Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.04

Grids of equidistant walks

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Abstract

We look at arithmetical progressions as equidistant walks, the difference being the steps' length. One can construct equidistant grids by putting equidistant walks to rows and columns. We introduce a notion of diagonal walks across grids. One specific diagonal walk is interesting because the sum of its elements gives polygonal numbers. We give one specific construction method for obtaining equidistant grids. It assumes alternating two equidistant walks placed in the first two columns, shifted up by the same shift whenever some of them repeat. We find conditions on the steps' lengths and the shift size that ensure that grids obtained in this way are equidistant. Also, we define a zig-zag pattern walks over a two-column grid. It is possible to identify them with horizontal walks in an infinite grid made of given columns by the mentioned construction method. Finally, we form equidistant walks of differences of consecutive products with overlapping odd and even factors and show that the distance between them is constant.

Keywords: *arithmetical progression, equidistant walks, equidistant grids, polygonal numbers, products with overlapping factors*

1 Introduction and preliminaries

An arithmetical progression or an arithmetical sequence is a sequence of numbers such that the difference between consecutive numbers is constant. If a_1 is the initial term and d is the common difference of consecutive numbers, then the n-th term a_n of the progression is given by: $a_n = a_1 + (n-1)d$. In general $a_n = a_m + (n-m)d$ for natural numbers n and m, $n > m$. We restrict here to sequences of positive integers, so $a_1, d \in \mathbb{N}^+$.

We consider arithmetical progressions as **equidistant walks** to infinity with a common difference as "the step length." Indeed, starting from the first term a_1 one obtains the next term from the previous by taking the step of the length d:

$$a_1, a_1 + d, (a_1 + d) + d, \ldots$$

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The arithmetical progression of odd numbers: $1, 3, 5, 7, 9, \ldots$ is an equidistant walk with the steps' length equal to 2, the same as the arithmetical progression of even numbers: $2, 4, 6, 8, 10, \ldots$. The only difference between those two equidistant walks is in the initial term.

Equidistant walks serve as building blocks for grids. To form a grid, we put one equidistant walk that starts with a_1 and has the steps' length d_c to the first column, and the other starting again with a_1 with the step length d_r to the first row. Then we add equidistant walks with step lengths equal to d_c (or d_r) to columns (or rows). So, equidistant grids are given by the first term a_1 and the pair of differences (d_r, d_c). There are different kinds of number grids in literature. For example, the most well-known is the Pascal triangle based on the usual Euclidean square grid

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \vdots & \vdots & \vdots & \ddots & . \end{array}$$

and the hyperbolic Pascal triangles based on the hyperbolic square grids, see [1]. Here we consider the Euclidean grid and new number grids: equidistant grids with equidistant walks in rows and columns.

Example 1. *Having $a_1 = 1$ and $d_c = d_r = 2$, we obtain the next grid of odd numbers*

$$\begin{array}{ccccc} 1 & 3 & 5 & 7 & \cdots \\ 3 & 5 & 7 & 9 & \cdots \\ 5 & 7 & 9 & 11 & \cdots \\ 7 & 9 & 11 & 13 & \cdots \\ \vdots & \vdots & \vdots & \ddots & . \end{array} \tag{1}$$

Notice also the other possible construction of this grid: besides odd numbers in the first column, we add new columns by shifting the walk from the previous column up by one and omitting the first number.

Here, we define walks over equidistant grids and a grid characteristic. We will show that equidistant grids are means to generate polygonal numbers. There are known problems of finding arithmetical progressions in polygonal numbers tackled in [2] and [3]. Polygonal numbers may not contain long arithmetical sequences. However, they constitute them. It is not new that polygonal numbers are sums of numbers that form an arithmetical sequence. However, equidistant grids provide more than one way to do this and give all possible ways. More precisely, we shall prove that s-gonal numbers are sums of elements of a diagonal walk in the grid of the characteristic s. Adding numbers in different directions of a number grid is an old idea. For example, it is known that the shallow diagonal sums of Pascal's triangle give Fibonacci numbers:

Some equidistant grids, like the (1) from above, are obtained by a specific construction method. It is the method of alternating columns of two equidistant walks of

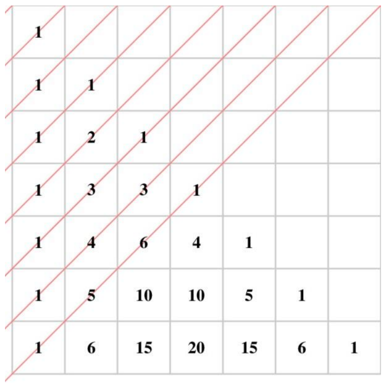

Figure 1: Sums in a Pascal triangle

the same steps' length, with each column shifted up by some fixed shift k in every new appearance. Some conditions on d_c, d_r, and k ensure the grid is equidistant.
Define an (R, D)**-walk** across the grid: for a pair of integers R and D, R gives a move to the right and D a move down that leads from one number of the walk to another. Here the negative value of R indicates a move to the left by $|R|$, and the negative value of D is a move up by $|D|$. Figure 1 illustrates directions of (R, D)-walks starting from the position A for different signs of R and D.

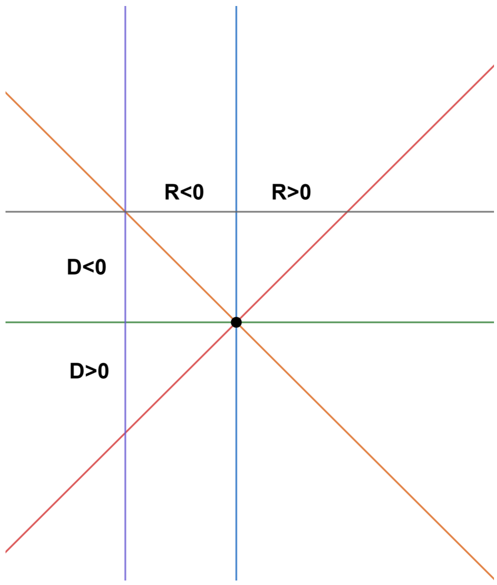

Figure 2: Directions of (R,D)-walks

Walks may start at any point in a grid. Observe that $(R, 0)$-walks are horizontal walks and $(0, D)$-walks are vertical walks. We call all other (R, D)-walks diagonal, not just the walk in the direction of the main diagonal for which $R = D$. Negative R and/or D results in a finite (R, D)-walk. Among infinite walks, we distinguish diagonal walks of the steep incline down for which $R < D$ and diagonal walks of the slight incline down if opposite.
Observe that each walk across the grid is an equidistant walk. We call such grid an **equidistant grid**. Being equidistant does not mean all walks through a grid

have the same step length. In the grid (1) any horizontal $(R,0)$-walk has the steps' length $2R$, for vertical $(0,D)$-walk it is $2D$, while a diagonal (R,D)-walk has the steps' length $2(R+D)$.
Notice that in a grid of odd numbers (1) any (R,D)-walk that starts with some number a_1 can be identified simply to an $(0,R+D)$-walk over the first column of the grid, i. e. a walk in which, starting from a_1, one steps on every $(R+D)$th number of the walk.
Observe that all the data necessary for the construction of the grid, (1) in particular, or any other equidistant grid in general, are given in the upper-left triangle of the grid, namely

$$\begin{array}{cc} 1 & 3 \\ 3 & \end{array} .$$

To an equidistant grid given by a_1, d_c and d_r **a grid characteristic** s is $s = 2a_1 + d_c + d_r$. This is the sum of elements $a_1 + d_c, a_1 + d_r$ of the first non-trivial $(1,-1)$-walk in an equidistant grid

$$\begin{array}{cc} a_1 & a_1 + d_r \\ a_1 + d_c & \end{array} .$$

Example 2. *Next grids have grid characteristics* 13 *and* 8*, respectively*

$$\begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 6 & 12 & 18 & 24 & \cdots \\ 11 & 17 & 23 & 29 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}, \quad \begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 1 & 7 & 13 & 19 & \cdots \\ 1 & 7 & 13 & 19 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} .$$

Notice that above, sums of elements of $(1,-1)$*-walks give tridecagonal and octagonal numbers, respectively:*

$$\begin{array}{rcl} 1 & = & 1 \\ 13 & = & 6+7 \\ 36 & = & 11+12+13 \\ \vdots & & \end{array}, \quad \begin{array}{rcl} 1 & = & 1 \\ 8 & = & 1+7 \\ 21 & = & 1+7+13 \\ \vdots & & \end{array} .$$

It happens by no chance: as we shall see, an equidistant grid with $a_1 = 1$ *and a characteristic* $s = 2 + d_c + d_r$ *generates* s*-gonal numbers as sums of members of* $(1,-1)$*-walks. In addition, observe that the first grid gives a non-standard decomposition of tridecagonal numbers; the standard one that describes the geometrical formation of* 13*-gonal numbers is:*

$$\begin{array}{rcl} 1 & = & 1 \\ 13 & = & 1+12 \\ 36 & = & 1+12+23 \\ \vdots & & \end{array} .$$

2 Equidistant grids and polygonal numbers

Polygonal numbers are numbers that, represented by dots, form a usual polygonal pattern. So, triangular numbers $1, 3, 6, \ldots$ form triangles, square numbers $1, 4, 9, \ldots$ form squares, and so.
Let s denote the number of polygon sides. There is a formula for the nth s-gonal number [2]

$$P_n{}^{(s)} = \frac{n}{2}\Big(2 + (n-1)(s-2)\Big).$$

Notice that the list of s-gonal numbers starts with 1 and then proceeds with s, since the second s-gonal number equals exactly s.

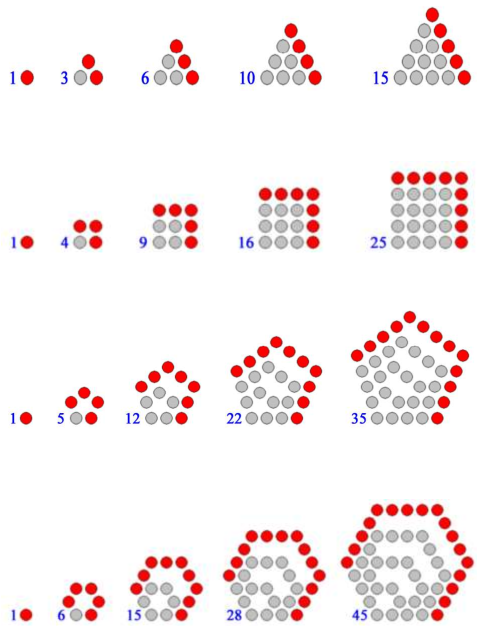

Figure 3: s-gonal numbers for $s = 3, 4, 5, 6$ (retrieved from Wikipedia.)

There are natural decompositions of nth s-gonal number to a sum of n members of an arithmetical sequence. These decompositions testify geometrical formation of s-gonal numbers. Namely, for triangular numbers, the list starts with 1, proceeds with $3 = 1 + 2$ because to one dot we must add two more dots to get the smallest non-trivial triangle, continues with $6 = 1 + 2 + 3$ because one has to add three more dots to get the third triangular number, and so on. In general, one enlarges the sides of an s-gon by the next member of an arithmetical sequence:

- for $s = 3$ addends are $\ldots(((1 + 2) + 3) + 4)\ldots$
- for $s = 4$ one has $\ldots(((1 + 3) + 5) + 7)\ldots$
- for $s = 5$, $\ldots(((1 + 4) + 7) + 10)\ldots$

These decompositions are justified geometrically: one can construct the $(n + 1)$st s-gonal number from the nth s-gonal number. Indeed, there is an initial point from which the s-gons begin to grow. Each of the two sides that meet in the initial point contains n points, so one adds $n-2$ points to each of the other $s-2$ sides to form the next s-gon. Since $(n-2)(s-2) = (n-1)(s-2) + (s-2)$ we see that the new s-gonal formation originates from the old one enlarged by $s-2$. Therefore $s-2$ is the difference of an arithmetical sequence whose consecutive sums give s-gonal numbers.

Besides the natural decomposition of s-gonal numbers as sums of members of equidistant walks, we can read others in the equidistant grids of a characteristic s. The total number of such decompositions is $\left\lfloor \frac{s}{2} \right\rfloor$. The following theorem holds.

Theorem 1. *For a given integer s, $s \geq 2$ there are exactly $\left\lfloor \frac{s}{2} \right\rfloor$ decompositions of s-gonal numbers as sums of members of $(1,-1)$-walks in equidistant grids of characteristic s.*

Proof. We claim that in every equidistant grid of the characteristics s sums of elements of $(1,-1)$-walks generate s-gonal numbers. Let $p \in \{1, 2, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\}$. Look at the equidistant grid of characteristics s:

$$\begin{array}{ll} 1 & s-p \\ p & \end{array} .$$

An equidistant walk in the first column starts with $a_1 = 1$ and has a step length equal to $p-1$. The one in the first row starts with $b_1 = 1$; its step length is $s-p-1$. So, $a_n = 1 + (n-1)(p-1)$ and $b_n = 1 + (n-1)(s-p-1)$. The sum of elements of a $(1,-1)$-walk is

$$s_n = \frac{n}{2}(a_n + b_n) = \frac{n}{2}\Big(2 + (n-1)(s-2)\Big)$$

and this is exactly the nth s-gonal number. □

Corollary 1. *For s even, every nth s-gonal number can be represented as the sum of n equal numbers.*

Proof. The decomposition follows from the equidistant grid of the characteristics s:

$$\begin{array}{ll} 1 & \frac{s}{2} \\ \frac{s}{2} & \end{array} .$$

□

Remark 1. Observe that s-gonal numbers are partial sums of elements of $(1,1)$-walk on the main diagonal of an equidistant grid of characteristic s. That is so because differences of consecutive s-gonal numbers form an equidistant walk with the step length $s-2$. Every $(1,1)$-walk has this step length, particularly the main diagonal walk. Thus, in the grid (2) from Example 3, nonagonal numbers are sums of elements of an $(1,-1)$-walk, but also partial sums of elements of an $(1,1)$-walk. Generally, the main diagonal is an invariant of all $\left\lfloor \frac{s}{2} \right\rfloor$ different equidistant grids of characteristic s. Indeed, for $p \in \{1, 2, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\}$ one has

$$\begin{array}{ll} 1 & s-p \\ p & s-1 \end{array} .$$

(The element $s-1$ in this grid is: $p + (s-p-1)$, in the horizontal direction, or $s-p+(p-1)$, in the vertical direction.) So, regardless of p, on the main diagonal is the walk $1, s-1, 2s-3, \ldots$ whose nth partial sum equals the nth s-gonal number.

3 Equidistant grids made by alternating and shifting columns

Example 3. *Look at the next equidistant grid in which* $a_1 = 1, d_r = 5$ *and* $d_c = 2$

$$\begin{array}{ccccc} 1 & 6 & 11 & 16 & \cdots \\ 3 & 8 & 13 & 18 & \cdots \\ 5 & 10 & 15 & 20 & \cdots \\ 7 & 12 & 17 & 22 & \cdots \\ 9 & 14 & 19 & 23 & \cdots \\ 11 & 16 & 21 & 26 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}. \tag{2}$$

The grid (2) is of a specific construction interest. Namely, we can start its construction with two equidistant walks having the same length of the step ($d_c = 2$) *given in the first two columns. We add additional columns by alternating existing columns and shifting them up by five in each new appearance. Indeed, the shift by five, and only by five, of each column ensures that all horizontal walks are equidistant. (Five is the difference of the first elements in given columns or* d_r*).*

We are interested in just-described equidistant grid construction. It starts by putting two walks of the same step length to the first two columns and then alternating columns shifted up by some fixed shift k. The grid (2) shows that not every shift size will result in an equidistant grid. The question is: how can one choose k to get it?

Assume there are equidistant walks with initial terms a_1 and $a_1 + d_r$ and the steps' lengths equal to d_c in the first two columns of a grid. Then add new columns in a way to alternate given walks shifted up by some k, $k > 0$ in each new appearance to obtain the grid

$$\begin{array}{ccccc} a_1 & a_1 + d_r & a_1 + kd_c & a_1 + d_r + kd_c & \cdots \\ a_1 + d_c & a_1 + d_r + d_c & a_1 + (k+1)d_c & a_1 + d_r + (k+1)d_c & \cdots \\ a_1 + 2d_c & a_1 + d_r + 2d_c & a_1 + (k+2)d_c & a_1 + d_r + (k+2)d_c & \cdots \\ a_1 + 3d_c & a_1 + d_r + 3d_c & a_1 + (k+3)d_c & a_1 + d_r + (k+3)d_c & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}.$$

Since d_r gives the steps' length of equidistant walks in rows, we have the requirement $a_1 + kd_c - (a_1 + d_r) = d_r$. That implies $d_c k = 2d_r$.

The condition $kd_c = 2d_r$ relates the shift k and the steps' lengths d_c and d_r of equidistant walks in columns and rows, respectively. Notice that for $d_c = 2$, the condition implies $k = d_r$ and therefore justifies that for $d_c = 2$ the only shift that gives an equidistant grid is the shift by d_r, the fact we already noticed in the grid (2).

Given d_r, each ordered pair (d_c, k) that satisfies $d_c k = 2d_r$ gives one equidistant grid. So, the number of all such grids equals the number of ordered pairs (d_c, k) such that $d_c k = 2d_r$. To investigate the total number of equidistant grids, we consider separate cases of prime and composite d_r.

In case of a prime d_r, $d_r \neq 2$ the condition $d_c k = 2d_r$ gives four possible choices of d_c and k: $d_c = 1, k = 2d_r$; $d_c = 2d_r, k = 1$; $d_c = 2, k = d_r$ and $d_c = d_r, k = 2$. For the first two, we have the following grids:

$$\begin{array}{cccc} a_1 & a_1 + d_r & a_1 + 2d_r & \cdots \\ a_1 + 1 & a_1 + d_r + 1 & a_1 + 2d_r + 1 & \cdots \\ a_1 + 2 & a_1 + d_r + 2 & a_1 + 2d_r + 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

and

$$\begin{array}{cccc} a_1 & a_1 + d_r & a_1 + 2d_r & \cdots \\ a_1 + 2d_r & a_1 + 3d_r & a_1 + 4d_r & \cdots \\ a_1 + 4d_r & a_1 + 5d_r & a_1 + 6d_r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} .$$

The case $d_c = 2, k = d_r$ gives already treated equidistant grids

$$\begin{array}{cccc} a_1 & a_1 + d_r & a_1 + 2d_r & \cdots \\ a_1 + 2 & a_1 + d_r + 2 & a_1 + 2d_r + 2 & \cdots \\ a_1 + 4 & a_1 + d_r + 4 & a_1 + 2d_r + 4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} .$$

Finally, the case $d_c = d_r, k = 2$ gives the equidistant grid of a specific (symmetric) layout

$$\begin{array}{cccc} a_1 & a_1 + d_r & a_1 + 2d_r & \cdots \\ a_1 + d_r & a_1 + 2d_r & a_1 + 3d_r & \cdots \\ a_1 + 2d_r & a_1 + 3d_r & a_1 + 4d_r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} .$$

If d_r is a composite number, besides the preceding four, there are other choices of d_c and k such that $d_c k = 2d_r$. We distinguish two cases: the case of odd and even d_r.
If d_r is odd, we can factorize it as $d_r = p_1^{n_1} p_2^{n_2} \ldots$. Here $p_1, p_2, \ldots$ are different (odd) prime numbers and $n_1, n_2, \ldots$ their respective multiplicities. The number $D(d_r)$ of all divisors of d_r equals $D(d_r) = (n_1 + 1)(n_2 + 1) \ldots$. Since $2d_r = 2p_1^{n_1} p_2^{n_2} \ldots$, the number of divisors of $2d_r$ is $D(2d_r) = 2D(d_r)$. That is exactly the number of all ordered pairs (d_c, k) such that $d_c k = 2d_r$ and also the number of all possible equidistant grids. Indeed, except for $d_r = 2$, the number of divisors of $2d_r$ is even. We can arrange them to an increasing sequence: $p_1, p_2, \ldots, p_{l-1}, p_l$ with $l = 2D(d_r)$, $p_1 = 1$ and $p_l = 2d_r$. Then $(p_1, p_l), (p_2, p_{l-1}), \ldots$ and $(p_l, p_1), (p_{l-1}, p_2), \ldots$ give all possible pairs of lengths of the step d_c of equidistant walks in columns and shifts k of columns that give equidistant grids.
For $d_r = 2$ there are three divisors of $2d_r = 4$, namely $1, 2$ and 4 as well as three possible ordered pairs (d_c, k): $(1, 4), (4, 1), (2, 2)$.
Even d_r has a prime factor decomposition $d_r = 2^{n_1} p_2^{n_2} p_3^{n_3} \ldots$, with different prime

factors $p_j \neq 2$. Then $2s = 2^{n_1+1} p_2^{n_2} p_3^{n_3} \ldots$. Consequently,

$$\begin{aligned} D(2d_r) &= (n_1+2)(n_2+1)(n_3+1)\cdots \\ &= (n_1+1)(n_2+1) + \cdots + (n_2+1)(n_3+1)\cdots \\ &= D(d_r) + D\left(\frac{d_r}{2^{n_1}}\right). \end{aligned}$$

Similarly as in the case of odd d_r, the number of all possible equidistant grids for even d_r is $D(d_r)+D\left(\frac{d_r}{2^{n_1}}\right)$. To summarize: up to the choice of a_1 and d_r, the number of equidistant grids equals the number of all divisors of $2d_r$. The sequence $D(n)$ is in the On-Line Encyclopedia of Integer Sequences [4] with the label A000005.

Example 4. *Let us look at some concrete examples of equidistant grids.*
Case $d_r = 5$
There are four divisors of $2d_r = 10$*:* $1, 2, 5, 10$*. There are also four possible choices for* (d_c, k)*:* $(1, 10), (10, 1), (2, 5)$ *and* $(5, 2)$*. If we take* $a_1 = 1$*, there are four equidistant grids*

$$\begin{array}{c|c} \begin{array}{ccccc} 1 & 6 & 11 & 16 & \cdots \\ 2 & 7 & 12 & 17 & \cdots \\ 3 & 8 & 13 & 18 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} & \begin{array}{ccccc} 1 & 6 & 11 & 16 & \cdots \\ 11 & 16 & 21 & 26 & \cdots \\ 21 & 26 & 31 & 36 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \\ \hline \begin{array}{ccccc} 1 & 6 & 11 & 16 & \cdots \\ 3 & 8 & 13 & 18 & \cdots \\ 5 & 10 & 15 & 20 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} & \begin{array}{ccccc} 1 & 6 & 11 & 16 & \cdots \\ 6 & 11 & 16 & 21 & \cdots \\ 11 & 16 & 21 & 26 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \end{array}$$

In given grids, we look at characteristics s *to see which* s*-gonal numbers they generate. We list here* s*-gonal numbers from the above grids together with their OEIS labels: octagonal numbers* ($s = 8$, *A000567*)*, heptadecagonal numbers* ($s = 17$, *A051869*)*, nonagonal numbers* ($s = 9$, *A001106*) *and dodecagonal numbers* ($s = 12$, *A051624*)*.*
Case $d_r = 6$
There are six divisors of $2d_r = 12$*:* $1, 2, 3, 4, 6, 12$ *and six possible ordered pairs* (d_c, k)*:* $(1, 12), (12, 1), (2, 6), (6, 2), (3, 4)$ *and* $(4, 3)$*. For* $a_1 = 1$*, we have the following equidistant grids*

$$\begin{array}{c|c|c} \begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 2 & 8 & 14 & 20 & \cdots \\ 3 & 9 & 15 & 21 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} & \begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 13 & 19 & 25 & 28 & \cdots \\ 25 & 31 & 37 & 40 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} & \begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 3 & 9 & 15 & 21 & \cdots \\ 5 & 11 & 17 & 23 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \\ \hline \begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 7 & 13 & 19 & 25 & \cdots \\ 13 & 19 & 25 & 31 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} & \begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 4 & 10 & 16 & 22 & \cdots \\ 7 & 13 & 19 & 25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} & \begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 5 & 11 & 17 & 23 & \cdots \\ 9 & 15 & 21 & 27 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \end{array}$$

Given grids again generate s-gonal numbers: nonagonal ($s = 9$, A001106), icosagonal ($s = 20$, A051872), decagonal ($s = 10$, A001107), tetradecagonal ($s = 14$, A051866), hendecagonal ($s = 11$, A051682) and dodecagonal ($s = 12$, A051624).

3.1 Zig-zag equidistant walks and bi-equidistant walks

Now we introduce zig-zag walks and investigate conditions under which they are equidistant. Zig-zag walks involve two equidistant walks of the same step length d_c placed in a two-column grid (the members of the walk are bolded)

$$\begin{array}{cc}
\mathbf{a_1} & a_1' \\
a_1 + d_c & a_1' + d_c \\
\vdots & \vdots \\
a_1 + rd_c & \mathbf{a_1' + rd_c} \\
\vdots & \vdots \\
\mathbf{a_1 + (r+l)d_c} & a_1' + (r+l)d_c \\
\vdots & \vdots \\
a_1 + (2r+l)d_c & \mathbf{a_1' + (2r+l)d_c} \\
\vdots & \vdots
\end{array}.$$

The zig-zag walk starts with a_1, the first term of an equidistant walk from the first column, continues to $a_1' + rd_c$ from the second column r places lower than its first term, then proceeds to the first column $r + l$ places below the first term, and so on. So, in each right turn zig-zag walk steps to the element of the second column r places lower than the present one, and in each left turn to the one of the first column l places below the present one. Look at conditions that ensure that a zig-zag walk is equidistant. Let $d_r = a_1' - a_1$ and assume $d_r > 0$. From

$$a_1' + rd_c - a_1 = a_1 + (r+l)d_c - (a_1' + rd_c)$$

it follows that the zig-zag walk is equidistant if and only if $2d_r = (l - r)d_c$. The condition $d_r > 0$ together with $d_c > 0$ (as assumed so far) forces $r < l$. Further, $l - r = \frac{2d_r}{d_c}$ provides the condition that the steps' length d_c must be a divisor of $2d_r$. There is a finite number of possible step lengths, but an infinite number of choices of l and r satisfying $r < l$.

Zig-zag walks can become horizontal walks through an infinite grid of given walks. Like above, we can construct an equidistant grid by alternating shifted column walks. Namely, the one from the first column stays as it is, and we shift the second up by r. After this, we alternate columns shifted up by $l+r$ in each new appearance. The original zig-zag walk now appears in the first row

$$a_1 \quad a_1' + rd_c \quad a_1 + (r+l)d_c \quad a_1' + (2r+l)d_c \quad a_1 + (2r+2l)d_c \quad \cdots$$

Example 5. *In a special case when $a_1 = a_1'$ (or $d_r = 0$) and $d_c = 1$, the previous horizontal walk turns into a walk we call a* **bi-equidistant walk**

$$a_1 \quad a_1 + r \quad a_1 + r + l \quad a_1 + 2r + l \quad a_1 + 2r + 2l \quad \cdots$$

It starts with a_1 with the first step length r and the second step length l and continues to infinity by alternating those two steps. There is also a bi-equidistant walk

$$a_1 \quad a_1 + l \quad a_1 + l + r \quad a_1 + 2l + r \quad a_1 + 2l + 2r \quad \cdots$$

that starts with a_1, and then alternates steps of lengths l and r to infinity. Notice that odd members of given two bi-equidistant walks coincide, while the distance between even members is constant and equals $|r - l|$.

3.2 Equidistant walks over progressions of products with overlapping odd and even factors

The sum and the difference of corresponding members of two arithmetical progressions is again an arithmetical progression. The same does not hold for a Hadamard product of two arithmetical progressions unless one is constant. For given n let us look at equidistant walks over odd and even numbers

$$\begin{array}{cc} 2n-1 & 2n \\ 2n+1 & 2n+2 \\ 2n+3 & 2n+4 \\ \vdots & \vdots \end{array}.$$

By a progression of **products with overlapping factors** of odd numbers we mean the sequence $(2n-1)(2n+1), (2n+1)(2n+3), (2n+3)(2n+5), \ldots$.
Let us further observe the progressions of products with overlapping factors starting with the odd number $(2n-1)(2n+1)$ and even number $2n(2n+2)$, respectively

$$\begin{array}{cc} (2n-1)(2n+1) & 2n(2n+2) \\ (2n+1)(2n+3) & (2n+2)(2n+4) \\ (2n+3)(2n+5) & (2n+4)(2n+6) \\ \vdots & \vdots \end{array}.$$

Since differences of consecutive members are not constant, none of the progressions is arithmetical. However, differences in products of consecutive elements do form arithmetical progressions. Indeed, look at progressions of differences of products of consecutive numbers

$$\begin{array}{cc} (2n+1)(2n+3) - (2n-1)(2n+1) & (2n+2)(2n+4) - 2n(2n+2) \\ (2n+3)(2n+5) - (2n+1)(2n+3) & (2n+4)(2n+6) - (2n+2)(2n+4) \\ \vdots & \vdots \end{array}.$$

They are both equidistant walks with a step length of 8

$$\begin{array}{cc} 8n+4 & 8n+8 \\ 8n+12 & 8n+16 \\ 8n+20 & 8n+24 \\ \vdots & \vdots \end{array}.$$

Further, the second walk is ahead of the first walk by 4 all the time. Namely, the difference of the corresponding numbers in sequences is constant and equals 4

$$\Big((2n+2)(2n+4)-2n(2n+2)\Big)-\Big((2n+1)(2n+3)-(2n-1)(2n+1)\Big)=4.$$

If we further shift the progression of even numbers up by one

$$\begin{array}{cc} & 2n \\ 2n-1 & 2n+2 \\ 2n+1 & 2n+4 \\ 2n+3 & 2n+6 \\ \vdots & \vdots \end{array}.$$

we may look at the progressions of respective products with overlapping factors as before

$$\begin{array}{cc} (2n-1)(2n+1) & (2n+2)(2n+4) \\ (2n+1)(2n+3) & (2n+4)(2n+6) \\ \vdots & \vdots \end{array}.$$

Distance between the corresponding differences is again constant; now it equals $12 = 2 \cdot 3!$.

$$\begin{aligned}&\Big((2n+4)(2n+6)-(2n+2)(2n+4)\Big)-\\ &-\Big((2n+1)(2n+3)-(2n-1)(2n+1)\Big)=12.\end{aligned}$$

In general, we may fix a natural number n and the progression of products with overlapping odd factors. Further, we shift the progression of products with overlapping even factors by k. Here $k > 0$ and $k < 0$ denote the shift up or down by $|k|$, respectively. Because $k = 1 - n$ is the maximal possible shift down, there is a condition $k \geq 1 - n$. After we shift the second sequence by k, we get

$$\begin{array}{cc} \vdots & \vdots \\ (2n-1)(2n+1) & (2n+2k)(2n+2(k+1)) \\ (2n+1)(2n+3) & \Big(2n+2(k+1)\Big)\Big(2n+2(k+2)\Big) \\ \vdots & \vdots \end{array}.$$

The following holds for the distance of differences between two consecutive products of numbers in sequences.

Proposition 1. *For a natural number n and an integer $k \geq 1 - n$,*

$$\begin{aligned}&\Bigg(\Big(2n+2(k+1)\Big)\Big(2n+2(k+2)\Big)\Bigg)-(2n+2k)\Big(2n+2(k+1)\Big)\Bigg)-\\ &\qquad -\Bigg((2n+1)(2n+3)-(2n-1)(2n+1)\Bigg)=8k+4.\end{aligned}$$

Remark 2. Products of overlapping factors can include more than two numbers that overlap in more than one factor. So we can speak of products of n consecutive odd or even numbers that overlap in m factors, $m < n$, i.e., of **m-overlapping n-products**. In this notation, the above overlapping products are 1-overlapping 2-products. As above, the distance between differences of two consecutive m-overlapping n-products of even numbers and odd numbers is constant.