Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.06

On a linear recurrence relation of the divide-and-conquer type

Daniele Parisse

Abstract

We study the six-parameter linear recurrence relation defined by

$$f(1) = \zeta,\ f(2n) = \alpha f(n) + \beta,\ f(2n+1) = \gamma f(n) + \delta f(n+1) + \varepsilon, \quad n \geq 1,$$

with $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{Z}$. We determine its ordinary generating function which shows that the sequence is of the divide-and-conquer type. Then we discuss some interesting special cases such as the Josephus problem, the number of 1's in the binary expression of $n \in \mathbb{N}$, the Gros sequence in the Tower of Hanoi with 3 pegs and n disks, $n \in \mathbb{N}$, the Prouhet-Thue-Morse sequence, Stern's diatomic sequence and others. We give its solution at first for the special case $\delta = 0$ and then for $\delta \neq 0$. Moreover, for $\delta \neq 0$, $\beta = 0 = \varepsilon$ and $\alpha \neq 0$, we conjecture that it satisfies a second-order recurrence relation. We prove this conjecture for three special cases and give the solution by means of continued fractions. Then, we generalize the recurrence relation by considering both β and ε as integer functions of n and discuss in detail the case $\delta = 0$ and a special case of this class. Finally, for $\delta \neq 0$, we only mention two known special cases.

Key words: Sequences of the divide-and-conquer type; Ordinary generating function; Stern's diatomic sequence; Prouhet-Thue-Morse sequence; Tower of Hanoi;

AMS subject classification (2010): Primary 05A10 ; Secondary 05A19.

1 Introduction

There are some problems such as the Tower of Hanoi puzzle, the Josephus problem, Stern's diatomic sequence, "the infinity sequence" of Per Nørgård, the Prouhet-Thue-Morse sequence and many others, which have apparently no connections among each other. However, they are all defined by the same six-parameter linear recurrence relation

$$f(1) = \zeta, \quad f(2n) = \alpha f(n) + \beta, \quad f(2n+1) = \gamma f(n) + \delta f(n+1) + \varepsilon, \quad n \geq 1 \quad (1)$$

(Daniele Parisse) Airbus Defence and Space GmbH, Manching, Germany, daniele.parisse@t-online.de

with $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \in \mathbb{Z}$. (Note that we begin the sequence by $n = 1$, although some of the sequences are also well-defined for all $n \in \mathbb{N}_0$. If possible, we shall define $f(0) := \eta \in \mathbb{Z}$ so that this value is consistent with (1), otherwise we set $f(0) := 0$.) Eq.(1) can also be written as follows:

$$f(1) = \zeta, \quad f(n) = a(n)f\Big(\lfloor n/2 \rfloor\Big) + b(n)f\Big(\lceil n/2 \rceil\Big) + c(n), \quad n \geq 2, \tag{2}$$

where

$$a(n) := \frac{\alpha + \gamma}{2} + (-1)^n \frac{\alpha - \gamma}{2}, \quad b(n) := \frac{1 - (-1)^n}{2}\delta, \quad c(n) := \frac{\beta + \varepsilon}{2} + (-1)^n \frac{\beta - \varepsilon}{2}.$$

Recurrence relations of this form are called *(binary) divide-and-conquer recurrences* and appear often in computer science, because algorithms based on the technique of *divide et impera* (divide and conquer) often reduce a problem of size n to the solution of two problems of approximately equal sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, where $n = \lfloor n/2 \rfloor + \lceil n/2 \rceil, n \in \mathbb{N}_0$. The solutions of the two subproblems are then used to solve the original problem.
In this paper we shall study the linear recurrence relation (1) and its ordinary generating function. We shall at first consider some known special cases of (1) and then we shall derive its general solution, first for $\delta = 0$ and then for $\delta \neq 0$. It will turn out that for $\delta = 0$ (1) is equivalent to a first-order linear recurrence relation with variable coefficients and that for $\delta \neq 0$, $\beta = 0 = \varepsilon$ and $\alpha \neq 0$ it seems to be equivalent to a second-order linear recurrence relation with variable coefficients. We prove this conjecture for three special cases. The solutions in both cases rely on the binary expansion of n. In the last chapter we study a generalization of (1) by considering instead of the constants β and ε the integer functions $g(n)$ and $h(n)$. Then, we give a solution for $\delta = 0$ and investigate in detail a special case. Finally, for $\delta \neq 0$, we only mention two known sequences.

2 Definitions and Examples

Definition 1. *A formal series $\varphi(s)$ is said to be of the* divide-and-conquer (DC) type *(see [31]), if it satisfies a functional equation of the form*

$$c_0(s)\varphi(s) + c_1(s)\varphi(s^2) + \cdots + c_n(s)\varphi(s^{2^n}) = b(s), \tag{3}$$

in which $b(s)$ is a formal series and $c_k(s), k \in [n+1]_0 := \{0, 1, \ldots, n\}$, are polynomials not all zero. If $b(s) = 0$, then the series $\varphi(s)$ is said to be Mahlerian*.*
A sequence is of the divide-and-conquer type*, if its ordinary generating function, defined by the power series $\sum_{n=0}^{\infty} f(n)s^n$, is of the divide-and-conquer type and* Mahlerian*, if its generating function is Mahlerian.*

We shall now prove that the generating function for the sequence $\Big(f(n)\Big)_{n \in \mathbb{N}_0}$ defined by (1) is of the DC type and, for some special values of the six parameters, even Mahlerian.

Proposition 1. *The generating function* $\varphi(s)$ *for the sequence* $\big(f(n)\big)_{n\in\mathbb{N}_0}$ *is of the DC type, since it satisfies the functional equation*

$$s\varphi(s) = (\delta+\alpha s+\gamma s^2)\varphi(s^2) - \delta\eta + \eta(1-\alpha)s + \Big(\zeta(1-\delta) - \eta\gamma\Big)s^2 + \frac{(\beta+\varepsilon s)s^3}{1-s^2} \quad (4)$$

Proof. Let $\varphi(s) := \sum_{n=0}^{\infty} f(n)s^n$ and $f(0) := \eta$, then by (1)

$$\begin{aligned}
\varphi(s) &= f(0) + \sum_{n=1}^{\infty} f(2n)s^{2n} + \sum_{n=0}^{\infty} f(2n+1)s^{2n+1} \\
&= \eta + \alpha\sum_{n=1}^{\infty} f(n)s^{2n} + \beta\sum_{n=1}^{\infty} s^{2n} + f(1)s + \gamma\sum_{n=1}^{\infty} f(n)s^{2n+1} + \\
&\quad + \delta\sum_{n=1}^{\infty} f(n+1)s^{2n+1} + \varepsilon\sum_{n=1}^{\infty} s^{2n+1} \\
&= \eta + \alpha\Big(\varphi(s^2)-\eta\Big) + \beta\frac{s^2}{1-s^2} + \zeta s + \gamma s\Big(\varphi(s^2)-\eta\Big) + \\
&\quad + \frac{\delta}{s}\Big(\varphi(s^2) - \eta - \zeta s^2\Big) + \varepsilon s\frac{s^2}{1-s^2},
\end{aligned}$$

since $\sum_{n=1}^{\infty} f(n+1)s^{2n+1} = \frac{1}{s}\Big(\varphi(s^2)-\eta-\zeta s^2\Big)$ and $\sum_{n=1}^{\infty} s^{2n} = \Big(\frac{1}{1-s^2}-1\Big) = \frac{s^2}{1-s^2}$. Simplifying and multiplying both sides by s we obtain formula (4), which shows that $f(n)$ is of the DC type. ☐

We shall now discuss some examples (see also [2, 4]).

- **E1** The Josephus problem ([13, pp. 8–13]).
  Let $n \in \mathbb{N}$ people, numbered 1 to n, stand around a circle and eliminate every second remaining person until one survives. The problem consists in determining the survivor's number $J(n)$. (The original and more difficult problem (for $n = 41$) was to eliminate every third remaining person.) It follows (see [13, p. 10])

  $$J(1) = 1, \quad J(2n) = 2J(n) - 1, \quad J(2n+1) = 2J(n) + 1, \quad n \geq 1 \quad (5)$$

  This is the special case $\alpha = 2 = \gamma$, $\beta = -1$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta := 0$ of (1). The generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

  $$\varphi(s) = 2(1+s)\varphi(s^2) + \frac{s}{1+s}$$

  and it is given by

  $$\varphi(s) = \frac{1}{1-s}\sum_{k=0}^{\infty} \frac{2^k \cdot s^{2^k}\Big(1-s^{2^k}\Big)}{1+s^{2^k}}, \quad (6)$$

since

$$\begin{aligned}
2(1+s)\varphi(s^2)+\frac{s}{1+s} &= \frac{2(1+s)}{1-s^2}\sum_{k=0}^{\infty}\frac{2^k\cdot s^{2^{k+1}}\left(1-s^{2^{k+1}}\right)}{1+s^{2^{k+1}}}+\frac{s}{1-s}\\
&= \frac{1}{1-s}\sum_{k=0}^{\infty}\frac{2^{k+1}\cdot s^{2^{k+1}}\left(1-s^{2^{k+1}}\right)}{1+s^{2^{k+1}}}+\frac{s}{1+s}\\
&= \frac{1}{1-s}\left(\sum_{k=0}^{\infty}\frac{2^{k+1}\cdot s^{2^{k+1}}\left(1-s^{2^{k+1}}\right)}{1+s^{2^{k+1}}}+\frac{s(1-s)}{1+s}\right)\\
&= \varphi(s).
\end{aligned}$$

The solution of **E1** can be given in several equivalent formulas in binary form.

1. Let $n=(b_m\dots b_0)_2 := \sum_{k=0}^{m} b_k\cdot 2^k$, $b_k\in\{0,1\}$, $k\in[m+1]_0$, $(b_m=1)$, be the binary expansion of $n\in\mathbb{N}$. Then (see [13, formula (1.10) at p. 11])

$$J\Big((b_m\dots b_0)_2\Big) = (b_{m-1}b_{m-2}\dots b_1b_0b_m)_2 \tag{7}$$

Since $(b_{m-1}b_{m-2}\dots b_1b_0b_m)_2 = \sum_{k=1}^{m} b_{k-1}2^k + b_m$, we have

$$J\Big((b_m\dots b_0)_2\Big) = \sum_{k=1}^{m} b_{k-1}2^k + b_m = 2\cdot\left(\sum_{k=0}^{m} b_k2^k - 2^m\right)+1 = 2n-2^{m+1}+1,$$

where $m=\lfloor\log_2 n\rfloor$. Further, since $-2^{m+1}+1 = -\sum_{k=0}^{m}2^k$ and $2b_k-1=(-1)^{1-b_k}$, we get

$$\begin{aligned}
J\Big((b_m\dots b_0)_2\Big) = (b_{m-1}b_{m-2}\dots b_1b_0b_m)_2 &= 2\sum_{k=0}^{m}b_k2^k - \sum_{k=0}^{m}2^k\\
&= \sum_{k=0}^{m}(2b_k-1)2^k = \sum_{k=0}^{m}(-1)^{1-b_k}\cdot 2^k.
\end{aligned} \tag{8}$$

2. Let $n=2^{a_0}+2^{a_1}+\dots+2^{a_\ell}$, $a_0>a_1>\dots>a_\ell\ge 0$, $\ell\ge 0$, then

$$J(2^{a_0}+2^{a_1}+\dots+2^{a_\ell}) = 1+\sum_{k=0}^{\ell-1}2^{a_{\ell-k}+1} = 2^{a_1+1}+2^{a_2+1}+\dots+2^{a_\ell+1}+2^0 \tag{9}$$

The first few values of $J(n)$ (the sequence A006257 in the On-Line Encyclopedia of Integer Sequences (OEIS ®) [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	0	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Table 1: $J(n)$, $0\le n\le 16$

- **E2** Number of 1's in the binary expansion of $n\in\mathbb{N}$.
  For $n=(b_m\dots b_0)_2$ we define the sum of the bits in the binary expansion of

n to be $s_2(n) := \sum_{k=0}^{m} b_k$. Since $b_k \in \{0,1\}$, $k \in [m+1]_0$, this sum gives the number of 1's in the binary expansion of n and satisfies the recurrence relation

$$s_2(1) = 1, \quad s_2(2n) = s_2(n), \quad s_2(2n+1) = s_2(n) + 1, \quad n \geq 1 \tag{10}$$

This is the special case $\alpha = 1 = \gamma$, $\beta = 0$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta = 0$ of (1), since 0 has no 1's in the binary expansion. The generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = (1+s)\varphi(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}}, \tag{11}$$

since

$$\begin{aligned}(1+s)\varphi(s^2) + \frac{s}{1-s^2} &= \frac{1+s}{1-s^2} \sum_{k=0}^{\infty} \frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1-s^2} \\ &= \frac{1}{1-s}\left(\sum_{k=0}^{\infty} \frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1+s} \right) = \varphi(s).\end{aligned}$$

The solution of **E2** is by definition $s_2\Big((b_m \ldots b_0)_2\Big) = \sum_{k=0}^{m} b_k$ or, alternatively, for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$,

$$s_2(2^{a_0} + 2^{a_1} + \cdots + 2^{a_l}) = l + 1. \tag{12}$$

Closely related to $s_2(n)$ is the sequence **E2'**, denoted by $e_0(n)$, giving the number of nonleading 0's in the binary expansion of n. It satisfies the recurrence relation

$$e_0(1) = 0, \quad e_0(2n) = e_0(n) + 1, \quad e_0(2n+1) = e_0(n), \quad n \geq 1 \tag{13}$$

This is the special case $\alpha = 1 = \gamma$, $\beta = 1$, $\delta = 0$, $\varepsilon = 0$, $\zeta = 0$ and $\eta = 0$ of (1). The generating function $\varphi_0(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi_0(s) = (1+s)\varphi_0(s^2) + \frac{s^2}{1-s^2}$$

and it is given by

$$\varphi_0(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{s^{2^{k+1}}}{1+s^{2^k}}, \tag{14}$$

since

$$\begin{aligned}(1+s)\varphi_0(s^2) + \frac{s^2}{1-s^2} &= \frac{1+s}{1-s^2} \cdot \sum_{k=0}^{\infty} \frac{s^{2^{k+2}}}{1+s^{2^{k+1}}} + \frac{s^2}{1-s^2} \\ &= \frac{1}{1-s}\left(\sum_{k=0}^{\infty} \frac{s^{2^{k+2}}}{1+s^{2^{k+1}}} + \frac{s^2}{1+s} \right) = \varphi_0(s).\end{aligned}$$

The solution of **E2'** for $n = (b_m \dots b_0)_2$ is given by

$$e_0(n) = m + 1 - \sum_{k=0}^{m} b_k = \lfloor \log_2 n \rfloor + 1 - s_2(n). \tag{15}$$

The first few values of $s_2(n)$ (the sequence A000120 in [24]) and $e_0(n)$ (the sequence A080791 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s_2(n)$	0	1	1	2	1	2	2	3	1	2	2	3	2	3	3	4	1
$e_0(n)$	0	0	1	0	2	1	1	0	3	2	2	1	2	1	1	0	4

Table 2: $s_2(n)$, $e_0(n)$, $0 \le n \le 16$

- **E3** The Tower of Hanoi with 3 pegs and n disks, $n \in \mathbb{N}$.
This puzzle has been invented in 1883 by the French mathematician Édouard Lucas (1842-1891). It consists of n disks, $n \in \mathbb{N}$, of different sizes and three vertical pegs. At the beginning, all disks are stacked in decreasing order on one of the three pegs (this is called a *tower*). The objective is to transfer the entire tower to another peg using the minimum number of legal moves, where a *legal move* is to move one topmost disk at a time and never moving a larger one onto a smaller one (this is the *divine rule*). For more details on the Tower of Hanoi with 3 or more pegs and on some variations of this puzzle we refer to the comprehensive monograph [19]. It can be shown (see [17, p. 281] and [19, Theorem 2.1, pp. 79–83]) that the number $d_2(n)$ of the disk to be moved at the nth step of the optimal solution of the Tower of Hanoi puzzle satisfies the recurrence relation

$$d_2(1) = 1, \quad d_2(2n) = d_2(n) + 1, \quad d_2(2n+1) = 1, \quad n \ge 1 \tag{16}$$

This is the special case $\alpha = 1$, $\beta = 1 = \varepsilon$, $\gamma = 0$, $\delta = 0$, $\zeta = 1$ and $\eta := 0$ of (1). This sequence is also known as the *Gros sequence* (see [19, pp. 79–83]). The generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = \varphi(s^2) + \frac{s}{1-s}$$

and it is given by

$$\varphi(s) = \sum_{k=0}^{\infty} \frac{s^{2^k}}{1 - s^{2^k}}, \tag{17}$$

since

$$\varphi(s^2) + \frac{s}{1-s} = \sum_{k=0}^{\infty} \frac{s^{2^{k+1}}}{1 - s^{2^{k+1}}} + \frac{s}{1-s} = \sum_{k=0}^{\infty} \frac{s^{2^k}}{1 - s^{2^k}} = \varphi(s).$$

The solution of **E3** for $n = 2^r(2k+1)$, $r, k \in \mathbb{N}_0$, is given by

$$d_2(n) = r + 1, \tag{18}$$

or, for $n = (b_m \dots b_0)_2$ (cf. (74))

$$d_2\Big((b_m \dots b_0)_2\Big) = \sum_{k=0}^{m} \Big(\prod_{j=0}^{k-1} (1 - b_j) \Big). \tag{19}$$

Alternatively, for $n = 2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}$, $a_0 > a_1 > \dots > a_\ell \geq 0$, $\ell \geq 0$,

$$d_2(n) = a_\ell + 1. \tag{20}$$

Other interpretations of this sequence based on (18) and (20) are

1. $d_2(n)$ gives the exponent of highest power of 2 dividing $2n$.
2. $d_2(n)$ gives the position of the first 1 (counting from right to left and beginning with 1) of the binary representation of n, e.g., $12 = (1100)_2 \Rightarrow d_2(12) = 3$.
3. $d_2(n)$ gives the position of the bit (counting from right to left and beginning with 1) to be changed in the Gray-code (see Table 3).

Therefore, the sequence $\tilde{d_2}(n) := d_2(n) - 1$ for all $n \in \mathbb{N}$ (called the *binary carry sequence*) gives the exponent of highest power of 2 dividing n or, equivalently, the index of the right-most non-zero bit in the binary representation of n. It is the sequence A007814 in [24].

n decimal	n binary	Gray-code	$d_2(n)$
1	1	000**1**	1
2	10	00**1**1	2
3	11	001**0**	1
4	100	0**1**10	3
5	101	011**1**	1
6	110	01**0**1	2
7	111	010**0**	1
8	1000	**1**100	4
9	1001	110**1**	1
10	1010	11**1**1	2
11	1011	111**0**	1
12	1100	1**0**10	3
13	1101	101**1**	1
14	1110	10**0**1	2
15	1111	100**0**	1

Table 3: Position of the nth bit to be changed in the Gray-code, $1 \leq n \leq 15$

Remark 1. The sequence $p_f(n)$ of the changed bit in the Gray-code above is the so-called *regular paper-folding* (or *dragon curve sequence).* $p_f(n)$ is the one's complement of the bit to the left of the least significant "1" in the binary expansion of n, e.g., $n = 7 = 111_2$, that is $p_f(7) = 0$. This sequence is defined recursively by the recurrence relation

$$p_f(1) = 1, \quad p_f(4n) = 1, \quad p_f(4n+2) = 0, \quad p_f(2n+1) = p_f(n), \quad n \geq 1 \tag{21}$$

The first few values of $d_2(n)$ (the sequence A001511 in [24]) and $p_f(n)$ (the sequence A014577 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_2(n)$	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1	5
$p_f(n)$	1	1	0	1	1	0	0	1	1	1	0	0	1	0	0	1

Table 4: $d_2(n)$, $p_f(n)$, $1 \leq n \leq 16$

Eq.(17) can be written in the form

$$\varphi(s) = \sum_{n=1}^{\infty} \rho(n) \cdot \frac{s^n}{1 - s^n}, \tag{22}$$

where $\rho(n)$ is the characteristic function of the powers of 2, that is $\rho(n) = 1$, if n is a power of 2 and zero otherwise. It shows that $\varphi(s)$ is a Lambert series (cf. (114)). Moreover, since $d_2(n)$ is the number of 2's that divides $2n$, or, equivalently, the number of the positive powers of 2 which divides n, we have

$$d_2(n) = \sum_{t|n} \rho(t), \quad n \in \mathbb{N}. \tag{23}$$

There is a relation between $d_2(n)$ and $s_2(n)$.

Proposition 2. *For all $n \in \mathbb{N}$ it is*

$$d_2(n) = 2 - \Delta s_2(n) \tag{24}$$

and, conversely,

$$s_2(n) = 2n - D_2(n) \tag{25}$$

where $\Delta s_2(n) := s_2(n) - s_2(n-1)$ is the difference sequence of $s_2(n)$ and

$$D_2(n) := \sum_{k=1}^{n} d_2(k) \tag{26}$$

is the sequence of the partial sums of $d_2(n)$.

Proof. Let $\alpha(n) := 2 - s_2(n) + s_2(n-1)$, then $\alpha(1) = 2 - s_2(1) + s_2(0) = 2 - 1 + 0 = 1 = d_2(1)$. Further, by (16) and noting that $s_2(2n-1) = s_2(2(n-1)+1) = s_2(n-1)+1$ we have $\alpha(2n) = 2 - s_2(2n) + s_2(2n-1) = 2 - s_2(n) + s_2(n-1) + 1 = \alpha(n) + 1$ and $\alpha(2n+1) = 2 - s_2(2n+1) + s_2(2n) = 2 - (s_2(n)+1) + s_2(n) = 1$.

For all $n \geq 1$ the sequence $\alpha(n)$ satisfies the same recurrence relation as $d_2(n)$ and has the same initial value, therefore the two sequences are equal and this proves (24).

Conversely, by repeated iteration we obtain $s_2(n) = s_2(n-1) + 2 - d_2(n) = s_2(n-2) + 2 \cdot 2 - (d_2(n-1) + d_2(n)) = \ldots = s_2(0) + 2n - \sum_{k=1}^{n} d_2(k) = 2n - D_2(n)$ and this is (25), since $s_2(0) = 0$. □

Note that since $s_2(n) \geq 1$ for all $n \in \mathbb{N}$ it follows that $D_2(n) \leq 2n$.

E4 Number of odd binomial coefficients in the nth row of Pascal's triangle. This sequence (also known as *Gould's sequence*) is given recursively by

$$G(1) = 2, \quad G(2n) = G(n), \quad G(2n+1) = 2G(n), \quad n \geq 1 \tag{27}$$

$G(n)$ gives the number of odd binomial coefficients in the nth row ($n \geq 0$) of Pascal's triangle. Since Pascal's triangle with 2^n rows modulo 2 is isomorphic to the graph of the Tower of Hanoi with 3 pegs and n disks [16], $G(n)$ gives also the number of regular states for which the distance to a given perfect state is equal to n. It is the special case $\alpha = 1$, $\beta = 0 = \varepsilon, \gamma = 2$, $\delta = 0$, $\zeta = 2$ and $\eta := 1$ of (1). The generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = (1+2s)\varphi(s^2)$$

which shows that $\varphi(s)$ is Mahlerian. It is given by the infinite product

$$\varphi(s) = \prod_{k=0}^{\infty}(1 + 2s^{2^k}), \tag{28}$$

since

$$(1+2s)\varphi(s^2) = (1+2s) \cdot \prod_{k=0}^{\infty}(1+2s^{2^{k+1}}) = \prod_{k=0}^{\infty}(1+2s^{2^k}) = \varphi(s).$$

The solution of **E4** for $n = (b_m \dots b_0)_2$ has been given by Glaisher [12, p. 10]

$$G((b_m \dots b_0)_2) = 2^{\sum_{k=0}^{m} b_k} = 2^{s_2(n)} \tag{29}$$

Alternatively, for $n = 2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}$, $a_0 > a_1 > \dots > a_\ell \geq 0$, $\ell \geq 0$,

$$G(2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}) = 2^{\ell+1}. \tag{30}$$

A slight modification of **E4** is the sequence **E4'** defined recursively by

$$G_1(1) = 1, \quad G_1(2n) = G_1(n), \quad G_1(2n+1) = 2G_1(n) + 1, \quad n \geq 1 \tag{31}$$

$G_1(n)$ is the smallest number with the same number of 1's in its binary representation as n. It is the special case $\alpha = 1$, $\beta = 0, \gamma = 2$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta = 0$ of (1) and it is $G_1(n) = G(n) - 1$ for all $n \in \mathbb{N}_0$. Its generating function $\varphi_1(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi_1(s) = (1+2s)\varphi_1(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi_1(s) = \varphi(s) - \sum_{n=0}^{\infty} s^n = \prod_{k=0}^{\infty}(1+2s^{2^k}) - \frac{1}{1-s}, \tag{32}$$

since

$$\begin{aligned}(1+2s)\varphi_1(s^2) + \frac{s}{1-s^2} &= (1+2s)\Big(\prod_{k=0}^{\infty}(1+2s^{2^{k+1}}) - \frac{1}{1-s^2}\Big) + \frac{s}{1-s^2} \\ &= \prod_{k=0}^{\infty}(1+2s^{2^k}) - \frac{1+2s}{1-s^2} + \frac{s}{1-s^2} = \varphi_1(s).\end{aligned}$$

The solution of (31) for $n = (b_m \dots b_0)_2$ is given by (cf. (74))

$$G_1((b_m \dots b_0)_2) = 2^{\sum_{k=0}^{m} b_k} - 1 = 2^{s_2(n)} - 1 = \sum_{k=0}^{m} b_k \cdot 2^{\sum_{j=0}^{k-1} b_j} \tag{33}$$

or, for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$,

$$G_1(2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}) = 2^{\ell+1} - 1. \tag{34}$$

The first few values of $G(n)$ (the sequence A001316 in [24]) and $G_1(n)$ (the sequence A038573 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$G(n)$	1	2	2	4	2	4	4	8	2	4	4	8	4	8	8	16	2
$G_1(n)$	0	1	1	3	1	3	3	7	1	3	3	7	3	7	7	15	1

Table 5: $G(n)$, $G_1(n)$, $0 \leq n \leq 16$

E5 Base 3 representation of n contains no 2's.
This sequence is given recursively by

$$f_2(1) = 1, \quad f_2(2n) = 3f_2(n), \quad f_2(2n+1) = 3f_2(n) + 1, \quad n \geq 1 \tag{35}$$

$f_2(n)$ gives the nth number that can be written as a sum of distinct powers of 3. It is the special case $\alpha = 3 = \gamma$, $\beta = 0$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta = 0$ of (1). Its generating function $\varphi_2(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi_2(s) = 3(1+s)\varphi_2(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi_2(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{3^k \cdot s^{2^k}}{1 + s^{2^k}}, \tag{36}$$

since

$$\begin{aligned} 3(1+s)\varphi_2(s^2) + \frac{s}{1-s^2} &= \frac{3(1+s)}{1-s^2} \sum_{k=0}^{\infty} \frac{3^k \cdot s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1-s^2} \\ &= \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{3^{k+1} \cdot s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1-s^2} \\ &= \frac{1}{1-s} \left(\sum_{k=0}^{\infty} \frac{3^{k+1} \cdot s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1+s} \right) = \varphi_2(s). \end{aligned}$$

The solution of **E5** for $n = (b_m \dots b_0)_2$ is given by

$$f_2((b_m \dots b_0)_2) = (b_m \dots b_0)_3 = \sum_{k=0}^{m} b_k \cdot 3^k. \tag{37}$$

Alternatively, for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$,

$$f_2(2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}) = 3^{a_0} + 3^{a_1} + \cdots + 3^{a_\ell}. \tag{38}$$

The first few values of $f_2(n)$ (the sequence A005836$(n+1)$ in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f_2(n)$	0	1	3	4	9	10	12	13	27	28	30	31	36	37	39	40	81

Table 6: $f_2(n)$, $0 \le n \le 16$

Remark 2. An interesting property of this sequence has been noted by D. Tseng in 2009, namely that $f_2(n) \equiv s_2(n) \pmod 2$, since $3^k \equiv 1 \pmod 2$ for all $k \in [m+1]_0$. By (37) it follows $f_2(n) \equiv \sum_{k=0}^{m} b_k = s_2(n) \pmod 2$. This is the Prouhet-Thue-Morse sequence $m(n)$ on $\{0,1\}$ which we shall now present as Example **E6**.

E6 The Prouhet-Thue-Morse sequence on $\{0,1\}$ (see [3, 14, 26]).
This famous sequence can be obtained in several ways:

1. consider the sequence of the natural numbers $(0,1,2,3,4,5,6,7,8,9,10\ldots)$ in binary form $(0,1,10,11,100,101,110,111,1000,1001,1010,\ldots)$ and build the sum of the digits of these numbers modulo 2, so we obtain the Prouhet-Thue-Morse sequence $(0,1,1,0,1,0,0,1,1,0,0,\ldots)$, that is $m(n) = s_2(n) \bmod 2$.
2. recursively, starting with $m(0) = 0$ and for all $k \in \mathbb{N}_0$, $m(0), m(1), \ldots, m(2^k - 1)$ is followed by its *one's complement*, i.e., $m(2^k) = \overline{m(0)}, \ldots, m(2^{k+1}) = \overline{m(2^k-1)}$. Hence, $0, 01, 0110, 01101001, 0110100110010110, \ldots$
3. by iteration according to the rules $0 \to 01$, $1 \to 10$, that is $0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \cdots$.
4. by the logical function XOR (exclusive or) (see Table 7 for $n = 2$)

x	y	$x\,XOR\,y$
0	0	0
0	1	1
1	0	1
1	1	0

Table 7: Truth table for XOR, x, y propositions, $1 = T(rue)$, $0 = F(alse)$

It satisfies the recurrence relation:

$$m(1) = 1, \quad m(2n) = m(n), \quad m(2n+1) = -m(n) + 1, \quad n \ge 1 \tag{39}$$

This is the special case $\alpha = 1$, $\beta = 0, \gamma = -1$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta := 0$ of (1). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = (1-s)\varphi(s^2) + \frac{s}{1-s^2}.$$

In order to get a simple expression for the generating function $\varphi(s)$ of $m(n)$ we use the transformation

$$m_1(n) := -2m(n) + 1 \tag{40}$$

which gives the Prouhet-Thue-Morse sequence on $\{-1, +1\}$ instead of $\{0, 1\}$. Note that by (40) we have $0 \to 1$ and $1 \to -1$, that is

$$m_1(n) = (-1)^{m(n)}. \tag{41}$$

This sequence **E6'** satisfies the recurrence relation:

$$m_1(1) = -1, \quad m_1(2n) = m_1(n), \quad m_1(2n+1) = -m_1(n), \quad n \geq 1 \tag{42}$$

This is the special case $\alpha = 1$, $\beta = 0 = \varepsilon, \gamma = -1$, $\delta = 0$, $\zeta = -1$ and $\eta = 1$ of (1). Its generating function $\psi(s)$ satisfies, by Proposition 1, the functional equation

$$\psi(s) = (1 - s)\psi(s^2).$$

$\psi(s)$ is Mahlerian and it is given by the infinite product

$$\psi(s) = \prod_{k=0}^{\infty}(1 - s^{2^k}) \tag{43}$$

known as *Euler's product*, since

$$(1 - s)\psi(s^2) = (1 - s)\prod_{k=0}^{\infty}(1 - s^{2^{k+1}}) = \prod_{k=0}^{\infty}(1 - s^{2^k}) = \psi(s).$$

By (41) and (43) we have

$$\sum_{n=0}^{\infty}(-1)^{m(n)}s^n = \prod_{k=0}^{\infty}(1 - s^{2^k}). \tag{44}$$

Using (40) and (43) we obtain the generating function $\varphi(s)$ of $m(n)$. Indeed, we have

$$\begin{aligned}
\varphi(s) &= \sum_{n=0}^{\infty} m(n)s^n = \sum_{n=0}^{\infty}\frac{1 - m_1(n)}{2}s^n \\
&= \frac{1}{2}\cdot\Big(\sum_{n=0}^{\infty}s^n - \sum_{n=0}^{\infty}m_1(n)s^n\Big) = \frac{1}{2}\cdot\Big(\frac{1}{1-s} - \psi(s)\Big) \\
&= \frac{1}{2}\cdot\Big(\frac{1}{1-s} - \prod_{k=0}^{\infty}(1 - s^{2^k})\Big)
\end{aligned}$$

The solution of **E6'** for $n = (b_m \dots b_0)_2$ is given by

$$m_1(n) = (-1)^{\sum_{k=0}^{m} b_k} = (-1)^{s_2(n)} \tag{45}$$

or, for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$, by

$$m_1(2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}) = (-1)^{\ell+1} \tag{46}$$

and that of **E6** is given by

$$m(n) = \frac{1}{2} \cdot \left(1 - (-1)^{s_2(n)}\right) \tag{47}$$

or, for $n = (b_m \dots b_0)_2$, by

$$m((b_m \dots b_0)_2) = \sum_{k=0}^{m} b_k (-1)^{\sum_{j=0}^{k-1} b_j} \tag{48}$$

or, for $n = 2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}$, $a_0 > a_1 > \dots > a_\ell \geq 0$, $\ell \geq 0$, by

$$m(2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}) = \sum_{k=0}^{\ell} (-1)^k = \frac{1}{2} \cdot \left(1 + (-1)^\ell\right) \tag{49}$$

Note that by (49) we obtain $|m(n)| = \left|\sum_{k=0}^{\ell} (-1)^k\right| \leq \sum_{k=0}^{\ell} |(-1)^k| = \sum_{k=0}^{\ell} 1 = \ell + 1 = s_2(n)$, for all $n \in \mathbb{N}$.
The first few values of $m(n)$ (the sequence A010060 in [24]) and $m_1(n)$ (the sequence A106400 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$m(n)$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1
$m_1(n)$	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	-1

Table 8: $m(n), m_1(n)$, $0 \leq n \leq 16$

E7 The *infinite sequence* of Per Nørgård (see [4, Example 23, pp. 21–22] and [5]).
This sequence is given recursively by

$$a(1) = 1, \quad a(2n) = -a(n), \quad a(2n+1) = a(n) + 1, \quad n \geq 1 \tag{50}$$

It is the special case $\alpha = -1$, $\beta = 0, \gamma = 1$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta := 0$ of (1). It is the *infinite sequence* of the Danish composer Per Nørgård (1932–), who invented it in an attempt to unify in a perfect way repetition and variation. Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = -(1-s)\varphi(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi(s) = \sum_{k=0}^{\infty} (-1)^k \frac{s^{2^k} \prod_{j=0}^{k-1} (1 - s^{2^j})}{1 - s^{2^{k+1}}}, \tag{51}$$

since

$$\begin{aligned} -(1-s)\varphi(s^2) + \frac{s}{1-s^2} &= -(1-s) \sum_{k=0}^{\infty} (-1)^k \frac{s^{2^{k+1}} \prod_{j=0}^{k-1} (1 - s^{2^{j+1}})}{1 - s^{2^{k+2}}} + \frac{s}{1-s^2} \\ &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{s^{2^{k+1}} \prod_{j=0}^{k} (1 - s^{2^j})}{1 - s^{2^{k+2}}} + \frac{s}{1-s^2} = \varphi(s). \end{aligned}$$

The solution of **E7** for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$, is given by

$$a(2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}) = \sum_{k=0}^{\ell} (-1)^k \cdot (-1)^{a_{\ell-k}}. \tag{52}$$

It is possible to generate this sequence by a procedure given by J. Mortensen: write n in binary form and read from left to right starting with 0 and interpreting 1 as "add 1" and 0 as "change sign". For example $23 = (10111)_2$ gives $0 \to 1 \to -1 \to 0 \to 1 \to 2$, so $a(23) = 2$. Note that by (52) we obtain $|a(n)| = \left|\sum_{k=0}^{\ell}(-1)^k \cdot (-1)^{a_{\ell-k}}\right| \leq \sum_{k=0}^{\ell} |(-1)^k| \cdot |(-1)^{a_{\ell-k}}| = \sum_{k=0}^{\ell} 1 = \ell+1 = s_2(n)$, for all $n \in \mathbb{N}$.

A variation of **E7**, **E2** and **E6** is the sequence **E7'** defined by

$$a'(1) = 1, \quad a'(2n) = -a'(n), \quad a'(2n+1) = -a'(n) + 1, \quad n \geq 1 \tag{53}$$

It is the special case $\alpha = -1 = \gamma$, $\beta = 0$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta := 0$ of (1). It can be defined as the alternating bit sum (adding from right to left and starting with a positive sign) of the binary expansion of n. For example: $n = 13 = (1101)_2$, hence $a'(13) = 1 - 0 + 1 - 1 = 1$. Its generating function $\varphi_1(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi_1(s) = -(1+s)\varphi_1(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi_1(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} (-1)^k \frac{s^{2^k}}{1+s^{2^k}}, \tag{54}$$

since

$$\begin{aligned} -(1+s)\varphi_1(s^2) + \frac{s}{1-s^2} &= -\frac{1+s}{1-s^2} \sum_{k=0}^{\infty} (-1)^k \frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1-s^2} \\ &= \frac{1}{1-s} \left(\sum_{k=0}^{\infty} (-1)^{k+1} \frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1+s} \right) = \varphi_1(s). \end{aligned}$$

The solution of **E7'** for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $\quad a_0 > a_1 > \cdots > a_\ell \geq 0$, $\quad \ell \geq 0$, is given by

$$a'(2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}) = \sum_{k=0}^{\ell} (-1)^{a_k} \tag{55}$$

This formula can be interpreted as follows: replace in the representation $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$ the terms 2^{a_k} by $(-1)^{a_k}$, $0 \leq k \leq \ell$. As in the case of $a(n)$ we have by (55) $|a'(n)| = \left|\sum_{k=0}^{\ell}(-1)^{a_k}\right| \leq \sum_{k=0}^{\ell} |(-1)^{a_k}| = \sum_{k=0}^{\ell} 1 = \ell + 1 = s_2(n)$, for all $n \in \mathbb{N}$.
The first few values of $a(n)$ (the sequence A004718 in [24]) and $a'(n)$ (the sequence A065359 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a(n)$	0	1	-1	2	1	0	-2	3	-1	2	0	1	2	-1	-3	4	1
$a'(n)$	0	1	-1	0	1	2	0	1	-1	0	-2	-1	0	1	-1	0	1

Table 9: $a(n), a'(n)$, $0 \le n \le 16$

Up to now all sequences were with $\delta = 0$. We shall now consider some example of (1) with $\delta \neq 0$. Perhaps the most simple sequence is

- **E8** Arithmetic progression
  An arithmetic progression is defined recursively for all $n \in \mathbb{N}_0$ by

$$a(0) = a_0, \quad (\Rightarrow a(1) = a_0 + d), \quad a(n+1) = a(n) + d \tag{56}$$

  The solution of (56) is given by

$$a(n) = a_0 + nd, \quad n \ge 0. \tag{57}$$

  This solution satisfies also the recurrence relation

$$\begin{aligned} a(1) = a_0 + d, \quad (\Rightarrow a(0) = a_0) \\ a(2n) = 2a(n) - a_0, \quad a(2n+1) = a(n) + a(n+1) - a_0 \end{aligned} \tag{58}$$

  It is the special case $\alpha = 2$, $\beta = -a_0, \gamma = 1$, $\delta = 1$, $\varepsilon = -a_0$, $\zeta = a_0 + d$ and $\eta = a_0$ of (1). Conversely, the difference between two consecutive terms of (58) is a constant, since for $d(n) := a(n+1) - a(n)$ we have

$$d(1) = d, \quad (\Rightarrow d(0) = d), \quad d(2n) = d(n), \quad d(2n+1) = d(n) \tag{59}$$

  It is the special case $\alpha = 1 = \gamma$, $\beta = 0 = \varepsilon$, $\delta = 0$, $\zeta = d$ and $\eta = d$ of (1) with the solution $d(n) = d$, $n \in \mathbb{N}_0$ (the constant sequence).
  The solution of (59) is given by (57) for $a_0 = -\beta$ and $d = \beta + \zeta$.

- **E9** Stern's diatomic sequence and two variations on it.
  The first $2^n + 1$, $n \in \mathbb{N}_0$, terms of this sequence form the rth row of the so-called *Stern's diatomic array* $\big((0,1)_r\big)_{r \in \mathbb{N}_0}$ defined as follows: $(0,1)_0(0) := 0$, $(0,1)_0(1) := 1$ (0 and 1 are called the *atoms* or the *seeds* of the array) and the next rows are obtained by successively intercalating the sum of two neighbouring numbers. It satisfies the recurrence relation ($r, n \in \mathbb{N}_0$):

$$\begin{aligned} (0,1)_0(0) = 0, \quad (0,1)_0(1) = 1, \\ (0,1)_{r+1}(2n) = (0,1)_r(n), \quad (0,1)_{r+1}(2n+1) = (0,1)_r(n) + (0,1)_r(n+1) \end{aligned} \tag{60}$$

  This array (originated by G. Eisenstein) has been studied in 1858 by M. A. Stern [32] and in 1860 (in a different form) by A. Brocot [7]. Later has been studied, among others, by É. Lucas [22], D. H. Lehmer [20] and D. A. Lind [21]. Stern's diatomic sequence can be defined by

$$s(n) := (0,1)_\infty(n), \quad n \in \mathbb{N}_0. \tag{61}$$

As a sequence it has been defined for the first time in 1947 by G. de Rham [29, p. 95] recursively as follows:

$$s(0) = 0, \quad s(1) = 1, \quad s(2n) = s(n), \quad s(2n+1) = s(n) + s(n+1), \quad n \geq 1 \tag{62}$$

It is the special case $\alpha = 1 = \gamma$, $\beta = 0 = \varepsilon$, $\delta = 1$, $\zeta = 1$ and $\eta = 0$ of (1). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$s\varphi(s) = (1 + s + s^2)\varphi(s^2)$$

which shows that $\varphi(s)$ is Mahlerian. It is given by the infinite product

$$\varphi(s) = s \cdot \prod_{k=0}^{\infty} (1 + s^{2^k} + s^{2^{k+1}}), \tag{63}$$

since

$$\begin{aligned} (1 + s + s^2)\varphi(s^2) &= (1 + s + s^2) \cdot s^2 \cdot \prod_{k=0}^{\infty} (1 + s^{2^{k+1}} + s^{2^{k+2}}) \\ &= s^2 \cdot \prod_{k=0}^{\infty} (1 + s^{2^k} + s^{2^{k+1}}) = s\varphi(s). \end{aligned}$$

This sequence has been studied by many mathematicians, see for example the recent publications by S. Northshield [23] and I. Urbiha [33], because it has remarkable combinatorial interpretations:

1. $s(n+1)$ is the third binary partition function $b(3;n)$, i.e., $s(n+1)$ is the number of representations of $n = \sum_{k=0}^{\infty} \xi \cdot 2^i$, $\xi \in \{0,1,2\}$ (see [28]).
2. in [9, 10] L. Carlitz defined the sequences $\Theta_0(n)$ as the number of odd Stirling numbers of second kind $\left\{ {n \atop m} \right\}$ with even k, $k \leq n$, and $\Theta_1(n)$ as the number of odd Stirling numbers of second kind $\left\{ {n \atop m} \right\}$ with odd k, $k \leq n$, and showed that $\Theta_1(n+1) = \Theta_0(n)$ and that the generating function of $\Theta_0(n)$ is given by $\prod_{k=0}^{\infty}(1 + s^{2^k} + s^{2^{k+1}})$ and consequently that of $\Theta_1(n)$ is given by (63). Moreover, he showed that $\Theta_0(n)$ satisfies the recurrence relation $\Theta_0(0) = 1$, $\Theta_0(1) = 1$, $\Theta_0(2n) = \Theta_0(n) + \Theta_0(n-1)$, $\Theta_0(2n+1) = \Theta_0(n)$, that is $\Theta_0(n) = s(n+1)$ and consequently $s(n) = \Theta_1(n)$.
3. $s(n)$ is the number of odd binomial coefficients in the nth subdiagonal (i.e. $\binom{\ell}{k}$ with $\ell + k = n$, $\ell \geq k \geq 0$) of Pascal's triangle.
4. $s(\mu) = z_n(2^n - \mu)$, $\mu = 0, 1, \ldots, 2^n$, where $z_n(\mu)$ (see [15, p. 305]) is the number of regular states in the graph of the Tower of Hanoi with 3 pegs and n disks, for which the difference of the distances to two distinct perfect states is equal to μ (see [18, Formula 2, p. 700]).

The first few values of $s(n)$ (the sequence A002487 in [24]) are given in Table 10.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s(n)$	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1

Table 10: $s(n)$, $0 \leq n \leq 16$

We consider now two variations on Stern's diatomic sequence. The first variation **E9'** has been introduced in [27] by B. Reznick and it is called *Reznick's sequence*. It is given recursively by

$$r(1) = 1, \quad r(2n) = r(n), \quad r(2n+1) = -r(n) + r(n+1), \quad n \geq 1 \tag{64}$$

It is the special case $\alpha = 1$, $\beta = 0 = \varepsilon, \gamma = -1$, $\delta = 1$, $\zeta = 1$ and $\eta := 0$ of (1). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$s\varphi(s) = (1 + s - s^2)\varphi(s^2)$$

which shows that $\varphi(s)$ is Mahlerian. It is given by the infinite product

$$\varphi(s) = s \prod_{k=0}^{\infty} (1 + s^{2^k} - s^{2^{k+1}}), \tag{65}$$

since

$$\begin{aligned}(1 + s - s^2)\varphi(s^2) &= (1 + s - s^2) \cdot s^2 \prod_{k=0}^{\infty} (1 + s^{2^{k+1}} - s^{2^{k+2}}) \\ &= s^2 \prod_{k=0}^{\infty} (1 + s^{2^k} - s^{2^{k+1}}) = s\varphi(s).\end{aligned}$$

The second variation **E9"** called *the twisted Stern sequence* has been introduced in [6] by R. Bacher and later studied by J. P. Allouche [1] and M. Coons [11]. It is given recursively by

$$t(1) = 1, \quad t(2n) = -t(n), \quad t(2n+1) = -t(n) - t(n+1), \quad n \geq 1 \tag{66}$$

It is the special case $\alpha = -1 = \gamma$, $\beta = 0 = \varepsilon$, $\delta = -1$, $\zeta = 1$ and $\eta := 0$ of (1). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$s\varphi(s) = -(1 + s + s^2)\varphi(s^2) + 2s^2$$

and it is given by

$$\varphi(s) = s \prod_{k=0}^{\infty} (-1)^k (1 + s^{2^k} + s^{2^{k+1}}) + 2s \sum_{n=0}^{\infty} (-1)^n \Big(\prod_{k=0}^{n-1} (1 + s^{2^k} + s^{2^{k+1}}) \Big), \tag{67}$$

since

$$-(1+s+s^2)\varphi(s^2)+2s^2=(1+s+s^2)\Bigg(s^2\prod_{k=0}^{\infty}(-1)^{k+1}(1+s^{2^{k+1}}+s^{2^{k+2}})+$$
$$+2s^2\sum_{n=0}^{\infty}(-1)^{n+1}\Big(\prod_{k=0}^{n-1}(1+s^{2^{k+1}}+s^{2^{k+2}})\Big)\Bigg)+2s^2$$
$$=s^2\prod_{k=0}^{\infty}(-1)^k(1+s^{2^k}+s^{2^{k+1}})+$$
$$+2s^2\Bigg(1+\sum_{n=0}^{\infty}(-1)^{n+1}\Big(\prod_{k=0}^{n}(1+s^{2^k}+s^{2^{k+1}})\Big)\Bigg)$$

and this is exactly $s\varphi(s)$, as the expression between the parentheses in the last term of the above equation is equal to $\sum_{n=0}^{\infty}(-1)^n\big(\prod_{k=0}^{n-1}(1+s^{2^k}+s^{2^{k+1}})\big)$.
The first few values of $r(n)$ (the sequence A005590 in [24]) and $t(n)$ (the sequence A213369 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$r(n)$	0	1	1	0	1	-1	0	1	1	-2	-1	1	0	1	1	0	1
$t(n)$	0	1	-1	0	1	1	0	-1	-1	-2	-1	-1	0	1	1	2	1

Table 11: $r(n)$, $t(n)$, $0 \le n \le 16$

- **E10** Tennis Tournament
  There are n participants to a (knock-out) tennis tournament. In the first round, all the losers are eliminated. The winners play in pairs in the second round, the losers are eliminated again, and so on. How many rounds $T(n)$ do we need to determine the winner?
  This sequence is given recursively for all $n \in \mathbb{N}$ by

$$T(1)=0,\quad T(2n)=T(n)+1,\quad T(2n+1)=T(n+1)+1 \tag{68}$$

It is the special case $\alpha=1$, $\beta=1=\varepsilon, \gamma=0$, $\delta=1$, $\zeta=0$ and $\eta:=0$ of (1). Equivalently, **E10** can be written as

$$T(1)=0,\quad T(n)=T\big(\lceil n/2\rceil\big)+1,\quad n\ge 2 \tag{69}$$

Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$s\varphi(s)=(1+s)\varphi(s^2)+\frac{s^3}{1-s}$$

and it is given by

$$\varphi(s)=\frac{s}{1-s}\sum_{k=0}^{\infty}s^{2^k}, \tag{70}$$

since

$$\begin{aligned}(1+s)\varphi(s^2)+\frac{s^3}{1-s}&=(1+s)\cdot\frac{s^2}{1-s^2}\cdot\sum_{k=0}^{\infty}s^{2^{k+1}}+\frac{s^3}{1-s}\\&=\frac{s^2}{1-s}\Big(\sum_{k=0}^{\infty}s^{2^{k+1}}+s\Big)=s\varphi(s)\end{aligned}$$

The solution of **E10** for $n=2^{a_0}+2^{a_1}+\cdots+2^{a_\ell}$, $a_0>a_1>\cdots>a_\ell\geq 0$, $\ell\geq 0$, is given by

$$T(2^{a_0})=a_0,\quad T(2^{a_0}+2^{a_1}+\cdots+2^{a_\ell})=a_0+1,\quad \ell\geq 1, \tag{71}$$

and since $a_0=\lfloor\log_2 n\rfloor$ we can also write

$$T(n)=\lceil\log_2 n\rceil,\quad n\geq 1. \tag{72}$$

The first few values of $T(n)$ (the sequence A029837 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$T(n)$	0	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4

Table 12: $T(n)$, $1\leq n\leq 16$

3 Solution of the special case $\delta=0$

We shall now study the special case $\delta=0$ of (1), that is

$$f(1)=\zeta,\quad f(2n)=\alpha f(n)+\beta,\quad f(2n+1)=\gamma f(n)+\varepsilon,\quad n\geq 1 \tag{73}$$

with $\alpha,\beta,\gamma,\varepsilon,\zeta\in\mathbb{Z}$, since the general case $\delta\neq 0$ can be reduced to (73).

Theorem 1. *The solution of (73) for $n=(b_m\dots b_0)_2$ is given by*

$$f(n)=\zeta\prod_{k=0}^{m-1}\Big(\gamma b_k+\alpha(1-b_k)\Big)+\sum_{k=0}^{m-1}\Big(\prod_{j=0}^{k-1}\Big(\gamma b_j+\alpha(1-b_j)\Big)\Big)\cdot\Big(\varepsilon b_k+\beta(1-b_k)\Big) \tag{74}$$

Proof. Let $n=(b_m\dots b_0)_2$. Then by (73):
if $b_0=0$ we have $f\Big((b_m\dots b_0)_2\Big)=\alpha f\Big((b_m\dots b_1)_2\Big)+\beta$ and
if $b_0=1$ we have $f\Big((b_m\dots b_0)_2\Big)=\gamma f\Big((b_m\dots b_1)_2\Big)+\varepsilon$.
Hence,

$$\begin{aligned}f\Big((b_m\dots b_0)_2\Big)&=b_0\cdot\Big(\gamma f\Big((b_m\dots b_1)_2\Big)+\varepsilon\Big)+(1-b_0)\cdot\Big(\alpha f\Big((b_m\dots b_1)_2\Big)+\beta\Big)\\&=\Big(\gamma b_0+\alpha(1-b_0)\Big)\cdot f\Big((b_m\dots b_1)_2\Big)+\Big(\varepsilon b_0+\beta(1-b_0)\Big).\end{aligned}$$

Similarly, $f\Big((b_m \dots b_1)_2\Big) = \Big(\gamma b_1 + \alpha(1-b_1)\Big) \cdot f\Big((b_m \dots b_2)_2\Big) + \Big(\varepsilon b_1 + \beta(1-b_1)\Big)$. Substituting this expression into the above equation gives

$$\begin{aligned} f\Big((b_m \dots b_0)_2\Big) &= \Big(\gamma b_0 + \alpha(1-b_0)\Big) \cdot \Big(\gamma b_1 + \alpha(1-b_1)\Big) \cdot f\Big((b_m \dots b_2)_2\Big) + \\ &+ \Big(\gamma b_0 + \alpha(1-b_0)\Big) \cdot \Big(\varepsilon b_1 + \beta(1-b_1)\Big) + \Big(\varepsilon b_0 + \beta(1-b_0)\Big) \end{aligned}$$

By iteration and noting that $f\Big((b_m)_2\Big) = f(1) = \zeta$ one obtains formula (74) and this proves the theorem. □

The next corollary gives the exact values of $f(n)$ for some special numbers.

Corollary 1. *We have*

$$f(2^m) = f\Big((10\dots0)_2\Big) = \zeta\alpha^m + \beta\sum_{k=0}^{m-1}\alpha^k, \quad m \geq 0 \tag{75}$$

$$f(2^m+1) = f\Big((10\dots01)_2\Big) = \zeta\gamma\alpha^{m-1} + \beta\gamma\sum_{k=0}^{m-2}\alpha^k + \varepsilon, \quad m \geq 1 \tag{76}$$

$$f(2^{m+1}-1) = f\Big((11\dots1)_2\Big) = \zeta\gamma^m + \varepsilon\sum_{k=0}^{m-1}\gamma^k, \quad m \geq 0 \tag{77}$$

In the next theorem we present an alternative expression of the solution of (73).

Theorem 2. *Let* $n = 2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}$, $a_0 > a_1 > \dots > a_\ell \geq 0$, $\ell \geq 0$, *a strictly decreasing sequence of positive integers. Then the solution of (73) for* $\alpha \neq 0$ *is given by*

$$f(n) = \Big(\frac{\gamma}{\alpha}\Big)^\ell \cdot \Big(\zeta\alpha^{a_0} + \beta\sum_{k=0}^{a_0-1}\alpha^k\Big) + \sum_{k=0}^{\ell-1}\Big(\frac{\gamma}{\alpha}\Big)^k \cdot \Big((\varepsilon-\beta)\alpha^{a_{\ell-k}} + \beta\Big(1-\frac{\gamma}{\alpha}\Big)\sum_{j=0}^{a_{\ell-k}}\alpha^j\Big) \tag{78}$$

and for $\alpha = 0$ *by*

$$f(n) = \beta + \gamma^\ell(\zeta-\beta)\prod_{k=0}^{\ell}[a_{\ell-k} = k] + (\beta\gamma + \varepsilon - \beta)\sum_{k=0}^{\ell-1}\gamma^k\Bigg(\prod_{j=0}^{k}[a_{\ell-j} = j]\Bigg), \tag{79}$$

where $[A]$ *is Iverson's convention meaning 1 if the assertion* A *is true and 0 if* A *is false.*

Proof. 1) Let $\alpha \neq 0$ and $A_i := f(2^{a_0} + \dots + 2^{a_i})$, $i \in [\ell+1]_0$, $a_0 > a_1 > \dots > a_\ell \geq 0$, a strictly decreasing sequence of positive integers. Then $f(2^{a_0+1}) = \alpha \cdot f(2^{a_0}) + \beta$ with the initial value $f(2^0) = f(1) = \zeta$. This is an arithmetic progression with the solution

$$f(2^{a_0}) = f(1) \cdot \alpha^{a_0} + \beta\sum_{k=0}^{a_0-1}\alpha^k = \zeta\alpha^{a_0} + \beta\sum_{k=0}^{a_0-1}\alpha^k.$$

Now let $i \geq 0$, then by repeated use of (1)

$$\begin{aligned} A_{i+1} &= f(2^{a_0} + \dots + 2^{a_{i+1}}) = f\Big(2^{a_{i+1}} \cdot (2^{a_0-a_{i+1}} + \dots + 2^{a_i-a_{i+1}} + 1)\Big) \\ &= \alpha^{a_{i+1}} \cdot f(2^{a_0-a_{i+1}} + \dots + 2^{a_i-a_{i+1}} + 1) + \beta\sum_{k=0}^{a_{i+1}-1}\alpha^k \\ &= \alpha^{a_{i+1}} \cdot \Big(\gamma \cdot f(2^{a_0-a_{i+1}-1} + \dots + 2^{a_i-a_{i+1}-1}) + \varepsilon\Big) + \beta\sum_{k=0}^{a_{i+1}-1}\alpha^k, \end{aligned}$$

that is

$$A_{i+1} = \gamma\alpha^{a_{i+1}} \cdot f(2^{a_0-a_{i+1}-1} + \cdots + 2^{a_i-a_{i+1}-1}) + \varepsilon\alpha^{a_{i+1}} + \beta \sum_{k=0}^{a_{i+1}-1} \alpha^k. \tag{80}$$

We have

$$\begin{aligned} f(2^{a_0} + \cdots + 2^{a_i}) &= f\Big(2^{a_{i+1}+1} \cdot (2^{a_0-a_{i+1}-1} + \cdots + 2^{a_i-a_{i+1}-1})\Big) \\ &= \alpha^{a_{i+1}+1} \cdot f(2^{a_0-a_{i+1}-1} + \cdots + 2^{a_i-a_{i+1}-1}) + \beta \sum_{k=0}^{a_{i+1}} \alpha^k \end{aligned}$$

or solving for $f(2^{a_0-a_{i+1}-1} + \cdots + 2^{a_i-a_{i+1}-1})$ and since $\alpha \neq 0$ it follows $f(2^{a_0-a_{i+1}-1} + \cdots + 2^{a_i-a_{i+1}-1}) = \Big(f(2^{a_0} + \cdots + 2^{a_i}) - \beta \sum_{k=0}^{a_{i+1}} \alpha^k\Big)/\alpha^{a_{i+1}+1}$. Inserting this formula into (80) yields

$$A_{i+1} = \frac{\gamma}{\alpha}A_i - \frac{\beta\gamma}{\alpha}\sum_{k=0}^{a_{i+1}} \alpha^k + \varepsilon\alpha^{a_{i+1}} + \beta\sum_{k=0}^{a_{i+1}} \alpha^k - \beta\alpha^{a_{i+1}} = \frac{\gamma}{\alpha}A_i - g(i+1),$$

where $g(i+1) := \beta\Big(1 - \frac{\gamma}{\alpha}\Big)\sum_{k=0}^{a_{i+1}} \alpha^k + (\varepsilon - \beta)\alpha^{a_{i+1}}$.

The solution of this arithmetic progression is given by

$$\begin{aligned} A_\ell &= A_0 \cdot \left(\frac{\gamma}{\alpha}\right)^\ell + \sum_{k=0}^{\ell-1}\left(\frac{\gamma}{\alpha}\right)^k \cdot g(\ell - k) \\ &= \left(\frac{\gamma}{\alpha}\right)^\ell \cdot \left(\zeta\alpha^{a_0} + \beta\sum_{k=0}^{a_0-1}\alpha^k\right) + \sum_{k=0}^{\ell-1}\left(\frac{\gamma}{\alpha}\right)^k \cdot \left((\varepsilon-\beta)\alpha^{a_{\ell-k}} + \beta\Big(1 - \frac{\gamma}{\alpha}\Big)\sum_{j=0}^{a_{\ell-k}}\alpha^j\right) \end{aligned}$$

and this is formula (78).
2) Let $\alpha = 0$, then we shall prove (79) by induction on $\ell \in \mathbb{N}_0$.
Base step: For $\ell = 0$ we have by (73) : if $a_0 = 0 :\ f(2^0) = \zeta$ and if $a_0 \neq 0 :\ f(2^{a_0}) = \beta$ or $f(2^{a_0}) = (a_0 = 0) \cdot \zeta + (1 - (a_0 = 0)) \cdot \beta = \beta + (\zeta - \beta) \cdot (a_0 = 0)$, and this is (79) for $\ell = 0$.
Induction step: By (73) we have: $f(2^{a_0} + \cdots + 2^{a_{\ell+1}}) = \gamma f(2^{a_0-1} + \cdots + 2^{a_\ell-1}) + \varepsilon$, if $a_{l+1} = 0$ and $f(2^{a_0} + \cdots + 2^{a_{\ell+1}}) = \beta$, if $a_{l+1} \neq 0$ or

$$\begin{aligned} f(2^{a_0} + \cdots + 2^{a_\ell}) &= (a_{\ell+1} = 0) \cdot \Big(\gamma f(2^{a_0-1} + \cdots + 2^{a_\ell-1}) + \varepsilon\Big) + (1 - (a_{\ell+1} = 0)) \cdot \beta \\ &= \beta + \Big(\gamma f(2^{a_0-1} + \cdots + 2^{a_\ell-1}) + \varepsilon\Big)(a_{\ell+1} = 0) \end{aligned} \tag{81}$$

By induction assumption (with $a_0 - 1, \ldots, a_\ell - 1$ instead of $a_0, \ldots, a_\ell$) we have

$$f(2^{a_0-1} + \cdots + 2^{a_\ell-1}) = \beta + \gamma^\ell(\zeta - \beta)\prod_{k=0}^{\ell}[a_{\ell-k} = k+1] + (\beta\gamma + \varepsilon - \beta)\sum_{k=0}^{\ell-1}\gamma^k\Big(\prod_{j=0}^{k}[a_{\ell-j} - 1 = j]\Big).$$

Hence, inserting this equation into (81) we obtain

$$f(2^{a_0}+\cdots+2^{a_{\ell+1}})=\beta+\Bigg(\gamma\cdot\Bigg(\beta+\gamma^\ell(\zeta-\beta)\prod_{k=0}^{\ell}[a_{\ell-k}=k]+ \\ +(\beta\gamma+\varepsilon-\beta)\sum_{k=0}^{\ell-1}\gamma^k\Big(\prod_{j=0}^{k}[a_{\ell-j}=j]\Big)\Bigg)+\varepsilon-\beta\Bigg)\cdot[a_{\ell+1}=0] \\ =\beta+\gamma^{\ell+1}(\zeta-\beta)\cdot[a_{\ell+1}=0]\cdot\prod_{k=0}^{\ell}[a_{\ell-k}=k+1]+ \\ +(\beta\gamma+\varepsilon-\beta)\cdot[a_{\ell+1}=0]\cdot\Bigg(1+\sum_{k=0}^{\ell-1}\gamma^{k+1}\Big(\prod_{j=0}^{k}[a_{\ell-j}=j+1]\Big)\Bigg)$$

and this is (79) for $\ell+1$ instead of ℓ, since

$$[a_{l+1}=0]\prod_{k=0}^{\ell}[a_{\ell-k}=k+1]=[a_{l+1}=0]\cdot[a_l=1]\cdot[a_{l-1}=2]\cdots[a_0=\ell+1] \\ =\prod_{k=0}^{\ell+1}[a_{\ell+1-k}=k]$$

and

$$[a_{l+1}=0]\cdot\Bigg(1+\sum_{k=0}^{\ell-1}\gamma^{k+1}\Big(\prod_{j=0}^{k}[a_{\ell-j}=j+1]\Big)\Bigg)= \\ =[a_{l+1}=0]\big(1+\gamma[a_l=1]+\gamma^2[a_l=1][a_{l-1}=2]+\cdots+\gamma^\ell[a_l=1][a_{l-1}=2]\cdots[a_1=\ell]\big) \\ =\gamma^0[a_{l+1}=0]+\gamma[a_{l+1}=0][a_l=1]+\gamma^2[a_{l+1}=0][a_l=1][a_{l-1}=2]+\cdots+ \\ +\gamma^\ell[a_{l+1}=0][a_l=1][a_{l-1}=2]\cdots[a_1=\ell]=\sum_{k=0}^{\ell}\gamma^k\Big(\prod_{j=0}^{k}[a_{\ell+1-j}=j]\Big).$$

By induction (79) is true and this proves Theorem 2. □

All sequences **E1** ..... **E8** considered in Chapter 2 are special cases of (73). We shall now mention some other interesting special cases of (73).

1. Choosing $\alpha=0$, $\beta=1=\varepsilon$, $\gamma=1$, $\zeta=1$ we obtain the recurrence relation

$$d_c(1)=1,\quad d_c(2n)=1,\quad d_c(2n+1)=d_c(n)+1,\quad n\geq 1 \tag{82}$$

with the solution

$$d_c\big((b_m\dots b_0)_2\big)=\sum_{k=0}^{m}\Big(\prod_{j=0}^{k-1}b_j\Big) \tag{83}$$

Note that by (19) and (83) we have $d_c(n)=d_2(c(n))$, $n\geq 1$, where $c(n)$ is the one's complement of $n=(b_m\dots b_0)_2$.
The first few values of $d_c(n)$ (the sequence A091090 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_c(n)$	1	1	2	1	2	1	3	1	2	1	3	1	2	1	4	1

Table 13: $d_c(n)$, $1 \le n \le 16$

2. Choosing $\alpha = 1$, $\beta = 1 = \varepsilon$, $\gamma = 2$, $\zeta = 1$ we obtain the recurrence relation

$$f(1) = 1, \quad f(2n) = f(n) + 1, \quad f(2n+1) = 2f(n) + 1, \quad n \ge 1 \tag{84}$$

with the solution given by (74)

$$f\Big((b_m \dots b_0)_2\Big) = \sum_{k=0}^{m} \left(\prod_{j=0}^{k-1} (b_j + 1) \right) = \sum_{k=0}^{m} 2^{\sum_{k=0}^{k-1} b_j}, \tag{85}$$

since $b_j + 1 = 2^{b_j}$, $b_j \in \{0, 1\}$.
Alternatively, by (78)

$$f\Big(2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}\Big) = 2^\ell a_0 + 1 - \sum_{k=0}^{\ell-1} 2^k a_{\ell-k}. \tag{86}$$

The first few values of $f(n)$ (the sequence A135533 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	1	2	3	3	5	4	7	4	7	6	11	5	11	8	15	5

Table 14: $f(n)$, $1 \le n \le 16$

3. The special case $\alpha = \gamma$ leads by (74) to the solution ($n = (b_m \dots b_0)_2$)

$$f(n) = \zeta\alpha^m + \sum_{k=0}^{m-1} \alpha^k \Big(\varepsilon b_k + \beta(1 - b_k)\Big) = \zeta\alpha^m + (\varepsilon - \beta) \sum_{k=0}^{m-1} \alpha^k b_k + \beta \sum_{k=0}^{m-1} \alpha^k \tag{87}$$

This formula shows that the solution of (73) for $\alpha = \gamma$ and $\beta = \varepsilon$ depends only on $m = \lfloor \log_2 n \rfloor$. Perhaps, one of the most important of these cases is for $\alpha = 1 = \gamma$, $\beta = 1 = \varepsilon$ and $\zeta = 1$, that is the recurrence relation

$$B(1) = 1, \quad B(2n) = B(n) + 1, \quad B(2n+1) = B(n) + 1, \quad n \ge 1 \tag{88}$$

or, equivalently,

$$B(1) = 1, \quad B(n) = B\Big(\lfloor n/2 \rfloor\Big) + 1, \quad n \ge 2 \tag{89}$$

with the solution

$$B(n) = 1 + \lfloor \log_2 n \rfloor. \tag{90}$$

This sequence is, for $n \ge 1$, equal to the number of bits in the binary representation of n, that is $B(n) = s_2(n) + e_0(n)$, $n \in \mathbb{N}_0$, (see Examples **E2** and **E2'**). It gives also the number of comparisons in the worst case with *binary search* in an (ordered) list of size n (see [30, pp. 64–65]). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = (1+s)\varphi(s^2) + \frac{s}{1-s}$$

and it is given by

$$\varphi(s) = \frac{1}{1-s} \cdot \sum_{k=0}^{\infty} s^{2^k}, \tag{91}$$

since

$$\begin{aligned}(1+s)\varphi(s^2) + \frac{s}{1-s} &= \frac{1+s}{1-s^2} \cdot \sum_{k=0}^{\infty} s^{2^{k+1}} + \frac{s}{1-s} \\ &= \frac{1}{1-s}\Big(\sum_{k=0}^{\infty} s^{2^{k+1}} + s\Big) = \varphi(s).\end{aligned}$$

The generating function (91) is the Cauchy product of two power series, namely

$$\frac{1}{1-s} \cdot \sum_{k=0}^{\infty} s^{2^k} = \Big(\sum_{k=0}^{\infty} s^n\Big) \cdot \Big(\sum_{k=1}^{\infty} \rho(n) s^n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=1}^{n} \rho(k)\Big) s^n,$$

that is $B(n) = \sum_{k=1}^{n} \rho(k)$ is the sequence of the partial sums of $\rho(n)$, where $\rho(n)$ is the characteristic function of the powers of 2 (cf. Example **E3** at p. 8). Note also that $B(n) = T(n+1)$, $n \geq 0$, where $T(n)$ is the sequence **E10**.
The first few values of $\rho(n)$ (the sequence A209229 in [24]) and $B(n)$ (the sequence A029837 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\rho(n)$	-	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1
$B(n)$	0	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4	5

Table 15: $\rho(n), B(n)$, $0 \leq n \leq 16$

4 Solution of the general case $\delta \neq 0$

Let $\vec{f}(n) := \begin{pmatrix} f(n) \\ f(n+1) \end{pmatrix}$, then $\vec{f}(1) = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix} = \begin{pmatrix} \zeta \\ \alpha\zeta+\beta \end{pmatrix}$, $\vec{f}(2n) = \begin{pmatrix} f(2n) \\ f(2n+1) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \vec{f}(n) + \begin{pmatrix} \beta \\ \varepsilon \end{pmatrix}$ and $\vec{f}(2n+1) = \begin{pmatrix} f(2n+1) \\ f(2n+2) \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ 0 & \alpha \end{pmatrix} \vec{f}(n) + \begin{pmatrix} \varepsilon \\ \beta \end{pmatrix}$.
In this way Eq.(1) is equivalent to

$$\vec{f}(1) = \begin{pmatrix} \zeta \\ \alpha\zeta+\beta \end{pmatrix}, \quad \vec{f}(2n) = A \cdot \vec{f}(n) + \vec{b}, \quad \vec{f}(2n+1) = B \cdot \vec{f}(n) + C\vec{b} \tag{92}$$

where $A := \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$, $B := \begin{pmatrix} \gamma & \delta \\ 0 & \alpha \end{pmatrix}$, $C := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{b} := \begin{pmatrix} \beta \\ \varepsilon \end{pmatrix}$ and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{Z}$.
Before giving the general solution of (92) we shall determine the values of this sequence for the numbers $n = 2^m$, $2^m + 1$, $2^{m+1} - 1$.

Lemma 1. *Let $m \geq 0$, then*

$$f(2^m) = \zeta\alpha^m + \beta \sum_{k=0}^{m-1} \alpha^k \tag{93}$$

$$f(2^m+1) = \delta^m(\alpha\zeta+\beta) + \sum_{k=0}^{m-1} \delta^k \Big(\gamma\zeta\alpha^{m-1-k} + \beta\gamma \sum_{j=0}^{m-2-k} \alpha^j + \varepsilon\Big) \tag{94}$$

$$f(2^{m+1}-1) = \zeta\gamma^m + \sum_{k=0}^{m-1} \gamma^k \Big(\delta\zeta\alpha^{m-k} + \beta\delta \sum_{j=0}^{m-1-k} \alpha^j + \varepsilon\Big) \tag{95}$$

Proof. 1) Let $A(m) := f(2^m)$, $A(0) = f(1) = \zeta$, then by (1):
$A(m) = \alpha f(2^{m-1}) + \beta = \alpha A(m-1) + \beta$. This is an arithmetic progression with constant coefficients. Its solution is given by $A(m) = A(0) \cdot \alpha^m + \sum_{k=0}^{m-1} \alpha^k \beta$ and this proves (93).

2) Let $B(m) := f(2^m+1)$, $B(0) = f(2) = \alpha\zeta + \beta$, then by (1):
$B(m) = \gamma f(2^{m-1}) + \delta f(2^{m-1}+1) + \varepsilon = \gamma A(m-1) + \delta B(m-1) + \varepsilon$.
Inserting (93) into this equation we obtain the arithmetic progression
$B(m) = \delta B(m-1) + g(m-1)$, where $g(m-1) := \gamma \cdot \Big(\zeta\alpha^{m-1} + \beta \sum_{k=0}^{m-2} \alpha^k\Big) + \varepsilon$ with the solution $B(m) = B(0) \cdot \delta^m + \sum_{k=0}^{m-1} \delta^k g(m-1-k)$ and this proves (94).

3) Let $C(m) := f(2^{m+1}-1) = f(2^m + \cdots + 2^1 + 2^0), C(0) = f(1) = \zeta$, then by (1):
$C(m) = \gamma f(2^{m-1} + \cdots + 2^0) + \delta f(2^{m-1} + \cdots + 2^0 + 1) + \varepsilon = \gamma C(m-1) + \delta f(2^m) + \varepsilon = \gamma C(m-1) + \delta A(m) + \varepsilon$.
Inserting (93) into this equation we obtain the arithmetic progression
$C(m) = \gamma C(m-1) + h(m)$, where $h(m) := \delta \cdot \Big(\zeta\alpha^m + \beta \sum_{j=0}^{m-1} \alpha^j\Big) + \varepsilon$ with the solution $C(m) = C(0) \cdot \gamma^m + \sum_{k=0}^{m-1} \gamma^k g(m-k)$ and this proves (95). □

We shall now determine the general solution of (92).

Theorem 3. *Let $n = (b_m \ldots b_0)_2$, then the solution of (92) is given by*

$$\begin{aligned}\vec{f}\Big((b_m \ldots b_0)_2\Big) &= M(b_0) \cdot M(b_1) \cdots M(b_{m-1}) \cdot \begin{pmatrix} \zeta \\ \alpha\zeta+\beta \end{pmatrix} + \\ &+ \Big(M(b_0) \cdots M(b_{m-2}) \cdot C^{b_{m-1}} + \cdots + M(b_0) \cdot C^{b_1} + C^{b_0}\Big) \cdot \begin{pmatrix} \beta \\ \varepsilon \end{pmatrix},\end{aligned} \tag{96}$$

where $M(b_k) := A^{1-b_k} \cdot B^{b_k} = (1-b_k)A + b_k B$, $k \in [m]_0$, i.e., $M(0) = A$, $M(1) = B$ and $C^0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 unit matrix E_2.

Proof. Let $n = (b_m \ldots b_0)_2$. Then by (92):
if $b_0 = 0$ we have $\vec{f}\Big((b_m \ldots b_0)_2\Big) = A \cdot \vec{f}\Big((b_m \ldots b_1)_2\Big) + \vec{b}$ and
if $b_0 = 1$ we have $\vec{f}\Big((b_m \ldots b_0)_2\Big) = B \cdot \vec{f}\Big((b_m \ldots b_1)_2\Big) + C \cdot \vec{b}$.
Hence,

$$\begin{aligned}\vec{f}\Big((b_m\dots b_0)_2\Big) &= (1-b_0)\cdot\Big(A\cdot\vec{f}\Big((b_m\dots b_1)_2\Big)+\vec{b}\Big)+b_0\cdot\Big(B\cdot\vec{f}\Big((b_m\dots b_1)_2\Big)+C\cdot\vec{b}\Big)\\ &= \Big((1-b_0)A+b_0B\Big)\cdot\vec{f}\Big((b_m\dots b_1)_2\Big)+\Big((1-b_0)\vec{b}+b_0\cdot C\vec{b}\Big)\\ &= M(b_0)\cdot\vec{f}\Big((b_m\dots b_1)_2\Big)+C^{b_0}\cdot\vec{b}.\end{aligned}$$

Similarly, $\vec{f}\Big((b_m\dots b_1)_2\Big) = M(b_1)\cdot\vec{f}\Big((b_m\dots b_2)_2\Big)+C^{b_1}\cdot\vec{b}$.
Substituting this expression into the above equation gives

$$\vec{f}\Big((b_m\dots b_0)_2\Big) = M(b_0)\cdot M(b_1)\cdot\vec{f}\Big((b_m\dots b_2)_2\Big)+\Big(M(b_0)\cdot C^{b_1}+C^{b_0}\Big)\cdot\vec{b}$$

By iteration and noting that $\vec{f}\Big((b_m)_2\Big)=\vec{f}(1)=\begin{pmatrix}\zeta\\ \alpha\zeta+\beta\end{pmatrix}$ one obtains formula (96) and this proves the theorem. □

We now apply formula (96) to the sequences **E9**, **E9'**and **E9"**.

1. For **E9** we have $\alpha=1=\gamma$, $\beta=0=\varepsilon$, $\delta=1$, $\zeta=1$. Hence $A=\begin{pmatrix}1&0\\1&1\end{pmatrix}$, $B=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $\vec{b}:=\begin{pmatrix}0\\0\end{pmatrix}$, $\vec{s}(1)=\begin{pmatrix}1\\1\end{pmatrix}$ and $M(b_k)=\begin{pmatrix}1&b_k\\1-b_k&1\end{pmatrix}$, $k\in[m]_0$. The solution is given by

$$\vec{s}\Big((b_m\dots b_0)_2\Big)=\begin{pmatrix}1&b_0\\1-b_0&1\end{pmatrix}\cdot\begin{pmatrix}1&b_1\\1-b_1&1\end{pmatrix}\cdots\begin{pmatrix}1&b_{m-1}\\1-b_{m-1}&1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}\quad(97)$$

   For example, if $n=13=(1101)_2$, then

$$\vec{s}(13)=\begin{pmatrix}s(13)\\s(14)\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2&3\\1&2\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$$

   that is $s(13)=5$ and $s(14)=3$.

   Remark 3. We have $B=CAC$ and since $C^2=E_2$ it follows $BC=CA=\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}F_2&F_1\\F_1&F_0\end{pmatrix}=:Q$, where Q is the Q-matrix that generates the Fibonacci numbers F_n, since $Q^n=\begin{pmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{pmatrix}$.

2. For **E9'** we have $\alpha=1$, $\beta=0=\varepsilon,\gamma=-1$, $\delta=1$, $\zeta=1$. Hence $A=\begin{pmatrix}1&0\\-1&1\end{pmatrix}$, $B=\begin{pmatrix}-1&1\\0&1\end{pmatrix}$, $\vec{b}:=\begin{pmatrix}0\\0\end{pmatrix}$, $\vec{r}(1)=\begin{pmatrix}1\\1\end{pmatrix}$ and $M(b_k)=\begin{pmatrix}1-2b_k&b_k\\b_k-1&1\end{pmatrix}$, $k\in[m]_0$. The solution is given by

$$\vec{r}\Big((b_m\dots b_0)_2\Big)=\begin{pmatrix}1-2b_0&b_0\\b_0-1&1\end{pmatrix}\cdot\begin{pmatrix}1-2b_1&b_1\\b_1-1&1\end{pmatrix}\cdots\begin{pmatrix}1-2b_{m-1}&b_{m-1}\\b_{m-1}-1&1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}\quad(98)$$

For example, if $n = 13 = (1101)_2$, then

$$\begin{aligned}\vec{r}(13) = \begin{pmatrix} r(13) \\ r(14) \end{pmatrix} &= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}\end{aligned}$$

that is $r(13) = 1$ and $r(14) = 1$.

3. For **E9”** we have $\alpha = -1 = \gamma$, $\beta = 0 = \varepsilon$, $\delta = -1$, $\zeta = 1$. Hence $A = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$, $\vec{b} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\vec{t}(1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $M(b_k) = \begin{pmatrix} -1 & -b_k \\ b_k - 1 & -1 \end{pmatrix}$, $k \in [m]_0$. The solution is given by

$$\vec{t}\big((b_m \dots b_0)_2\big) = \begin{pmatrix} -1 & -b_0 \\ b_0 - 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & -b_1 \\ b_1 - 1 & -1 \end{pmatrix} \cdots \begin{pmatrix} -1 & -b_{m-1} \\ b_{m-1} - 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{99}$$

For example, if $n = 13 = (1101)_2$, then

$$\begin{aligned}\vec{t}(13) = \begin{pmatrix} t(13) \\ t(14) \end{pmatrix} &= \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -3 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\end{aligned}$$

that is $t(13) = 1$ and $t(14) = 1$.

For $\beta = 0 = \varepsilon$ and $\alpha \neq 0$ the sequence (92) seems to satisfy a second-order linear recurrence relation with variable coefficients. We conjecture the following statement.

Conjecture 1. *Let $\beta = 0 = \varepsilon$, $\alpha \neq 0$ in (92) and $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$, be a strictly decreasing sequence of positive integers. Then*

$$f(2^{a_0}) = \zeta \cdot \alpha^{a_0}, \quad f(2^{a_0} + 2^{a_1}) = \zeta \cdot \alpha^{a_0} \left(\alpha \left(\frac{\delta}{\alpha} \right)^{a_0 - a_1} + \frac{\gamma}{\alpha} \sum_{k=0}^{a_0 - a_1 - 1} \left(\frac{\delta}{\alpha} \right)^k \right)$$
$$f(2^{a_0} + \cdots + 2^{a_\ell}) = A \cdot f(2^{a_0} + \cdots + 2^{a_{\ell-1}}) + B \cdot f(2^{a_0} + \cdots + 2^{a_{\ell-2}}), \quad \ell \geq 2 \tag{100}$$

where $A := \left(\frac{\delta}{\alpha}\right)^{a_{\ell-1} - a_\ell} + \frac{\gamma}{\alpha} \cdot \sum_{k=0}^{a_{\ell-1} - a_\ell - 1} \left(\frac{\delta}{\alpha}\right)^k$ *and* $B := (\alpha - \delta - \frac{\gamma}{\alpha}) \cdot \left(\frac{\delta}{\alpha}\right)^{a_{\ell-2} - a_\ell - 1}$.

Indeed, for the cases discussed in Section 2, namely **E9** (Stern’s diatomic sequence $s(n)$), **E9’** (Reznick sequence $r(n)$), and **E9”** (twisted Stern sequence $t(n)$) we can prove three special cases of this Conjecture.

Theorem 4. *Let $\beta = 0 = \varepsilon$ in (92) and $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$, be a strictly decreasing sequence of positive integers. Then*

1. For $\alpha = 1$, $\gamma = 1$, $\delta = 1$ we have for $\ell \geq 2$

$$s(2^{a_0}) = 1, \quad s(2^{a_0} + 2^{a_1}) = a_0 - a_1 + 1,$$
$$s(2^{a_0} + \cdots + 2^{a_\ell}) = (a_{\ell-1} - a_\ell + 1) \cdot s(2^{a_0} + \cdots + 2^{a_{\ell-1}}) - s(2^{a_0} + \cdots + 2^{a_{\ell-2}}) \tag{101}$$

2. For $\alpha = 1$, $\gamma = -1$, $\delta = 1$ we have for $\ell \geq 2$

$$r(2^{a_0}) = 1, \quad r(2^{a_0} + 2^{a_1}) = -a_0 + a_1 + 1,$$
$$r(2^{a_0} + \cdots + 2^{a_\ell}) = (-a_{\ell-1} + a_\ell + 1) \cdot r(2^{a_0} + \cdots + 2^{a_{\ell-1}}) + r(2^{a_0} + \cdots + 2^{a_{\ell-2}}) \tag{102}$$

3. For $\alpha = -1$, $\gamma = -1$, $\delta = -1$ we have for $\ell \geq 2$

$$t(2^{a_0}) = (-1)^{a_0}, \quad t(2^{a_0} + 2^{a_1}) = (-1)^{a_0}(a_0 - a_1 - 1),$$
$$t(2^{a_0} + \cdots + 2^{a_\ell}) = (a_{\ell-1} - a_\ell + 1) \cdot t(2^{a_0} + \cdots + 2^{a_{\ell-1}}) - t(2^{a_0} + \cdots + 2^{a_{\ell-2}}) \tag{103}$$

Proof. By induction on $\ell \in \mathbb{N}_0$.
1) Case $\ell = 0$: the first initial value of (101) follows from (93) for $m = a_0$ and $\zeta = 1$.
2) Case $\ell = 1$: Since $s(2^{a_0} + 2^{a_1}) = s\Big(2^{a_1} \cdot (2^{a_0 - a_1} + 1)\Big) = s(2^{a_0 - a_1} + 1)$, we obtain from (94) for $m = a_0 - a_1$ and $\zeta = 1$ the second initial value of (101).
Let now $\ell \geq 1$ and $A_{\ell+1} := s(2^{a_0} + \cdots + 2^{a_{\ell+1}})$, then by definition
$A_{\ell+1} = s\Big(2^{a_{\ell+1}} \cdot (2^{a_0 - a_{\ell+1}} + \cdots + 2^{a_\ell - a_{\ell+1}} + 1)\Big) = s(2^{a_0 - a_{\ell+1}} + \cdots + 2^{a_\ell - a_{\ell+1}} + 1)$.
Let $b_i := a_i - a_{\ell+1}$, $i \in [\ell+2]_0$, then $b_0 > b_1 > \cdots > b_\ell > b_{\ell+1} = 0$. Hence, by definition

$$\begin{aligned} A_{\ell+1} &= s(2^{b_0} + \cdots + 2^{b_\ell} + 1) = s(2 \cdot (2^{b_0 - 1} + \cdots + 2^{b_\ell - 1}) + 1) \\ &= s(2^{b_0 - 1} + \cdots + 2^{b_\ell - 1}) + s(2^{b_0 - 1} + \cdots + 2^{b_\ell - 1} + 1) \\ &= s(2^{b_0} + \cdots + 2^{b_\ell}) + s(2^{b_0 - 1} + \cdots + 2^{b_\ell - 1} + 1). \end{aligned}$$

Repeating this procedure b_ℓ times we obtain by induction

$$\begin{aligned} A_{\ell+1} &= b_\ell \cdot s(2^{b_0} + \cdots + 2^{b_\ell}) + s(2^{b_0 - b_\ell} + \cdots + 2^{b_{\ell-1} - b_\ell} + 2^{b_\ell - b_\ell} + 1) \\ &= b_\ell \cdot s(2^{b_0} + \cdots + 2^{b_\ell}) + s(2^{b_0 - b_\ell - 1} + \cdots + 2^{b_{\ell-1} - b_\ell - 1} + 1) \\ &= (a_\ell - a_{\ell-1}) \cdot s(2^{a_0 - a_{\ell+1}} + \cdots + 2^{a_\ell - a_{\ell+1}}) + s(2^{a_0 - a_\ell - 1} + \cdots + 2^{a_{\ell-1} - a_\ell - 1} + 1) \end{aligned}$$

and this is by definition and since $b_i - b_\ell - 1 = a_i - a_{\ell+1} - (a_\ell - a_{\ell+1}) - 1 = a_i - a_\ell - 1$ equal to

$$A_{\ell+1} = (a_\ell - a_{\ell-1}) \cdot A_\ell + s(2^{a_0 - a_\ell - 1} + \cdots + 2^{a_{\ell-1} - a_\ell - 1} + 1) \tag{104}$$

We distinguish now two cases: 1.) $a_{\ell-1} - a_\ell - 1 > 0$ and 2.) $a_{\ell-1} - a_\ell - 1 = 0$.
First case: $a_{\ell-1} - a_\ell - 1 > 0$.
Applying by assumption the assertion to the sequence of $\ell + 1$ integers $a_0 - a_\ell - 1 >$

$a_1-a_\ell-1 > \ldots > a_{\ell-1}-a_\ell-1 > 0$ and by definition we obtain for $c_i := a_i-a_\ell-1$, $i \in [\ell]_0$, $c_\ell := 0$,

$$\begin{aligned} s(2^{c_0}+\cdots+2^{c_{\ell-1}}+1) &= (c_{\ell-1}-c_\ell+1)\cdot s(2^{c_0}+\cdots+2^{c_{\ell-1}}) - s(2^{c_0}+\cdots+2^{c_{\ell-2}}) \\ &= (a_{\ell-1}-a_\ell)\cdot s(2^{a_0}+\cdots+2^{a_{\ell-1}}) - s(2^{a_0}+\cdots+2^{a_{\ell-2}}) \\ &= (a_{\ell-1}-a_\ell)\cdot A_{\ell-1} - A_{\ell-2}, \end{aligned}$$

since $s(2^{c_0}+\cdots+2^{c_{\ell-1}}) = s(2^{a_0-a_\ell-1}+\cdots+2^{a_{\ell-1}-a_\ell-1}) = s(2^{a_0}+\cdots+2^{a_{\ell-1}})$. Inserting the above expression into (104) we obtain

$$\begin{aligned} A_{\ell+1} &= (a_\ell-a_{\ell-1})\cdot A_\ell + (a_{\ell-1}-a_\ell)\cdot A_{\ell-1} - A_{\ell-2} \\ &= (a_\ell-a_{\ell-1}+1-1)\cdot A_\ell + (a_{\ell-1}-a_\ell+1-1)\cdot A_{\ell-1} - A_{\ell-2} \\ &= (a_\ell-a_{\ell-1}+1)\cdot A_\ell - A_\ell + (a_{\ell-1}-a_\ell+1)\cdot A_{\ell-1} - A_{\ell-1} - A_{\ell-2} \\ &= (a_\ell-a_{\ell-1}+1)\cdot A_\ell - A_{\ell-1} \end{aligned}$$

and this is the assertion for $\ell+1$ instead of ℓ, since by assumption $-A_\ell + (a_{\ell-1} - a_\ell+1)\cdot A_{\ell-1} - A_{\ell-2} = 0$.
<u>Second case</u>: $a_{\ell-1}-a_\ell-1 = 0$.
In this case the sequence $c_i := a_i - a_\ell - 1$, $i \in [\ell]_0$, $c_\ell := 0$ is no longer strictly decreasing, since $c_{\ell-1}-c_\ell = a_{\ell-1}-a_\ell-1 = 0$. We define $n := 2^{c_0}+\cdots+2^{c_{\ell-1}}$ and using the rule $s(n+1) = s(2n+1) - s(n)$ we obtain $s(2^{c_0}+\cdots+2^{c_{\ell-1}}+1) = s(2^{c_0+1}+\cdots+2^{c_{\ell-1}+1}+1) - s(2^{c_0}+\cdots+2^{c_{\ell-1}})$ or

$$\begin{aligned} s(2^{c_0}+\cdots+2^{c_{\ell-1}}+1) &= s(2^{a_0-a_\ell}+\cdots+2^{a_{\ell-1}-a_\ell}+1) - s(2^{a_0-a_\ell-1}+\cdots+2^{a_{\ell-1}-a_\ell-1}) \\ &= s(2^{a_0}+\cdots+2^{a_\ell}) - s(2^{a_0}+\cdots+2^{a_{\ell-1}}) \\ &= A_\ell - A_{\ell-1}. \end{aligned}$$

Inserting this expression into (104) we get

$$\begin{aligned} A_{\ell+1} &= (a_\ell-a_{\ell-1})\cdot s(2^{a_0}+\cdots+2^{a_\ell}) + s(2^{a_0}+\cdots+2^{a_\ell}) - s(2^{a_0}+\cdots+2^{a_{\ell-1}}) \\ &= (a_\ell-a_{\ell-1}+1)\cdot A_\ell - A_{\ell-1} \end{aligned}$$

and this is again the assertion for $\ell+1$ instead of ℓ. This proves (101).
The proofs of (102) and (103) are similar and will be omitted. □
By means of Theorem 4 (and using the Euler-Wallis relations) the solution of Stern's diatomic sequence can be given by the numerator of the finite continued fraction

$$\frac{A_\ell}{B_\ell} = a_0 - a_1 + 1 - \frac{1\,|}{|\, a_1-a_2+1} - \cdots - \frac{1\,|}{|\, a_{\ell-1}-a_\ell+1}, \quad \text{for} \quad \ell > 0$$

and $s(2^{a_0}) = 1$ for $\ell = 0$, that is $s(n) = A_\ell$ for $\ell > 0$ and $s(2^{a_0}) = 1$ for $\ell = 0$.

We recall that $b_0 + \frac{a_1\,|}{|\, b_1} + \frac{a_2\,|}{|\, b_2} + \cdots + \frac{a_{n-1}\,|}{|\, b_{n-1}} + \frac{a_n\,|}{|\, b_n}$ is the notation of Pringsheim of a finite generalized continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots \cfrac{a_{n-1}}{b_{n-1} + \cfrac{a_n}{b_n}}}}$$

If some or all a_k, $k \in [n]$, are provided with minus signs, then these are also written in front of the a_k instead of the plus signs; thus instead of $b_0 + \frac{-a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{-a_n|}{|b_n}$ we will write $b_0 - \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots - \frac{a_n|}{|b_n}$.

If $a_k = 1$ and $b_k \in \mathbb{N}$ for all $k \in [n]$, then the continued fraction is said to be *simple*.

Remark 4. Other equivalent solutions of Stern's diatomic sequence have been given in [29, pp. 95–96], in [13, Exercise 6.50 at p. 300 and its solution at p. 531] and in [20] by different representations of n.

Similarly, the solution of Reznick's sequence can be given by the numerator of the finite simple continued fraction

$$\frac{A_\ell}{B_\ell} = -a_0 + a_1 + 1 + \frac{1|}{|-a_1 + a_2 + 1} + \cdots + \frac{1|}{|-a_{\ell-1} + a_\ell + 1}, \quad \text{for} \quad \ell > 0$$

and $r(2^{a_0}) = 1$ for $\ell = 0$, that is $r(n) = A_\ell$ for $\ell > 0$ and $r(2^{a_0}) = 1$ for $\ell = 0$.

Finally, the solution of the twisted Stern sequence can be given by the numerator of the finite continued fraction

$$\frac{A_\ell}{B_\ell} = (-1)^{a_0} \cdot \left(a_0 - a_1 - 1 - \frac{1|}{|a_1 - a_2 + 1} - \cdots - \frac{1|}{|a_{\ell-1} - a_\ell + 1} \right), \quad \text{for} \quad \ell > 0$$

and $t(2^{a_0}) = (-1)^{a_0}$ for $\ell = 0$, that is $t(n) = A_\ell$ for $\ell > 0$ and $t(2^{a_0}) = (-1)^{a_0}$ for $\ell = 0$.

5 Generalization

An obvious generalization of (1) is given by

$$f(1) = \zeta, \quad f(2n) = \alpha f(n) + g(n), \quad f(2n+1) = \gamma f(n) + \delta f(n+1) + h(n), \quad n \geq 2, \tag{105}$$

with $\alpha, \gamma, \delta, \zeta \in \mathbb{Z}$ and $g, h : \mathbb{N} \longrightarrow \mathbb{Z}$ two arbitrary integer functions.

5.1 The case $\delta = 0$

At first we shall give the general solution of (105) for the case $\delta = 0$.

Theorem 5. *The solution of (105) for $\delta = 0$ and for $n = (b_m \dots b_0)_2$ is given by*

$$f(n) = \zeta \cdot \prod_{k=0}^{m-1} \Big(\gamma b_k + \alpha(1 - b_k)\Big) + \\ + \sum_{k=0}^{m-1} \left(\prod_{j=0}^{k-1} \Big(\gamma b_j + \alpha(1 - b_j)\Big) \right) \cdot \Big(b_k \cdot h\big((b_m \dots b_{k+1})_2\big) + (1 - b_k) \cdot g\big((b_m \dots b_{k+1})_2\big) \Big) \tag{106}$$

Proof. Let $n = (b_m \dots b_0)_2$. Then by (105):
if $b_0 = 0$ we have $f\Big((b_m \dots b_0)_2\Big) = \alpha f\Big((b_m \dots b_1)_2\Big) + g\Big((b_m \dots b_1)_2\Big)$ and
if $b_0 = 1$ we have $f\Big((b_m \dots b_0)_2\Big) = \gamma f\Big((b_m \dots b_1)_2\Big) + h\Big((b_m \dots b_1)_2\Big)$.
Hence,

$$\begin{aligned} f\Big((b_m \dots b_0)_2\Big) &= b_0 \cdot \Big(\gamma f\Big((b_m \dots b_1)_2\Big) + h\Big((b_m \dots b_1)_2\Big)\Big) + \\ &\quad + (1-b_0) \cdot \Big(\alpha f\Big((b_m \dots b_1)_2\Big) + g\Big((b_m \dots b_1)_2\Big)\Big) \\ &= \Big(\gamma b_0 + \alpha(1-b_0)\Big) \cdot f\Big((b_m \dots b_1)_2\Big) + \Big(b_0 \cdot h\Big((b_m \dots b_1)_2\Big) + \\ &\quad + (1-b_0) \cdot g\Big((b_m \dots b_1)_2\Big)\Big). \end{aligned}$$

Similarly, $f\Big((b_m \dots b_1)_2\Big) = \Big(\gamma b_1 + \alpha(1-b_1)\Big) \cdot f\Big((b_m \dots b_2)_2\Big) + \Big(b_1 \cdot h\Big((b_m \dots b_2)_2\Big) + (1-b_1) \cdot g\Big((b_m \dots b_2)_2\Big)\Big)$.
Substituting this expression into the above equation gives

$$\begin{aligned} f\Big((b_m \dots b_0)_2\Big) &= \Big(\gamma b_0 + \alpha(1-b_0)\Big) \cdot \Big(\gamma b_1 + \alpha(1-b_1)\Big) \cdot f\Big((b_m \dots b_2)_2\Big) + \\ &\quad + \Big(\gamma b_0 + \alpha(1-b_0)\Big) \cdot \Big(b_1 \cdot h\Big((b_m \dots b_2)_2\Big) + (1-b_1) \cdot g\Big((b_m \dots b_2)_2\Big)\Big) + \\ &\quad + \Big(b_0 \cdot h\Big((b_m \dots b_1)_2\Big) + (1-b_0) \cdot g\Big((b_m \dots b_1)_2\Big)\Big) \end{aligned}$$

By iteration and noting that $f(b_m) = f(1) = \zeta$ one obtains formula (106) and this proves the theorem. □

An interesting special case of (105) is for $\gamma = \alpha$ and $g(n) := 2n$, $h(n) := 2n+1$, that is the recurrence relation

$$f(1) = \zeta, \quad f(2n) = \alpha f(n) + 2n, \quad f(2n+1) = \alpha f(n) + 2n + 1, \quad n \geq 1 \tag{107}$$

or, equivalently,

$$f(1) = \zeta, \quad f(n) = \alpha f\Big(\lfloor n/2 \rfloor\Big) + n, \quad n \geq 2 \tag{108}$$

Proposition 3. *The solution of (107) for $n = (b_m \dots b_0)_2$ is given by*

$$f\Big((b_m \dots b_0)_2\Big) = \begin{cases} (\zeta - 1) \cdot 2^m + \sum_{k=1}^m k \cdot b_k \cdot 2^k + (b_m \dots b_0)_2, & \textit{if } \alpha = 2 \\ (\zeta - 1) \cdot \alpha^m + \frac{\alpha}{\alpha - 2}(b_m \dots b_0)_\alpha - \frac{2}{\alpha-2}(b_m \dots b_0)_2, & \textit{if } \alpha \neq 2 \end{cases} \tag{109}$$

where $(b_m \dots b_0)_\alpha := \sum_{k=0}^m b_k \cdot \alpha^k$.

Proof. By (106) with

$$h\Big((b_m \dots b_{k+1})_2\Big) = 2 \cdot (b_m \dots b_{k+1})_2 + 1 = (b_m \dots b_{k+1}1)_2,$$

$$g\Big((b_m \dots b_{k+1})_2\Big) = 2 \cdot (b_m \dots b_{k+1})_2 = (b_m \dots b_{k+1}0)_2$$

and

$$b_k \cdot (b_m \dots b_{k+1})_2 + (1 - b_k) \cdot (b_m \dots b_{k+1})_2 = (b_m \dots b_{k+1} b_k)_2$$

we obtain

$$f\Big((b_m \dots b_0)_2\Big) = \zeta \cdot \alpha^m + \sum_{k=0}^{m-1} \alpha^k \cdot (b_m \dots b_{k+1} b_k)_2.$$

Since $(b_m \dots b_{k-1} b_k)_2 = \sum_{j=k}^{m} b_j 2^{j-k}$ it follows

$$\begin{aligned}
\sum_{k=0}^{m-1} \alpha^k \cdot (b_m \dots b_{k+1} b_k)_2 &= \sum_{k=0}^{m-1} \alpha^k \Big(\sum_{j=k}^{m} b_j 2^{j-k} \Big) \\
&= \alpha^0 \cdot (b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{m-1} 2^{m-1} + b_m 2^m) + \\
&+ \quad \alpha^1 \cdot (\qquad b_1 2^0 + b_2 2^1 + \dots + b_{m-1} 2^{m-2} + b_m 2^{m-1}) + \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
&+ \quad \alpha^{m-1} \cdot (\qquad\qquad\qquad b_{m-1} 2^0 + b_m 2^1) \\
&= b_0(\alpha^0 2^0) + b_1(\alpha^0 2^1 + \alpha^1 2^0) + b_2(\alpha^0 2^1 + \alpha^1 2^1 + \alpha^2 2^0) + \\
&+ \dots + b_m(\alpha^0 2^m + \alpha^1 2^{m-1} + \dots + \alpha^{m-1} 2^1 + \alpha^m 2^0 - \alpha^m 2^0) \\
&= \sum_{k=0}^{m} b_k \cdot 2^k \cdot \Big(\sum_{j=0}^{k} (\alpha/2)^j \Big) - b_m \alpha^m
\end{aligned}$$

Substituting the value of the finite geometric series

$$\sum_{j=0}^{k} (\alpha/2)^j = \begin{cases} k+1, & \text{if } \alpha = 2 \\ \frac{1}{2^k} \frac{\alpha^{k+1} - 2^{k+1}}{\alpha - 2}, & \text{if } \alpha \neq 2 \end{cases}$$

into the last formula and simplifying one obtains the formula (109) and this proves the proposition. □

An important example of (107) is for $\alpha = 1, \zeta = 1$. Its solution $f(n) = 2n - \sum_{k=0}^{n} b_k = 2n - s_2(n)$ shows, that this is the sequence $\big(D_2(n)\big)_{n \in \mathbb{N}}$ defined in (26). Besides, it is not difficult to show directly that by (16) this sequence satisfies the recurrence relation

$$D_2(1) = 1, \quad D_2(2n) = D_2(n) + 2n, \quad D_2(2n+1) = D_2(n) + 2n + 1, \quad n \geq 1 \quad (110)$$

We now derive some properties of this sequence.

Proposition 4. *1. The solution of (110) is given by*

$$D_2(n) = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \lfloor n/2^k \rfloor, \quad (111)$$

that is, $D_2(n)$ is the number of positive powers of 2 which divide numbers $\leq n$.

2. The generating function $\varphi(s) := \sum_{n=1}^{\infty} D_2(n) s^n$ satisfies the functional equation

$$\varphi(s) = (1+s)\varphi(s^2) + \frac{s}{(1-s)^2}, \quad (112)$$

which shows that $D_2(n)$ is of the DC type.

3. The solution of (112) is given either by

$$\varphi(s) = \frac{1}{1-s}\sum_{k=0}^{\infty}\frac{s^{2^k}}{1-s^{2^k}} \tag{113}$$

or by the Lambert series

$$\varphi(s) = \sum_{n=1}^{\infty} a_n \frac{s^n}{1-s^n} \tag{114}$$

where $a_n := 2\phi(n) - \sum_{t|n}\mu(n/t)s_2(t)$, $\mu(n)$ *is the Möbius function and* $\phi(n)$ *is Euler's totient function.*

Proof. By (23) we have $d_2(n) = \sum_{t|n}\rho(t)$, where $\rho(n) = 1$, if n is a power of 2 and zero otherwise. Hence,

$$D_2(n) = \sum_{k=1}^{n} d_2(k) = \sum_{k=1}^{n}\Big(\sum_{t|k}\rho(t)\Big) \tag{115}$$

We collect terms with equal values of $\rho(t)$ in the above double sum. For every $j \le n$, $\rho(j)$ appears in $\sum_{t|k}\rho(t)$ if and only if $j \mid k$. Since each integer has itself as a divisor, the right-hand side of (115) includes $\rho(j)$ at least once. Furthermore, there are exactly $\lfloor n/j \rfloor$ integers among $1, 2, \ldots, n$ which are divisible by j, namely: $j, 2j, 3j, \ldots, \lfloor n/j\rfloor j$. Hence, $\sum_{k=1}^{n}\Big(\sum_{t|k}\rho(t)\Big) = \sum_{j=1}^{n}\rho(j)\cdot\lfloor n/j\rfloor = \sum_{k=0}^{\lfloor \log_2 n\rfloor}\lfloor n/2^k\rfloor$ and this proves (111) (see [8, Theorem 6-11, pp. 119–120]).

Let $\varphi(s) := \sum_{n=1}^{\infty} D_2(n)s^n$, then by (110)

$$\begin{aligned}
\varphi(s) &= \sum_{n=1}^{\infty} D_2(2n)s^{2n} + \sum_{n=0}^{\infty} D_2(2n+1)s^{2n+1}\\
&= \sum_{n=1}^{\infty} D_2(n)s^{2n} + \sum_{n=1}^{\infty}(2n)s^{2n} + s\sum_{n=1}^{\infty} D_2(n)s^{2n} + \sum_{n=0}^{\infty}(2n+1)s^{2n+1}\\
&= (1+s)\varphi(s^2) + \sum_{n=1}^{\infty}(2n)s^{2n} + \sum_{n=0}^{\infty}(2n+1)s^{2n+1}
\end{aligned}$$

and this is (112), since $\sum_{n=1}^{\infty}(2n)s^{2n} + \sum_{n=0}^{\infty}(2n+1)s^{2n+1} = \sum_{n=1}^{\infty} ns^n = \frac{s}{(1-s)^2}$.

It is well-known that the generating function for the sequence of the partial sums of a sequence $a(n)$ is equal to the generating function of $a(n)$ divided by $1-s$. Hence,

$$\sum_{n=1}^{\infty} D_2(n)s^n = \frac{1}{1-s}\sum_{n=1}^{\infty} d_2(n)s^n \tag{116}$$

Substituting (17) into this equation gives immediately (113).
By (22) the generating function for $d_2(n)$ is given by a Lambert series. Therefore, it is reasonable to assume that the generating function of $D_2(n)$ is given by a Lambert series, too. Let $\varphi(s) = \sum_{n=1}^{\infty} a_n\frac{s^n}{1-s^n}$, where (a_n) is an unknown sequence to be

determined. Since $\frac{s^n}{1-s^n} = s^n(1 + s^n + s^{2n} + \cdots) = s^n + s^{2n} + s^{3n} + \cdots = \sum_{m=1}^{\infty} s^{m\cdot n}$ we obtain $\sum_{n=1}^{\infty} a_n \frac{s^n}{1-s^n} = \sum_{n=1}^{\infty} a_n \left(\sum_{m=1}^{\infty} s^{m\cdot n} \right)$. Summing by rows we obtain

$$\sum_{n=1}^{\infty} a_n \frac{s^n}{1-s^n} = \sum_{n=1}^{\infty} \Big(\sum_{t|n} a_t \Big) s^n,$$

that is $\sum_{t|n} a_t = D_2(n)$. By the Möbius Inversion Formula (see [8, Theorem 6-7, pp. 113–114]) it follows $a_n = \sum_{t|n} \mu(n/t) \cdot D_2(t)$, where

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = \prod_{j=1}^{k} p_j, \quad p_1, \ldots, p_k \text{ distinct primes} \\ 0, & \text{otherwise} \end{cases}$$

is the Möbius function. Since by (25) $D_2(t) = 2t - s_2(t)$ we obtain

$$a_n = 2 \sum_{t|n} \mu(n/t) t - \sum_{t|n} \mu(n/t) s_2(t) = 2\phi(n) - \sum_{t|n} \mu(n/t) s_2(t)$$

(see [8, Theorem 7-8, pp. 138–139]), where $\phi(n)$ is Euler's totient function giving the number of positive integers not exceeding n that are relative prime to n, and this proves (114). □

The first few values of $D_2(n)$ (the sequence A005187 in [24]) and a_n (the sequence A035532 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$D_2(n)$	1	3	4	7	8	10	11	15	16	18	19	22	23	25	26	31
a_n	1	2	3	4	7	4	10	8	12	8	18	8	22	12	15	16

Table 16: $D_2(n)$, a_n, $1 \le n \le 16$

Note that for all primes p we have

$$D_2(p) - a_p = 1, \tag{117}$$

since $a_p = 2\phi(p) - \sum_{t|p} \mu(p/t)s_2(t) = 2(p-1) - \mu(p)s_2(1) - \mu(1)s_2(p) = 2p - 2 - (-1) \cdot 1 - 1 \cdot s_2(p) = D_2(p) - 1$.

The next proposition lists some more properties of the sequences $(d_2(n))$, $(s_2(n))$ and $(D_2(n))$.

Proposition 5. *Let $n \in \mathbb{N}$, then*

1. *(Highest power of 2 that divides $n!$)*

$$D_2(n) - n = n - s_2(n) = \sum_{k=1}^{\lfloor \log_2 n \rfloor} \lfloor n/2^k \rfloor \tag{118}$$

is the highest power of 2 dividing $n!$, that is

$$n! = 2^{n - s_2(n)} \cdot (2k+1), \text{ for a } k \in \mathbb{N}_0 \tag{119}$$

2. *(Asymptotic behavior)*

$$\lim_{n\to\infty} \frac{D_2(n)}{n} = 2, \tag{120}$$

consequently,

$$\lim_{n\to\infty} \frac{s_2(n)}{n} = 0 \tag{121}$$

3. *(Another representation of the generating function of $s_2(n)$)*

$$\sum_{n=1}^{\infty} s_2(n)s^n = \frac{2s}{(1-s)^2} - \frac{1}{1-s}\sum_{k=0}^{\infty}\frac{s^{2^k}}{1-s^{2^k}} \tag{122}$$

4. *(Another representation of the generating function of $d_2(n)$)*

$$\sum_{n=1}^{\infty} d_2(n)s^n = \frac{2s}{1-s} - \sum_{k=0}^{\infty}\frac{s^{2^k}}{1+s^{2^k}} \tag{123}$$

5. *(Another representation of the generating function of $D_2(n)$)*

$$\sum_{n=1}^{\infty} D_2(n)s^n = \frac{2s}{(1-s)^2} - \frac{1}{1-s}\sum_{k=0}^{\infty}\frac{s^{2^k}}{1+s^{2^k}} \tag{124}$$

Proof. By (2) and (111) one obtains the formula (118). The right-hand side of this (double) equation gives the exponent of the highest power of 2 that divides $n!$, since there are $\lfloor n/2\rfloor$ numbers (namely, $2, 2\cdot 2, 3\cdot 2, \ldots, \lfloor n/2\rfloor \cdot 2$) which are divisible by 2, $\lfloor n/2^2\rfloor$ numbers which are divisible by 2^2, etc. This proves (119).
By definition it holds that $x - 1 < \lfloor x\rfloor \le x$ for all $x \in \mathbb{R}$. Choosing $x := n/2^k$ we obtain $n/2^k - 1 < \lfloor n/2^k\rfloor \le 2/n^k$. Summing up for $k = 0, 1, \ldots, \lfloor \log_2 n\rfloor$ it follows

$$\sum_{k=0}^{\lfloor \log_2 n\rfloor}\Big(n/2^k - 1\Big) < \sum_{k=0}^{\lfloor \log_2 n\rfloor}\lfloor n/2^k\rfloor \le \sum_{k=0}^{\lfloor \log_2 n\rfloor} n/2^k$$

or by (111) and using the formula for finite geometric series we obtain

$$2n\Big(1 - 1/2^{\lfloor \log_2 n\rfloor+1}\Big) - (\lfloor \log_2 n\rfloor + 1) < D_2(n) \le 2n\Big(1 - 1/2^{\lfloor \log_2 n\rfloor+1}\Big)$$

or dividing by n

$$2\Big(1 - \frac{1}{2^{\lfloor \log_2 n\rfloor+1}}\Big) - \frac{\lfloor \log_2 n\rfloor + 1}{n} < \frac{D_2(n)}{n} \le 2\Big(1 - \frac{1}{2^{\lfloor \log_2 n\rfloor+1}}\Big)$$

Assertion (120) follows by noting that $1/2^{\lfloor \log_2 n\rfloor+1} \to 0$ and $(\lfloor \log_2 n\rfloor + 1)/n \to 0$ for $n \to \infty$.
Since $s_2(n) = 2n - D_2(n)$ we conclude by (120) that $\frac{s_2(n)}{n} = 2 - \frac{D_2(n)}{n} \to 0$ for $n \to \infty$ and this proves (121).
By (25) we have

$$\sum_{n=1}^{\infty} s_2(n)s^n = \sum_{n=1}^{\infty}(2n - D_2(n))s^n = 2\sum_{n=1}^{\infty} ns^n - \sum_{n=1}^{\infty} D_2(n)s^n$$

and this is formula (122) by (118) and noting that $\sum_{n=1}^{\infty} ns^n = \frac{s}{(1-s)^2}$.
By (24) and noting that the generating function for a difference sequence $\Delta a(n)$ is equal to $1-s$ times the generating function for the sequence $a(n)$ we have

$$\begin{aligned}\sum_{n=1}^{\infty} d_2(n)s^n &= \sum_{n=1}^{\infty} \Big(2 - \Delta s_2(n)\Big)s^n \\ &= 2\sum_{n=1}^{\infty} s^n - \sum_{n=1}^{\infty} \Delta s_2(n)s^n \\ &= \frac{2s}{1-s} - (1-s)\sum_{n=1}^{\infty} s_2(n)s^n\end{aligned}$$

and this is formula (123) by (11) and noting that $\sum_{n=1}^{\infty} s^n = \frac{s}{1-s}$.
Substituting (123) into (116) gives immediately the formula (124). □
Note that formula (118) has been obtained for the first time in 1808 by A. M. Legendre in the form (cf. Example **E3** at p. 7)

$$\sum_{k=1}^{n} \tilde{d}_2(k) = \sum_{k=1}^{n} \Big(d_2(k) - 1\Big) = D_2(n) - n = n - s_2(n).$$

By equating the two representations (11) and (122) of the generating function for $s_2(n)$ and simplifying one obtains the identities

Corollary 2.

$$\sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}} = \frac{2s}{1-s} - \sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^k}} \tag{125}$$

$$\sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^{k+1}}} = \frac{s}{1-s}. \tag{126}$$

5.2 The case $\delta \neq 0$

In this subsection we just mention two known sequences, namely

1. The Bodlaender sequence, that is the special case $\alpha = 1$, $\gamma = 0$, $\delta = 1$, $\zeta = -1$, $g(n) = 0$, $h(n) = n$ of (105)

$$f(1) = -1, \quad f(2n) = f(n), \quad f(2n+1) = f(n+1) + n, \quad n \geq 1 \tag{127}$$

   already studied in [25] with the solution

$$f\Big((b_m \dots b_0)_2 + 1\Big) = -1 + \sum_{k=0}^{m-1} (1-b_k)(b_m \dots b_{k+1})_2 \tag{128}$$

2. Merge sort (see [13, p. 79, Exercise 34 at p. 98])
   The total number of comparisons for merge sort of n records is at most $C(n)$, where

$$C(1) = 0, \quad C(n) = C\Big(\lfloor n/2 \rfloor\Big) + C\Big(\lceil n/2 \rceil\Big) + n - 1, \quad n \geq 2, \tag{129}$$

or, equivalently, for $n \geq 1$

$$C(1) = 0, \quad C(2n) = 2C(n)+2n-1, \quad C(2n+1) = C(n)+C(n+1)+2n \quad (130)$$

This is the special case $\alpha = 2$, $\gamma = 1$, $\delta = 1$, $\zeta = 0$, $g(n) = 2n - 1$, $h(n) = 2n$ of (105) with the solution (see [13, p. 496])

$$C(n) = \sum_{k=1}^{n} \lceil \log_2 k \rceil = n \cdot \lceil \log_2 n \rceil - 2^{\lceil \log_2 n \rceil} + 1. \quad (131)$$

Acknowledgments

I would like to thank Andreas M. Hinz (Munich) for some useful comments.

References

[1] J. P. Allouche, On the Stern sequence and its twisted version, *Integers* **12** (2012) #A58, available electronically at http://www.integers-ejcnt.org/vol12.html.

[2] J. P. Allouche, J. O. Shallit, The ring of k-regular sequences, *Theoret. Comput. Sci.* **98** (1992) 163–197.

[3] J. P. Allouche, J. O. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in: C. Ding, T. Helleseth, H. Niederreiter (eds.), *Sequences and their Applications*, Springer, London, 1999, 1–16.

[4] J. P. Allouche, J. O. Shallit, The ring of k-regular sequences, II, *Theoret. Comput. Sci.* **307** (2003) 3–29.

[5] Y. H. Au, C. Drexler-Lemire, J. O. Shallit, Notes and note pairs in Nørgård's infinity series, *Journal of Mathematics and Music* Volume **11**, Issue 1, (2017) 1–19.

[6] R. Bacher, Twisting the Stern sequence, arXiv preprint (2010), available electronically at http://arxiv.org/abs/1005.5627.

[7] A. Brocot, Calcul de rouages par approximation, *Revue Chronométrique* **3** (1859-61) 186–194. (Reprinted in: R. Chavigny, *Les Brocot, une dynastie d'horlogers*, Antoine Simonin, Neuchâtel, 1991, 187–197.)

[8] D. M. Burton, *Elementary Number Theory*, 3rd ed., McGraw-Hill, New York, 1997.

[9] L. Carlitz, Single variable Bell polynomials, *Collect. Math.* **14** (1962) 13–25.

[10] L. Carlitz, A problem in partitions related to the Stirling numbers, *Bull. Amer. Math. Soc.* **70** (1964) 275–278.

[11] M. Coons, On some conjectures concerning Stern's sequence and its twist, *Integers* **11** (2011) #A35, available electronically at http://www.integers-ejcnt.org/vol11.html.

[12] H. L. Glaisher, On the residue of a binomial coefficient theorem with respect to a prime modulus, *Quarterly J. Pure and Appl. Math.* **30** (1899) 150–156.

[13] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading MA, 1989.

[14] W. Hilberg, Mehrdimensionale Morse-Thue-Folgen, *Physik in unserer Zeit 22 Jahrg.* Nr. 1 (1991) 24–28.

[15] A. M. Hinz, The Tower of Hanoi, *L'Enseignement Mathématique* (2) **35** (1989) 289–321.

[16] A. M. Hinz, Pascal's Triangle and the Tower of Hanoi, *Amer. Math. Monthly* **99** (1992) 538–544.

[17] A. M. Hinz, The Tower of Hanoi, in: *Algebras and Combinatorics (An International Congress, ICAC '99, Hong Kong (Kar-Ping Shum, Earl J. Taft, and Zhe-Xian Wan, eds.))*, Springer, Singapore (1999) 277–289.

[18] A. M. Hinz, S. Klavžar, U. Milutinović, D. Parisse, C. Petr, Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence, *European J. Combin.* **26** (2005) 693–708.

[19] A. M. Hinz, S. Klavžar, C. Petr, *The Tower of Hanoi − Myths and Maths*, 2nd edition, Birkhäuser, Cham, 2018.

[20] D. H. Lehmer, On Stern's diatomic series, *Amer. Math. Monthly* **36** (1929) 59–67.

[21] D. A. Lind, An extension of Stern's diatomic series, *Duke Math. J.* **36** (1969) 55–60.

[22] É. Lucas, Sur le suites de Farey, *Bulletin de la Société mathématique de France* **6** (1878) 118–119.

[23] S. Northshield, Stern's diatomic sequence $0, 1, 1, 2, 1, 3, 2, 3, 1, 4, \ldots$, *Amer. Math. Monthly* **117** (2010) 581–598.

[24] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/`

[25] D. Parisse, On the Bodlaender Sequence, *Proceedings of the 2nd Croatian Combinatorial Days, Zagreb, Sept. 27-28, 2018* (Eds. T. Došlić and I. Martinjak), Fac. Civil Eng., Univ. Zagreb, (2018) 105–121.

[26] J. C. Puchta, J. Spilker, Die Thue-Morse-Folge, *Elem. Math.* **55** No. 3 (2000) 110–122.

[27] B. Reznick, A new sequence with many properties, *Abs. Amer. Math. Soc.* **5** (1984) 16.

[28] B. Reznick, Some binary partition functions, in: B. C. Berndt et al. (Eds.), *Analytic Number Theory (Proceedings of a Conference in Honor of Paul T. Bateman),* Birkhäuser, Boston (1990) 451-477.

[29] G. de Rham, Un peu de mathématiques à propos d'une courbe plane, *Elem. der Math.* Band II, **4**, **5** (1947) 73–88, 89–97.

[30] R. Sedgewick, P. Flajolet, *An Introduction to the Analysis of Algorithms,* Addison-Wesley, 1996.

[31] R. Stephan, Divide-and-conquer generating functions. I. Elementary sequences, available electronically at arXiv:math/0307027 [math.CO], 2003.

[32] M. A. Stern, Über eine zahlentheoretische Funktion, *J. Reine Angew. Math.* **55** (1858) 193-220.

[33] I. Urbiha, Some properties of a function studied by De Rham, Carlitz and Dijkstra and its relation to the (Eisenstein-)Stern's diatomic sequence, *Math. Commun.* **6** (2001) 181–198.