq_class** from the GMP library which is a multiprecision rational type. It is supported in GAP as well as `C++`. In order to compute with polytopes related to the Coxeter groups H_3 and H_4 one needs to allow for the ring $\mathbb{Q}[\sqrt{5}]$. Implementing the arithmetic operations $(+, -, *, /)$ is relatively easy but the sign determinations require more care.
Testing if $a + b\sqrt{5}$ is positive can be done in the following way. If a and b are of the same sign it is easy to conclude. Otherwise, a and b are of opposite sign and we write

$$a + b\sqrt{5} = \frac{a - b\sqrt{5}}{a^2 - 5b^2}$$

The sign of $a - b\sqrt{5}$ can be decided and together with the sign of the rational number $a^2 - 5b^2$ we can conclude. The same strategy allows one to decide the sign in mixed cubic rings. For other real fields of algebraic numbers, different approaches would have to be used.
Also, for some subroutines like `lrs` one needs only to use ring operations. In that case one can reduce to the case of $\mathbb{Z}[\sqrt{5}]$. For the kind of computations we are doing here there is no need for algebraic closures or such kind of constructions.

3.3 Canonicalization strategies

In preceding works, when we had two orbits of facets in order to check isomorphism, we used the [12] implementation of the partition backtrack. See [16, 17] and [15] for accounts of this class of algorithms. That is we encode facets by the subset of their incident vertices and then use the partition backtrack for set equivalence.
However, in [14] an algorithm for finding a canonical representative of a subset for a permutation group action was found. This greatly simplifies the code since for N orbits instead of having to compute N equivalences, we simply have to do one canonicalization and one string comparison.

4 Results

4.1 Results for F_4

For the group F_4 we use the following two generators:

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

There are exactly two orbits of facets for conv(F_4):

1. Orbit 1 of Birkhoff tensor with incidence 288. Stabilizer has order 4608. One representative inequality is $Tr(XA) \le 1$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

2. Orbit 2 of facets with incidence 288. Stabilizer has order 48. One representative inequality is $Tr(XA) \le 1$ with

$$A = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

The total number of facets is 55872.

4.2 Results for H_4

For the group H_4 we use the following two generators:

$$\frac{1}{4}\begin{pmatrix} 1 & -2 & -1 & 0 \\ 2 & 2 & -2 & 2 \\ 1 & -2 & 0 & 1 \\ 0 & -2 & -1 & 1 \end{pmatrix} + \frac{\sqrt{5}}{4}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We found 1063 orbits of facets of conv H_4. If we express such facets in the form $Tr(AX) \le 1$ then one 1, 4, 130, 928 orbits of facets of rank 1, 2, 3 and 4.
Tables 1 and 2 give the statistics about the incidence of orbits and about the size of their stabilizers. The full list of orbits is presented in [7]. See below one matrix of incidence 120 with a stabilizer of size 120. This suffices to show that the conjecture is false.

$$A = \frac{1}{4}\begin{pmatrix} -6 & -11 & -7 & 4 \\ 0 & -2 & 3 & 1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{\sqrt{5}}{4}\begin{pmatrix} 2 & 5 & 3 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

p	Nr.	p	Nr.	p	Nr.	p	Nr.
16	376	17	282	18	116	19	85
20	48	21	30	22	12	23	5
24	44	25	3	26	31	28	10
30	7	32	5	36	4	38	1
48	1	100	1	120	1	480	1

Table 1: For each incidence p the number of orbits of incidence p is given.

s	Nr.	s	Nr.	s	Nr.	s	Nr.
1	800	2	189	4	50	6	1
8	8	12	4	16	3	24	1
36	1	40	1	48	2	120	1
576	1	2880	1				

Table 2: For each size s the number of orbits of orbits having a stabilizer of size s is given.

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