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On divisibility properties of some binomial sums connected with the Catalan and Fibonacci numbers

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Abstract

We show that an alternating binomial sum which is connected with the Catalan numbers is divisible by n. A natural generalization of this sum is connected with the generalized Catalan numbers and also divisible by n. A new class of binomial sum is used. In Appendix A, we consider a positive binomial sum connected with Fibonacci and Lucas numbers. In Appendix B, we consider an alternating binomial sum which is also connected with Catalan numbers and divisible by $(a+1)n+1$. Similar reasoning was already used by the author to reprove more simply Calkin's result for divisibility of the alternating sum of powers of binomials coefficients by the central binomial coefficient.

Key words: Catalan number, generalized Catalan number, Fibonacci number, Lucas number, M sum, alternating binomial sum.

AMS subject classification (2000): 11B37, 05C15

1 Introduction

Let us consider the following alternating binomial sum:

$$S_1(n,m) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{2n+k}{2n+1}; \tag{1}$$

where n and m are natural numbers.
Let x and y be non-negative numbers. It is well-known [5, Eq. (10.15), p. 47] that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{y} = (-1)^n \binom{x}{y-n}. \tag{2}$$

The Vandermonde convolution formula [4, Eq. (5.24)] is equivalent to Eq. (2). Also there are, at least, two combinatorial proofs of Eq. (2) by using sign-reversing involutions [2].

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Let C_n denote [7, Section 5, p. 103] the n-th Catalan number. For $m = 1$, by the Eq. (2), it follows that

$$S_1(n,1) = (-1)^n n C_n. \tag{3}$$

We assert that:

Theorem 1. *The sum $S_1(n,m)$ is always divisible by n for all natural numbers n and m.*

The sum $S_1(n,m)$ has a natural generalization:

$$S_1(n,m;a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{an+k}{an+1}; \tag{4}$$

where a is a natural number.
Obviously, for $a = 2$, $S_1(n,m;2) = S_1(n,m)$.
For $m = 1$, by the Eq. (2), it can be shown that

$$S_1(n,1;a) = (-1)^n \frac{n}{(a-1)n+1} \binom{an}{n}. \tag{5}$$

Due to $gcd(n,(a-1)n+1) = 1$, by the Eq. (5), it follows that $S_1(n,1;a)$ is divisible by n. Note that the number $C(n,a) = \frac{1}{(a-1)n+1}\binom{an}{n}$ is known [7, Section 17, Eq. (17.1), p. 375] as generalized Catalan number or Fuss-Catalan number. See also [1, Eq. (2.2)]. For $a = 2$, $C(n,2) = C_n$.
We assert that:

Theorem 2. *Let a be a fixed natural number. The sum $S_1(n,m;a)$ is always divisible by n for all natural numbers n and m.*

Furthermore, let us consider the following alternating binomial sum:

$$S_2(n,m) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{2n+1+k}{2n}; \tag{6}$$

where n and m are natural numbers.
For $m = 1$, by the Eq. (2), it follows that

$$S_2(n,1) = (-1)^n (2n+1) C_n. \tag{7}$$

We assert that:

Theorem 3. *The sum $S_2(n,m)$ is always divisible by $2n+1$ for all natural numbers n and m.*

The sum $S_2(n,m)$ has a natural generalization:

$$S_2(n,m;a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{an+1+k}{an}; \tag{8}$$

where a is a natural number.

For $m = 1$, by the Eq. (2), it can be shown that

$$S_2(n,1;a) = (-1)^n \frac{an+1}{n+1}\binom{an}{n}. \tag{9}$$

Note that the integers $an+1$ and $n+1$ are not relatively prime in general.
We assert that:

Theorem 4. *Let a be a fixed natural number. The sum $S_2(n,m;a)$ is always divisible by $\frac{an+1}{\gcd(a-1,n+1)}$ for all natural numbers n and m.*

We prove Thms. (2) and (4) by using a new class of binomial sums. Theorem 1 is a special case of a Theorem 2 for $a = 2$. Similarly, Theorem 3 is a special case of a Theorem 4 for $a = 2$.
Let us consider the following sum:

$$S(n,m;a) = \sum_{k=0}^{n} \binom{n}{k}^m F(n,k,a); \tag{10}$$

where n, m and a are natural numbers, and $F(n,k,a)$ is an integer-valued function. Our goal is to investigate some divisibility properties of the sum $S(n,m;a)$. In order to do so, we use a new class of binomial sums which we called M sums.

Definition 1. *Let $S(n,m;a)$ be a sum from the Eq. (10) . Then*

$$M_S(n,j,t;a) = \binom{n-j}{j} \sum_{k=0}^{n-2j} \binom{n-2j}{k}\binom{n}{j+k}^t F(n,j+k,a); \tag{11}$$

where j and t are non-negative integers such that $j \le \lfloor \frac{n}{2} \rfloor$.

See [9, Def. 7, Eq. (28), p. 9].
Obviously, by setting $j = 0$ in the Eq. (11), it follows that [9, Eq. (29), p. 9]

$$S(n,t+1;a) = M_S(n,0,t;a). \tag{12}$$

Due to Eq. (12), we can see $M_S(n,j,t;a)$ sum as a generalization of $S(n,m;a)$. Furthermore, M sums satisfy [9, Thm. 8, p. 9] the following recurrence:

$$M_S(n,j,t+1;a) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n,j+u,t;a). \tag{13}$$

Eqns. (12) and (13) have a simple consequence which is important to us. Let us suppose that an integer $q(n,a)$ divides $M_S(n,j,t_0;a)$ for all $0 \le j \le \lfloor \frac{n}{2} \rfloor$, where t_0 is a fixed non-negative integer. By using the Eq. (13) and the induction principle, it can be shown that $q(n,a)$ must divide $M_S(n,j,t;a)$, for all integers t such that $t \ge t_0$. By the Eq. (12), it follows that $q(n,a)$ divides $S(n,t;a)$ for all integers t such that $t \ge t_0 + 1$.
By setting $t := 0$ in the Eq. (11), we obtain that

$$M_S(n,j,0;a) = \binom{n-j}{j} \sum_{k=0}^{n-2j} \binom{n-2j}{k} F(n,j+k,a). \tag{14}$$

The M sums give an elementary proof of Calkin result [3, Thm. 1]. See also [6, Thm. 1.2, Thm. 1.3, p. 2]. Note that there are also another applications [10, Section 2, p. 4], [8, 9] of M sums.

2 The Main Lemmas

We present four lemmas.
We calculate M sums for the sum $S_1(n,m;a)$ for $t=0$ and $t=1$.

Lemma 1.

$$M_{S_1}(n,j,0;a) = \frac{(-1)^{n-j}(n-j)\binom{an+j}{j}\binom{an+1}{n-2j}}{an+1}. \tag{15}$$

Lemma 2.

$$M_{S_1}(n,j,1;a) = n\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor} \frac{(-1)^{n-j-u}\binom{n-1}{j}\binom{j+u}{u}\binom{an+j+u}{j+u}\binom{an+1}{n-2j-2u}}{an+1}. \tag{16}$$

Also we calculate M sums for the sum $S_2(n,m;a)$ for $t=0$ and $t=1$.

Lemma 3.

$$M_{S_2}(n,j,0;a) = \frac{(-1)^{n-j}(an+1)\binom{an+1+j}{j}\binom{an}{n-2j}}{n+1-j}. \tag{17}$$

Lemma 4.

$$M_{S_2}(n,j,1;a) = \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor} \frac{(-1)^{n-j-u}(an+1)\binom{n+1}{j+u}\binom{j+u}{j}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}}{n+1}. \tag{18}$$

3 A Proof of Lemma 1

Proof. Obviously, the sum $S_1(n,m;a)$ is an instance of the sum $S(n,m;a)$ from the Eq. (10), where $F_1(n,k,a) = (-1)^k\binom{an+k}{an+1}$.
By the Eq. (14), we have:

$$\begin{aligned} M_{S_1}(n,j,0;a) &= \binom{n-j}{j}\sum_{k=0}^{n-2j}\binom{n-2j}{k}F_1(n,j+k,a) \\ &= (-1)^j\binom{n-j}{j}\sum_{k=0}^{n-2j}(-1)^k\binom{n-2j}{k}\binom{an+j+k}{an+1} \end{aligned} \tag{19}$$

By the Eq. (2), it follows that

$$\sum_{k=0}^{n-2j}(-1)^k\binom{n-2j}{k}\binom{an+j+k}{an+1} = (-1)^{n-2j}\binom{an+j}{n-j-1}. \tag{20}$$

By the Eq. (20), the Eq. (19) becomes

$$M_{S_1}(n,j,0;a) = (-1)^{n-j}\binom{an+j}{n-j-1}\binom{n-j}{j}. \tag{21}$$

Due to well-known [7, Section 1, Eq. (1.3), p. 5] formula:

$$\binom{n}{k-1} = \frac{k}{n-k}\binom{n}{k},$$

we have that:
$$\binom{an+j}{n-j-1} = \frac{n-j}{(a-1)n+2j+1}\binom{an+j}{n-j}, \tag{22}$$
By the Eq. (22), the Eq. (21) becomes:
$$M_{S_1}(n,j,0;a) = (-1)^{n-j}\frac{n-j}{(a-1)n+2j+1}\binom{an+j}{n-j}\binom{n-j}{j}. \tag{23}$$
It is well-known [7, Section 1, Eq. (1.4), p. 5] that:
$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}; \tag{24}$$
where a, b, and c are non-negative integers such that $a \geq b \geq c$.
By the Eq. (24), it follows that
$$\binom{an+j}{n-j}\binom{n-j}{j} = \binom{an+j}{j}\binom{an}{n-2j}. \tag{25}$$
By the Eq. (25), the Eq. (23) becomes
$$M_{S_1}(n,j,0;a) = (-1)^{n-j}\frac{n-j}{(a-1)n+2j+1}\binom{an+j}{j}\binom{an}{n-2j}. \tag{26}$$
Due to another well-known [7, Section 1, Eq. (1.2), p. 5]formula:
$$\frac{1}{n+1-k}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k},$$
it follows that:
$$\frac{1}{(a-1)n+2j+1}\binom{an}{n-2j} = \frac{1}{an+1}\binom{an+1}{n-2j}. \tag{27}$$
By using Eqns. (26) and (27), we obtain that
$$M_{S_1}(n,j,0;a) = (-1)^{n-j}\frac{n-j}{an+1}\binom{an+j}{j}\binom{an+1}{n-2j}. \tag{28}$$
The Eq. (28) completes the proof of Lemma 1. □

Note that that Eq. (21) leads directly to Eq. (28) by expressing binomial coefficients in terms of factorials and cancellation of equal factors.

4 A Proof of Lemma 2

Proof. We use the Eq. (13) and Lemma 1.
By setting $t := 0$ in the Eq. (13), we have that:

$$M_{S_1}(n,j,1;a) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_{S_1}(n,j+u,0;a). \tag{29}$$

By Lemma 1, it follows that:

$$M_{S_1}(n,j+u,0;a) = (-1)^{n-j-u} \frac{n-j-u}{an+1} \binom{an+j+u}{j+u} \binom{an+1}{n-2j-2u}. \tag{30}$$

By the Eq. (24), it follows that

$$\binom{n}{j}\binom{n-j}{u} = \binom{n}{j+u}\binom{j+u}{u}. \tag{31}$$

By using Eqns. (30) and (31) in the Eq. (29), we obtain that $M_{S_1}(n,j,1;a)$ is equal to the following sum:

$$\sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} (-1)^{n-j-u} \binom{n}{j+u}\binom{j+u}{u} \frac{n-j-u}{an+1} \binom{an+j+u}{j+u} \binom{an+1}{n-2j-2u}. \tag{32}$$

It is readily verified that:

$$(n-j-u)\binom{n}{j+u} = n\binom{n-1}{j+u}. \tag{33}$$

By using the Eq. (33) in the Eq. (32), we obtain that $M_{S_1}(n,j,1;a)$ is equal to

$$(-1)^{n-j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} (-1)^u \frac{n\binom{n-1}{j+u}\binom{j+u}{u}\binom{an+j+u}{j+u}\binom{an+1}{n-2j-2u}}{an+1}. \tag{34}$$

This completes the proof of Lemma 2.

□

5 A Proof of Theorem 2

Proof. Let n and a be fixed natural numbers.
By the Eq. (5), we know that $S_1(n,m;a)$ is divisible by n for $m = 1$.
Due to fact that $F_1(n,m,a)$ is an integer-valued function, we know that $M_{S_1}(n,j,0;a)$ is an integer. Also, see the Eq. (21).
The number $\binom{n}{j}\binom{n-j}{u}M_{S_1}(n,j+u,0;a)$ is also an integer; and, by the Eq. (34), it is equal to the number $\frac{n\binom{n-1}{j+u}\binom{j+u}{u}\binom{an+j+u}{j+u}\binom{an+1}{n-2j-2u}}{an+1}$

By Lemma 2 and the fact that $\gcd(n, an+1) = 1$, it follows that the sum $M_{S_1}(n, j, 1; a)$ is divisible by n for all non-negative integers j such that $j \leq \lfloor \frac{n}{2} \rfloor$. Therefore, we can take $q_1(n, a) = n$. By the Eq. (13) and the induction principle on t, it follows that $M_{S_1}(n, j, t; a)$ is divisible by n for all natural numbers t. By the Eq. (12), it follows that $S_1(n, t+1; a)$ is divisible by n for all natural numbers t. Hence, $S_1(n, m; a)$ is divisible by n for all natural numbers m such that $m \geq 2$. This completes the proof of Theorem 2.

□

6 A Proof of Lemma 3

Proof. Obviously, the sum $S_2(n, m; a)$ is an instance of the sum $S(n, m; a)$ from the Eq. (10), where $F_2(n, k, a) = (-1)^k \binom{an+1+k}{an}$.
By the Eq. (14), we have:

$$\begin{aligned} M_{S_2}(n, j, 0; a) &= \binom{n-j}{j} \sum_{k=0}^{n-2j} \binom{n-2j}{k} F_2(n, j+k, a) \\ &= (-1)^j \binom{n-j}{j} \sum_{k=0}^{n-2j} (-1)^k \binom{n-2j}{k} \binom{an+1+j+k}{an} \end{aligned} \tag{35}$$

By the Eq. (2), it follows that

$$\sum_{k=0}^{n-2j} (-1)^k \binom{n-2j}{k} \binom{an+1+j+k}{an} = (-1)^{n-2j} \binom{an+1+j}{n-j+1}. \tag{36}$$

By the Eq. (36), the Eq. (35) becomes

$$M_{S_2}(n, j, 0; a) = (-1)^{n-j} \binom{an+1+j}{n-j+1} \binom{n-j}{j}. \tag{37}$$

It is readily verified that

$$\binom{an+1+j}{n-j+1} = \frac{(an+1+j}{n-j+1} \binom{an+j}{n-j}. \tag{38}$$

By the Eq. (38), the Eq. (37) becomes

$$M_{S_2}(n, j, 0; a) = (-1)^{n-j} \frac{an+1+j}{n-j+1} \binom{an+j}{n-j} \binom{n-j}{j}. \tag{39}$$

By the Eq. (25), it follows that

$$M_{S_2}(n, j, 0; a) = (-1)^{n-j} \frac{an+1+j}{n-j+1} \binom{an+j}{j} \binom{an}{n-2j}. \tag{40}$$

By the Eq. (24), it follows that

$$(an+1+j) \binom{an+j}{j} = \binom{an+1+j}{j} (an+1). \tag{41}$$

By the Eq. (41), the Eq. (40) becomes:

$$M_{S_2}(n,j,0;a) = (-1)^{n-j}\frac{an+1}{n-j+1}\binom{an+1+j}{j}\binom{an}{n-2j}. \tag{42}$$

The Eq. (42) completes the proof of Lemma 3.

□

7 A Proof of Lemma 4

Proof. We use the Eq. (13) and Lemma 3.
By setting $t := 0$ in the Eq. (13), we have that:

$$M_{S_2}(n,j,1;a) = \binom{n}{j}\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n-j}{u}M_{S_2}(n,j+u,0;a). \tag{43}$$

By Lemma 3, it follows that:

$$M_{S_2}(n,j+u,0;a) = (-1)^{n-j-u}\frac{an+1}{n-j-u+1}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}. \tag{44}$$

By using Eqns. (31) and (44) in the Eq. (43), we obtain that $M_{S_2}(n,j,1;a)$ is equal to the following sum:

$$\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}(-1)^{n-j-u}\binom{n}{j+u}\frac{an+1}{n-j-u+1}\binom{j+u}{u}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}. \tag{45}$$

It is readily verified that

$$\binom{n}{j+u}\frac{1}{n-j-u+1} = \frac{1}{n+1}\binom{n+1}{j+u}. \tag{46}$$

By using the Eq. (46) in the Eq. (45), it follows that $M_{S_2}(n,j,1;a)$ is equal to:

$$\frac{an+1}{n+1}\cdot\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}(-1)^{n-j-u}\binom{n+1}{j+u}\binom{j+u}{u}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}. \tag{47}$$

The Eq. (47) proves Lemma 4.

□

8 A Proof of Theorem 4

Proof. Let n and a be fixed natural numbers.
By the Eq. (9), we know that $S_2(n,m;a)$ is divisible by $\frac{an+1}{\gcd(an+1,n+1)}$ for $m=1$.
Due to fact that $F_2(n,m,a)$ is an integer-valued function, we know that $M_{S_2}(n,j,0;a)$ is an integer. Also, see the Eq. (37).

The number $\binom{n}{j}\binom{n-j}{u}M_{S_2}(n, j+u, 0; a)$ is an integer; and, by the Eq. (47), it is equal to the number $(-1)^{n-j-u} \cdot \frac{an+1}{n+1}\binom{n+1}{j+u}\binom{j+u}{u}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}$
By Lemma 4, it follows that the sum $M_{S_2}(n, j, 1; a)$ is divisible by $\frac{an+1}{\gcd(an+1,n+1)}$ for all non-negative integers j such that $j \leq \lfloor \frac{n}{2} \rfloor$.
Therefore, we can take $q_2(n, a) = \frac{an+1}{\gcd(an+1,n+1)}$. By the Eq. (13) and the induction principle on t, it follows that $M_{S_2}(n, j, t; a)$ is divisible by $q_2(n, a)$ for all natural numbers t. By the Eq. (12), it follows that $S_1(n, t+1; a)$ is divisible by $q_2(n, a)$ for all natural numbers t. Hence, $S_1(n, m; a)$ is divisible by $\frac{an+1}{\gcd(an+1,n+1)}$ for all natural numbers m such that $m \geq 2$.
Note that $\gcd(an+1, n+1) = \gcd(a-1, n+1)$. This completes the proof of Theorem 4.

□

Remark 1. By setting $a := 2$ in Theorem 4 and by using the fact $\gcd(2n+1, n+1) = 1$, we obtain the proof of Theorem 3.

9 Concluding Remarks

Let us consider the following alternating sum:

$$S_1(n, m; a, b) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{an+k}{an+b}; \tag{48}$$

where n, m, a, and b are natural numbers.
Obviously, $S_1(n, m; a, 1) = S_1(n, m; a)$. Furthermore, by using M sums, it can be shown that:

$$M_{S_1}(n, j, 0; a, b) = \frac{(-1)^{n-j}\binom{an+j}{j}\binom{n-j}{b}\binom{an+b}{n-2j}}{\binom{an+b}{b}}, \tag{49}$$

$$M_{S_1}(n, j, 1; a, b) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \frac{(-1)^{n-j-u}\binom{n}{b}\binom{n-b}{j+u}\binom{j+u}{j}\binom{an+j+u}{j+u}\binom{an+b}{n-2j-2u}}{\binom{an+b}{b}}. \tag{50}$$

The Eq. (50) suggests that $q_1(n, a, b) = \frac{\binom{n}{b}}{\gcd(\binom{n}{b}, \binom{an+b}{b})}$. We assert that:

Theorem 5. *Let n, a, and b be fixed natural numbers. Then the sum $S_1(n, m; a, b)$ is always divisible by $\frac{\binom{n}{b}}{\gcd(\binom{n}{b}, \binom{an+b}{b})}$ for all natural numbers m.*

By setting $b := 1$ in the Eq. (49), we obtain Lemma 1. Similarly, by setting $b := 1$ in the Eq. (50), we obtain Lemma 2. Finally, by setting $b := 1$ in Theorem 5, we obtain Theorem 2.
Let us now consider the following sum:

$$S_2(n, m; a, b) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{an+b+k}{an}; \tag{51}$$

where n, m, a, and b are natural numbers.
Clearly, $S_2(n,m;a,1) = S_2(n,m;a)$. Furthermore, by using M sums, it can be shown that:

$$M_{S_2}(n,j,0;a,b) = \frac{(-1)^{n-j}\binom{an+b+j}{j}\binom{an}{n-2j}\binom{an+b}{b}}{\binom{n-j+b}{b}}, \tag{52}$$

$$M_{S_2}(n,j,1;a,b) = \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor} \frac{(-1)^{n-j-u}\binom{an+b}{b}\binom{n+b}{j+u}\binom{j+u}{j}\binom{an+b+j+u}{j+u}\binom{an}{n-2j-2u}}{\binom{n+b}{n}}. \tag{53}$$

The Eq. (53) suggests that $q_2(n,a,b) = \frac{\binom{an+b}{b}}{\gcd(\binom{an+b}{b},\binom{n+b}{n})}$. We assert that:

Theorem 6. *Let n, a, and b be fixed natural numbers. Then the sum $S_2(n,m;a,b)$ is always divisible by $\frac{\binom{an+b}{b}}{\gcd(\binom{an+b}{b},\binom{n+b}{n})}$ for all natural numbers m.*

By setting $b := 1$ in the Eq. (52), we obtain Lemma 3. Similarly, by setting $b := 1$ in the Eq. (53), we obtain Lemma 4. Finally, by setting $b := 1$ in Theorem 6, we obtain Theorem 4.
For the sake of brevity and clarity, we omit proofs of Thms. (5) and (6).

10 Appendix A

We give an example with the positive binomial sum with Fibonacci numbers.
Let F_n denote the n-th Fibonacci number, where n is a non-negative integer.
Let us consider the following binomial identity:

$$\sum_{k=0}^{n}\binom{n}{k}5^{\lfloor\frac{k}{2}\rfloor} = 2^n F_{n+1}. \tag{54}$$

It is well-known that the Binet formula for Fibonacci numbers states:

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}. \tag{55}$$

The Binet formula is equivalent with the following binomial identity:

$$\sum_{k=0}^{n}\binom{n}{2k+1}5^k = 2^{n-1}F_n. \tag{56}$$

Let L_n denote the n-th Lucas number. Formula for Lucas numbers is also well-known:

$$L_n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n}. \tag{57}$$

The formula for Lucas numbers is equivalent with the following binomial identity:

$$\sum_{k=0}^{n}\binom{n}{2k}5^k = 2^{n-1}L_n. \tag{58}$$

Note that Eq. (54) follows by Eqns. (56) and (58), and the fact that $L_n = F_{n+1} - F_{n-1}$.
Let us consider the following sum:

$$S_3(n,m) = \sum_{k=0}^{n} \binom{n}{k}^m 5^{\lfloor \frac{k}{2} \rfloor}. \tag{59}$$

Obviously the sum $S_3(n,m)$ is an instance of the sum S from the Eq. (10), where $F_3(n,k) = 5^{\lfloor \frac{k}{2} \rfloor}$.
By the Eq. (54), for $m = 1$, we have that

$$S_3(n,1) = 2^n F_{n+1}.$$

By using Eqns. (13), (56), and (58), it can be shown that:

$$M_{S_3}(n,j,0) = \begin{cases} \binom{n-j}{j} 5^{\frac{j}{2}} \cdot 2^{n-2j} \cdot F_{n-2j+1}, & \text{if } j \text{ is even;} \\ \binom{n-j}{j} 5^{\frac{j-1}{2}} \cdot 2^{n-2j} \cdot L_{n-2j+1}, & \text{if } j \text{ is odd.} \end{cases} \tag{60}$$

We see that the formula for the $M_{S_3}(n,j,0)$ sum appear both Fibonacci and Lucas numbers. Since the sum $M_{S_3}(n,j,0)$ is a slight generalization of the sum $S_3(n,1)$, this is expected, due to Eqns. (56) and (58).

Remark 2. By setting $t = 0$ in the Eq. (11), and by using Eqns. (12) and (60), we can calculate the sum $S_3(n,2)$. It can be shown that $S_3(n,2)$ is equal to:

$$\sum_{\substack{u=0 \\ u \text{ is even}}}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{u} \binom{n-u}{u} 5^{\frac{u}{2}} 2^{n-2u} F_{n-2u+1} + \sum_{\substack{u=0 \\ u \text{ is odd}}}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{u} \binom{n-u}{u} 5^{\frac{u-1}{2}} 2^{n-2u} L_{n-2u+1}. \tag{61}$$

11 Appendix B

Let $S_4(n,m;a)$ denote the following alternating binomial sum:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \left(\binom{an+k}{an} + 2\binom{an+k}{an+1} \right); \tag{62}$$

where n, m, and a are natural numbers.
Obviously the sum $S_4(n,m;a)$ is an instance of the sum S from the Eq. (10), where $F_4(n,k,a) = (-1)^k(\binom{an+k}{an} + 2\binom{an+k}{an+1})$.
By the Eq. (13) and the Eq. (2), it can be shown that:

$$M_{S_4}(n,j,0;a) = (-1)^{n-j} \binom{an+j}{j} \binom{an+1}{n-2j} \frac{(a+1)n+1}{an+1}. \tag{63}$$

Note that the integers $(a+1)n+1$ and $an+1$ are relatively prime. By the Eq. (63), it follows that the sum $M_{S_4}(n,j,0;a)$ is always divisible by $(a+1)n+1$. Hence, we can take $q_4(n,a) = (a+1)n+1$.
By using M sums, we can prove the following theorem:

Theorem 7. *Let n and a be fixed natural numbers. Then the sum $S_4(n,m;a)$ is always divisible by $(a+1)n+1$ for all natural integers m.*

Remark 3. By the Eq. (63), it follows that an integer $T(n,j,a)=\binom{an+j}{j}\binom{an+1}{n-2j}$ is always divisible by $an+1$ for all non-negative integers j such that $j \le \lfloor \frac{n}{2} \rfloor$. These numbers also appear in Lemma 1 and Lemma 2. Note that the Eq. (15) can be also written as

$$M_{S_1}(n,j,0;a) = (-1)^{n-j}(n-j)T(n,j,a). \tag{64}$$

Similarly, the Eq. (16) can be written as

$$M_{S_1}(n,j,1;a) = n \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} (-1)^{n-j-u} \binom{n-1}{j} \binom{j+u}{u} T(n,j+u,a). \tag{65}$$

Obviously,

$$M_{S_4}(n,j,0;a) = (-1)^{n-j}((a+1)n+1)T(n,j,a). \tag{66}$$

It would be interesting to find a combinatorial interpretation for numbers $T(n,j,a)$.

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