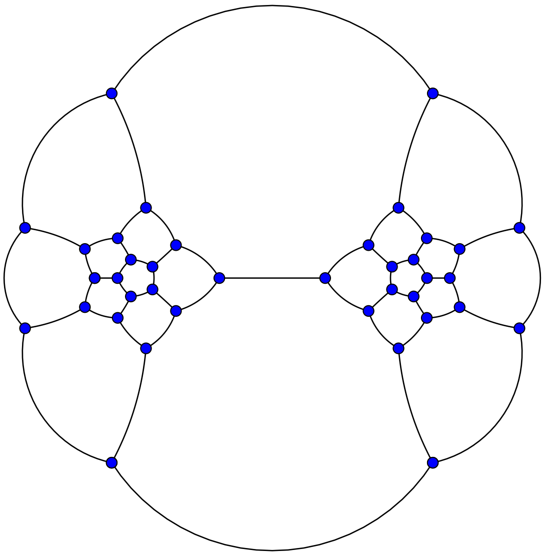
Tomislav Došlić, Snježana Majstorović, Luka Podrug (Eds.)

Proceedings

of the



4th Croatian Combinatorial Days

Zagreb, September 22–23, 2022

Faculty of Civil Engineering, University of Zagreb

Impressum

Editors: Tomislav Došlić, Snježana Majstorović, Luka Podrug

Technical editor: Luka Podrug

Publisher: Faculty of Civil Engineering, University of Zagreb

Copies: 70

Zagreb, 2023

ISBN: 978-953-8168-63-5

DOI: 10.5592/CO/CCD.2022

 The cover art is the planar Lombardi drawing of a 38-vertex graph with girth five and cyclic edge connectivity three by David Eppstein (arxiv:1206.6142v1), used with kind permission of the author.

CIP zapis dostupan u računalnom katalogu Nacionalne i sveučilišne knjižnice u Zagrebu pod brojem 001178214.

Sponsors:
Faculty of Civil Engineering, University of Zagreb
Croatian Academy of Sciences and Art
Zagreb Tourist Board

Name of the conference: CroCoDays 2022 – 4$^{\text{th}}$ Croatian Combinatorial Days
Organizer: Faculty of Civil Engineering, University of Zagreb
Place: Kačićeva 26, 10000 Zagreb, CROATIA
Dates: September 22–23, 2022

Organizing and program committee:
Tomislav Došlić, Zagreb, Croatia (Chairman)
Snježana Majstorović, Osijek, Croatia
László Németh, Sopron, Hungary
Luka Podrug, Zagreb, Croatia
Dragutin Svrtan, Zagreb, Croatia
Riste Škrekovski, Ljubljana, Slovenia
Damir Vukičević, Split, Croatia
Ivana Zubac, Mostar, Bosnia and Herzegovina
Petra Žigert Pleteršek, Maribor, Slovenia

Local organizing committee:
Tomislav Došlić
Luka Podrug
Dragutin Svrtan

Technical support:
Neven Bratolić
Berislav Cvetnić

Contents

Preface

Writing a preface is always a pleasure to me. At the risk of repeating myself – it is my third CroCoDays Proceedings preface starting with the same sentence – there is no better way to describe my feelings at presenting you yet another volume of CroCoDays Proceedings. As in the previous three cases, this volume is a result of concerted efforts of our authors, referees and sponsors, and it is my pleasure to thank them all – to thank you all – for helping to prepare a nice publication and thus bring the story of our fourth CroCoDays meeting to its successful conclusion.
First and foremost, I thank all participants for sharing with us their latest results. Then, I thank the authors who contributed to this volume. My special thanks go to our referees – you know who you are – for their constructive and timely reports. Last, but not the least, I thank the members of the Organizing and program committee and my colleagues co-editors for all their help.
The conference and this volume would not be possible without generosity of our sponsors. It is a pleasant duty to acknowledge the financial, logistical and technical support of the Faculty of Civil Engineering. The publication of the Proceedings was partially supported by the grant of the Foundation of the Croatian Academy of Science and Arts. The Zagreb Tourist Board kindly provided promotional material for participants.
The present volume is freely available at the conference website. Please spread a word and forward it to any colleagues who might be interested in participating in our future meetings. I am looking forward to meeting you all, and many others, at the next, semi-jubilary, fifth CroCoDays, scheduled for September 2024.

Zagreb, May 19, 2023 Tomislav Došlić

Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.01

Balanced simplicial complex associated with $1 \times p$ polyomino

Đorđe Baralić and Edin Liđan

Abstract

Balanced simplicial complexes are important objects in combinatorics and commutative algebra. A d-dimensional simplicial complex is balanced if its vertices can be coloured into $d+1$ colors, so there is no monochromatic edge. In this article, we establish two results concerning balanced simplicial complexes assigned to tilings of $m \times n$ board in a plane and a torus by I p-omino tile.

1 Introduction

Simplicial complexes are abstract objects that acquire most topological, geometrical, logical, combinatorial and algebraic features of spaces. It permits us to study complex spaces, and their application in topological data analysis, neural networks and modern physics is irreplaceable. A particular class of simplicial complexes, balanced complexes, is significant in combinatorics, commutative algebra and computer science. They are intensively studied since their introduction by Richard Stanley in [10]. Among the researchers who made significant progress in understanding these complexes in the last twenty years are Izmestiev, Klee, Juhnke-Kubitzke, Novik, Juhnke-Kubitzke and Murai (see [5], [2], [6], [3] and [4]).
Recreational mathematics often pays the way for establishing new mathematical disciplines. One of the most exciting puzzles is those with polyominoes. A polyomino is a polyform whose cells are squares. Solomon Golomb, who wrote the first book on polyominoes [1], gave them this name in 1953. In combinatorics, challenges are often posed for tiling a prescribed region, or the entire plane, with polyominoes; see [8] for some problems investigated in mathematics and computer science.
Recently, Baralić and Liđan come out with the idea to assign a simplicial complex to a polyomino tiling problem. Their work is related to studying simplicial complexes of graph hereditary properties that were investigated in topological combinatorics.

(Đorđe Baralić) Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia, djbaralic@mi.sanu.ac.rs

(Edin Liđan) International Burch University, Sarajevo, Bosnia and Herzegovina, edin.lidan@ibu.edu.ba

Some results were published in Liđan's PhD thesis [7] from 2022. Their approach open the way for applying topological and commutative algebra methods to study these questions. In this article, we present two exciting results regarding the tiling of the board $m \times n$ in a plane and on a torus by $1 \times p$ polyomino and the balanced property of the corresponding simplicial complexes.

2 Preliminaries

This section reviews basic facts about simplicial complexes associated with polyomino tilings.

Definition 1. *An abstract simplicial complex K on a vertex set $[m] = \{1, 2, \ldots, m\}$ is a collection of subsets of $[m]$ such that*

(i) for each $i \in [m]$, $\{i\} \in K$,

(ii) for every $\sigma \in K$, if $\tau \subset \sigma$ then $\tau \in K$.

We assume that $\varnothing \in K$.

The elements of K are called *faces*. The dimension of a face σ of a simplicial complex K is defined as $\dim \sigma \colon = |\sigma| - 1$, where $|\sigma|$ denotes the cardinality of σ. Faces of dimension 0 are called *vertices* while faces of dimension one are called edges. *The dimension of* K, denoted by $\dim K$, is the maximal dimension of a face of K.
A face of K is called *maximal* or *a facet* if it is not a face of any other face of K. A subset τ of $[m]$ is called a *missing face* of K if $\tau \notin K$. A missing face τ is called *minimal* if any proper set $\sigma \subset \tau$ is a face K. A simplicial complex is completely determined by the list of its facets or minimal missing faces. Softwares like SAGE and Macaulay 2 take advantage of both approaches.
Let K be a simplicial complex and d its dimension.

Definition 2. *K is called balanced if there exists a function $k \colon V \mapsto [d+1]$ such that, if for every edge $\{x, y\} \in K$ holds $k(x) \neq k(y)$.*

In other words, Definition 2 says that a d-dimensional simplicial complex if its vertices can be colored into $d+1$ colors so that there is no monochromatic edge. Balanced simplicial complexes were introduced by Stanley in [10]. Notable examples are Coxeter complexes, Tits buildings, and the order complex of a graded poset, with the vertex set partition given by the rank function.
Simplicial complexes arise in many exciting ways. They are essential for applications of topology, commutative algebra and combinatorics. Baralić and Liđan studied a class of simplicial complexes arising from polyomino tilings. Recall that a polyomino is a plane geometric figure formed by joining one or more equal squares edge to edge. Some of their results are available in Liđan's PhD thesis [7].
Let M be a given finite subset of a plane or a torus square grid and $\mathcal{T}$ be a given finite set of polyomino tiles. Polyomino tiling problem asks whether M can be tiled using the tiles from $\mathcal{T}$ without overlapping or placing a tile so that it covers a cell outside of M. The problem that originated from recreational mathematics is an NP problem; it motivated further research in combinatorics and computer science. We assign a simplicial complex to the situation in the following way.

Definition 3. *$K(M;\mathcal{T})$ is a simplicial complex whose i-faces correspond to a placement of $i+1$ polyomino tiles from $\mathcal{T}$ onto M without overlapping.*

Definition of $K(M;\mathcal{T})$ implies the next property.

Proposition 1. *$\dim(K(M;\mathcal{T}))+1$ is maximal number of polyomino tiles from $\mathcal{T}$ that may be placed on M without overlapping.*

Geometrical features of simplicial complexes $K(M;\mathcal{T})$ can be visualized only in a handful of cases, such as the following example.

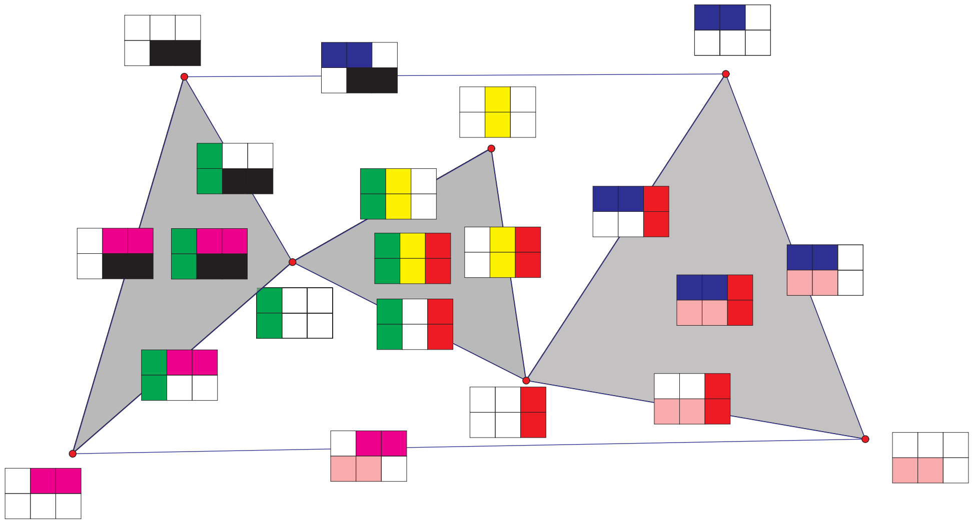


Figure 1: A geometrical realization of $K(M,\mathcal{T})$ when M is 2×3 plane board and $\mathcal{T}$ consists of dominoes.

Example 1. *If M is 2×3 plane board and $\mathcal{T}$ contains only dominoes, then a geometrical realization of the corresponding simplicial complex is presented in Figure 1.*

In the next section, we establish two results about balanced simplicial complexes associated with polyomino tilings.

3 Balanced simplicial complex associated with polyomino tilings

Let $\mathcal{T}$ consist of only I p-omino tile. Recall that I p-omino is $1\times p$ tile. Let $K(p;m,n)$ and $T(p;m,n)$ be the simplicial complexes of polyomino tilings for planar $m\times n$ board and torus $m\times n$ board, respectively.
I-omino tiles are sensitive to diagonal colorings of square grids. Indeed, if we label the cells by the numbers $1,2,\dots,p$ as in Figure 2, I p-omino will always cover p different numbers, regardless of its placement. Therefore, we cannot put more than d_i tiles on the $m\times n$ board, where d_i is the number of cells labeled with i in Figure 2.

1	2	3	4	5	6	7	8	9	$\cdots$	p	1	2	3	4	5	$\cdots$
p	1	2	3	4	5	6	7	8	$\cdots$	$p-1$	p	1	2	3	4	$\cdots$
$p-1$	p	1	2	3	4	5	6	7	$\cdots$	$p-2$	$p-1$	p	1	2	3	$\cdots$
$p-2$	$p-1$	p	1	2	3	4	5	6	$\cdots$	$p-3$	$p-2$	$p-1$	p	1	2	$\cdots$
$p-3$	$p-2$	$p-1$	p	1	2	3	4	5	$\cdots$	$p-4$	$p-3$	$p-2$	$p-1$	p	1	$\cdots$
$p-4$	$p-3$	$p-2$	$p-1$	p	1	2	3	4	$\cdots$	$p-5$	$p-4$	$p-3$	$p-2$	$p-1$	p	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
1	2	3	4	5	6	7	8	9	$\cdots$	p	1	2	3	4	5	$\cdots$
p	1	2	3	4	5	6	7	8	$\cdots$	$p-1$	p	1	2	3	4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 2: Labeling of the cells in $m \times n$ board.

Lemma 1. $\dim K(p;m,n) = d_{a+1} - 1$ *for integers* $m = m_1 p + a$ *and* $n = n_1 p + b$ *such that* $m_1, n_1 \geq 1$ *and* $0 \leq a, b \leq p-1$.

Proof: If $p \mid m$ or $p \mid n$, the corresponding horizontal or vertical side can be tiled by I p-omino in an obvious way. Therefore, the board can be tiled with $\frac{mn}{p}$ tiles, and each number in Figure 2 appears the same number of times. In this case, the statement is valid.

Let us analyze a challenging situation when $a, b > 0$. We will distinguish two cases: $a + b \leq p - 1$ and $a + b \geq p$.

If $a + b \leq p - 1$ we can conclude that $d_{a+1} = m_1 n_1 p + a n_1 + b m_1$, since there is no square labeled by $a + 1$ in the shaded region in Figure 3. The same figure depicts the positioning of $m_1 n_1 p + a n_1 + b m_1$ I p-ominoes in the board in this case.

Let us now consider the case $a + b \geq p$. We already know that $a + b \leq 2p - 2$.

In this case, from Figure 4, we can find that $d_{a+1} = m_1 n_1 p + a n_1 + b m_1 + a + b - p$ and that there is a tiling covering all squares labeled by $a + 1$. Similarly, as in the previous case, the critical argument is that in the shaded region in Figure 4, there is no $a + 1$ cell. It finishes our proof. ■

Theorem 1. *Simplicial complex* $K(p;m,n)$, $m, n \geq p$ *is balanced.*

Proof: Consider the cells labeled by $a + 1$ in Figure 2 and denote them by x_1, x_2, $\ldots$, $x_{d_{a+1}}$ as in Figure 5. Each I p-omino tile has to cover exactly one of these cells. Indeed, this induces coloring of the vertices of $K(p;m,n)$ in d_{a+1} colors since two vertices are colored by the same color exactly when they cover the same square denoted by x_i for some i. By definition of $K(p;m,n)$, they do not span an edge. Because of Lemma 1, it follows that $K(p;m,n)$ is balanced.

■

The following theorem establishes the analogues result for the simplicial complex $T(p;m,n)$ associated with the tiling of torus $m \times n$ board by I p-ominoes.

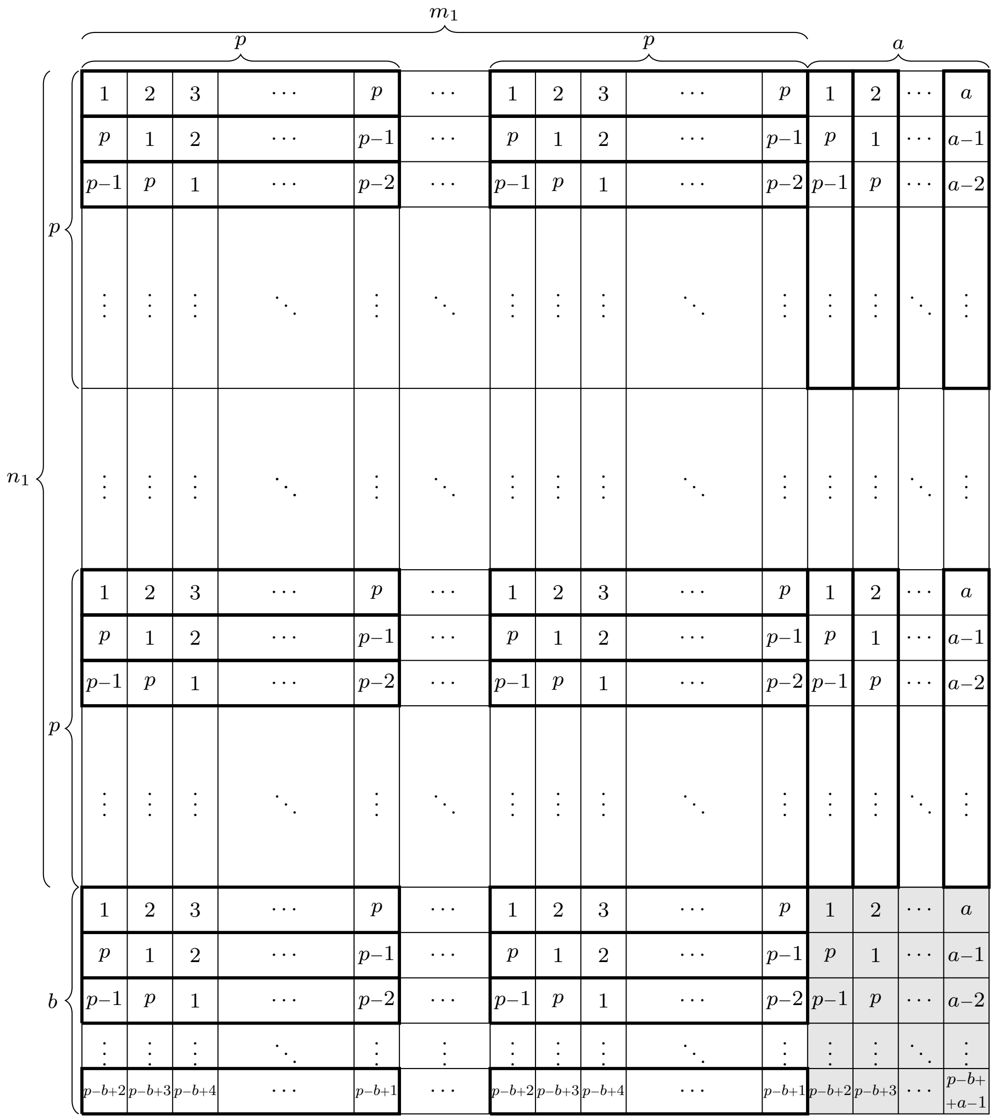


Figure 3: A placement of the maximal number of I p-ominoes in the board $m \times n$ in the case $a + b \leq p - 1$.

Theorem 2. *Simplicial complex $T(p; m, n)$ is a balanced complex if and only if $p \mid m$ and $p \mid n$.*

Proof: We can place at most $\lfloor \frac{mn}{p} \rfloor$ p-omino tiles on any $m \times n$ board so $\dim T(p; m, n) \leq \lfloor \frac{mn}{p} \rfloor - 1$. Observe that we can place mn vertical and mn horizontal I p-ominoes on torus $m \times n$ board, so $T(p; m, n)$ has $2mn$ vertices.
Assume that the vertices of $T(p; m, n)$ are properly colored. Two vertices can be colored in same way if the corresponding I p-omino placements overlap. Moreover, three vertices sharing the color correspond to the placements of tiles that cover a common square thanks to the tile of I-omino, which is convex. Celebrated Helly's theorem [9] implies that all vertices colored by the same color correspond to the tile placements having a joint cell. This implies that there are at most $2p$ vertices of $T(p; m, n)$ having same color.
We will examine the following cases separately.

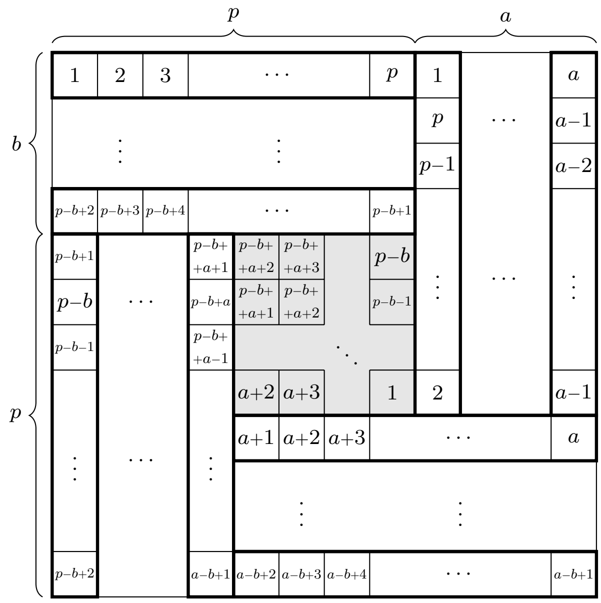


Figure 4: A placement of the maximal number I p-ominoes in the board $m \times n$ in the case $a + b \geq p$.

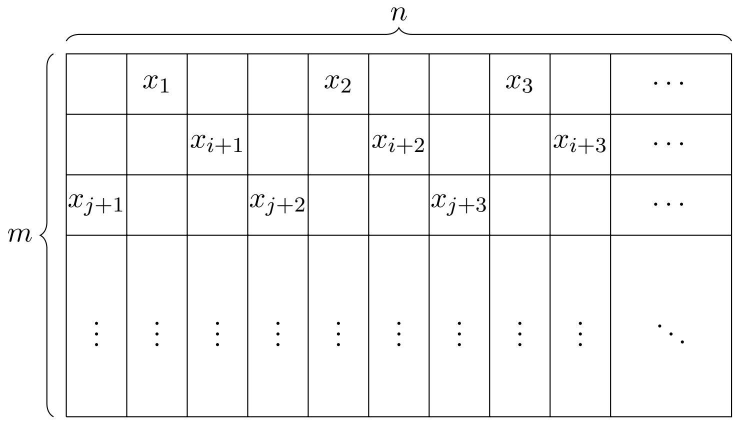


Figure 5: Coloring of vertices in simplicial complex $K(p; m, n)$.

1) Let $p|m$ and $p|n$. We can use the coloring used in the proof of Theorem 1 and verify that the complex is balanced.

2) Without loss of generality let assume that $p \nmid m$ and $p|n$. Obviously, $p|m \cdot n$. Then we used $\frac{m \cdot n}{p}$ colors, and there must be $2p$ vertices of $T(p; m, n)$ of each color. For each of $\frac{m \cdot n}{p}$ colors take a common cell for the corresponding placements of I p–omino tiles. Assume that these $\frac{m \cdot n}{p}$ cells are marked on the torus board.

   By placing one I p-omino tile on the board, we precisely cover one of the marked cells, so the coloring of each vertex is determined by the marked cell and the corresponding tile covers. On the other hand, in a row, we can have at most $\lfloor \frac{m}{p} \rfloor$ marked squares. We can show this in the following way. Between two consecutive marked cells in a row, we have to have precisely $p - 1$ empty cells to achieve that each vertex has its color and that I-omino cannot cover two marked squares. Since we have the torus board, we have at most $k + k(p-1)$ squares in each row, where k is the number of the marked cell in the row. The inequality is now proven.

However, it implies that the total number of marked cells in the board is at most $n \cdot \lfloor \frac{m}{p} \rfloor < n \cdot \frac{m}{p} = \frac{mn}{p}$ and that is the contradiction. Therefore, $T(p; m, n)$ is not balanced in this case.

3) Let $p \nmid m$ and $p \nmid n$. Then on the board we can put at most $\left\lfloor \frac{mn}{p} \right\rfloor - 1$ I p–ominoes, implying that

$$\left\lfloor \frac{mn}{p} \right\rfloor - 1 \geq \dim T(p; m, n).$$

If we used $\left\lfloor \frac{mn}{p} \right\rfloor$ colors for a proper coloring , then we would have at most $2p \cdot \left\lfloor \frac{mn}{p} \right\rfloor < 2p \cdot \frac{mn}{p} = 2mn$ vertices colored. But, this contradicts the fact that $T(p; m, n)$ has $2mn$ colored vertices.

Based on the considered cases, the statement has been proved. ■

Acknowledgements

The first author was supported by the Serbian Ministry of Science, Innovations and Technological Development through the Mathematical Institute of the Serbian Academy of Sciences and Arts. The authors are greatly indebted to the anonymous referees and Editor for careful reading and valuable comments and remarks.

References

[1] S. Golomb, *Polyominoes* (2nd ed.). Princeton, New Jersey: Princeton University Press (1994).

[2] I. Izmestiev, S. Klee and I. Novik, *Simplicial moves on balanced complexes*, Advances in Mathematics **320** (2017), 82–114.

[3] M. Juhnke-Kubitzke and S. Murai, *Balanced generalized lower bound inequality for simplicial polytopes*, Sel. Math. New Ser. **24** (2018), 1677–1689.

[4] M. Juhnke-Kubitzke and L. Venturello, *Graded Betti Numbers of Balanced Simplicial Complexes*, Acta Math. Vietnam. **46** (2021), 839–871.

[5] S. Klee, *The Fundamental Group of Balanced Simplicial Complexes and Posets*, The Electronic Journal of Combinatorics **16** (2009), no.2., 7.

[6] S. Klee and I. Novik, *Lower bound theorems and a generalized lower bound conjecture for balanced simplicial complexes*, Mathematika **62** (2016), no.2.. 441–477.

[7] E. Liđan, *Topological characterization of generalized polyomino tilings*, PhD thesis, University of Montenegro, (2022).

[8] G. Martin, *Polyominoes: A guide to puzzles and problems in tiling* (2nd ed.). Mathematical Association of America (1996).

[9] E. Helly (1923), *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jahresbericht der Deutschen Mathematiker-Vereinigung, **32** (1923), 175–176.

[10] R. P. Stanley, *Balanced Cohen-Macaulay complexes*, Trans. Amer. Math. Soc. **249** (1979), no.1., 139-157.

Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.02

Solving the dimer problem on Apollonian gasket

Tomislav Došlić and Luka Podrug

Abstract

For any three circles in the plane where each circle is tangent to the other two, the Descartes' theorem yields the existence of a fourth circle tangent to the starting three. Continuing this process by adding a new circle between any three tangent circles leads to Apollonian packings. The fractal structures resulting from infinite continuation of such processes are known as Apollonian gaskets. Close-packed dimer configurations on such structures are well modeled by perfect matchings in the corresponding graphs. We consider Apollonian gaskets for several types of initial configurations and present explicit expressions for the number of perfect matchings in such graphs.

1 Introduction

Configurations of close-packed dimers are among the most important objects of study in statistical physics. Also called Kekulé structures in chemical, and perfect matchings in mathematical literature, they serve as useful and (in may cases) tractable models of, among other things, adsorption processes of diatomic molecules on various types of substrates. In that context, the most interesting cases are two-dimensional surfaces with regular structure which are best described by finite portions of two-dimensional infinite lattices. For the simplest case of the square lattice, the exact solution has been known since the early works of Kasteleyn [5] and Temperley and Fisher [4, 12]. (Here by exact solution we mean explicit expressions for all quantities of interest in such problems, such as their growth-rate, entropy, and per-dimer molecular freedom.) For a brief survey of results for some other 2D lattices we refer the reader to [13].

In recent years much attention has been given to another class of lattices or networks, answering thus a need to model processes on fractal-like substrates. The best known example of such lattices is, certainly, the Serpiński gasket. For an accessible

(Tomislav Došlić) University of Zagreb Faculty od Civil Engineering, Croatia, tomislav.doslic@grad.unizg.hr

(Luka Podrug) University of Zagreb Faculty od Civil Engineering, Croatia, luka.podrug@grad.unizg.hr

introduction we refer the reader to [2] and references therein. In this paper we are concerned with related structures which arise from Apollonian packings of circles in the plane and which are known as Apollonian gaskets. Apollonian gaskets are fascinating objects; see, for example, [10] for some beautiful illustrations and for their geometric and number-theoretic properties. They have already attracted significant attention of researchers both in graph theory and in statistical physics. For example, their Tutte polynomials and spanning trees were investigated in [8, 14], while Ising and magnetic models on them were studied in [1, 11] by employing transfer matrices and numerical methods. In this paper we study them from the graph-theoretical point of view and present explicit formulas for the number of close-packed dimer configurations by computing the number of perfect matchings in the corresponding graphs.

In the next section we start from Apollonian packings and introduce the Apollonian gaskets as fractals arising from iterating the packings. Along the way we introduce the necessary graph-theoretic terminology and state some preliminary results. In Section 3 we use those results to present explicit formula for the number of close-packed dimer configurations in a given stage of one particular Apollonian gasket which is, in a sense, canonical. In Section 4 we generalize our findings to some other Apollonian gaskets. The paper is concluded by a short section in which we establish universality of entropy in all Apollonian gaskets and indicate some possible directions for future research.

2 Definitions and preliminary results

Apollonian gasket is a fractal that starts with a number of circles, usually arranged so that each circle is tangent to at least three, but sometimes more, other circles. Construction of the n^{th} level, or the n^{th} iteration, of an Apollonian gasket consists of adding more circles so that every new circle touches exactly three circles in the $(n-1)^{\text{st}}$ level. The number of circles inscribed depends on the number of starting circles and their initial configuration. Radius of each new circle is determined by Descartes' theorem which states that four circles that are tangent to each other at six distinct points with curvatures k_1, k_2, k_3 and k_4 satisfy the following relation:

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2). \tag{1}$$

Here the curvature of a circle is the reciprocal value of its radius, i.e., $k = \frac{1}{r}$. Note that equation (1) has two solutions. If one of them is negative, then the corresponding circle touches the other three from inside, otherwise, from outside of a circle. As an example, Figure 1 demonstrates one possible starting configuration of an Apollonian gasket. The curvatures of starting circles uniquely determine curvatures of circles in all subsequent iterations.

Graphs are one of the most fundamental mathematical objects. They naturally appear in many mathematical models, and in this paper we present one such model. More precisely, we consider close-packed dimer configurations on Apollonian gaskets and model them by perfect matching in the corresponding graphs. For the reader's convenience, we state some basic definitions. A *graph* is an ordered pair $G = (V, E)$,

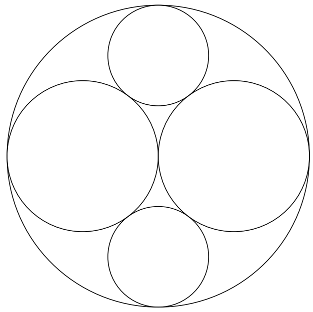


Figure 1: Initial configuration of an Apollonian gasket with circle curvatures -1, 2 and 3.

where $V = V(G)$ stands for a non-empty set of vertices, and $E = E(G)$ is the set of edges. We say that vertices u and v are *adjacent* if there is an edge connecting them, i.e., edge $uv \in E(G)$. For a given vertex v, the number of other vertices adjacent to it is called the *degree* of v. The largest degree in G is denoted by $\Delta(G)$. If all vertices of a graph G have the same degree k, we say that G is *k-regular*. A *perfect matching* in a graph G is a subset $M \subset E(G)$ such that every vertex of G is end-point of exactly one edge in M. The number of perfect matchings of graph G is denoted by $\Phi(G)$. Clearly, a necessary condition for G to have a perfect matching is that the number of vertices must be even. We refer the reader to the classical monograph by Lovász and Plummer [9] for a comprehensive introduction to almost all aspects of matching theory.

Perfect matchings are perfect models for close-packed dimer configurations: The fact that each dimer covers exactly two neighboring sites on the considered substrate is reflected in the fact that each edge of a perfect matching covers two adjacent vertices in the corresponding graph. Further, the non-overlapping requirement for dimers corresponds to the fact that each vertex is incident to exactly one edge of the perfect matching. Finally, the fact that the matching covers every vertex of the graph describes the condition that dimers cover all sites on the substrate, i.e., that they are no empty sites. So the configuration is close-packed.

In order to study dimer configurations on Apollonian gaskets, to each level of an Apollonian gasket we associate a graph obtained in the following manner: each point of tangency is a vertex of the graph and two vertices are adjacent if there is an arc connecting them, provided there are no other tangent points between them. One should note that not every Apollonian gasket produces a graph which has a perfect matching: The graph obtained from Apollonian gasket in Figure 1 has 9 vertices, and thus cannot have a perfect matching. All Apollonian gaskets considered from now on give rise to graphs with perfect matchings. Also, throughout this paper, we will assume that the outside circle in our starting configurations has radius 1.

The *line graph* of a graph G is defined as the graph $L(G)$, where $V(L(G)) = E(G)$, and two edges in $V(L(G))$ are adjacent if they have a common end-vertex in G. The line graph of a cycle is another cycle of the same length, while the line graph of a path is a path with one vertex less. Figures 2 and 3 show two polyhedra, the tetrahedron and the four-sided prism, respectively, and their respective line graphs. Here the edge a in G corresponds to vertex A in $L(G)$, etc.

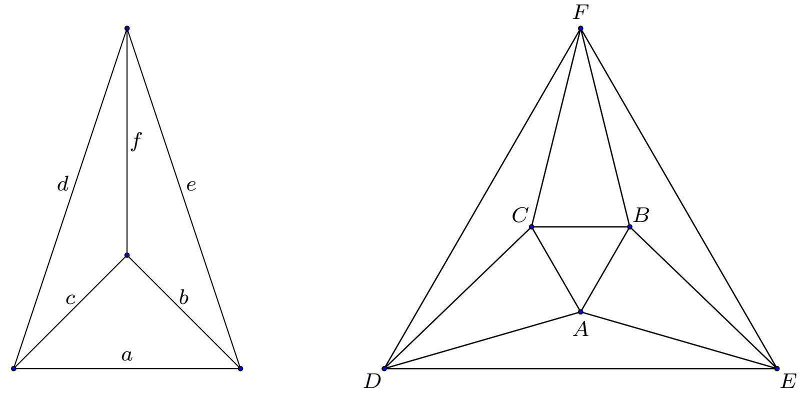


Figure 2: Tetrahedron and its line graph.

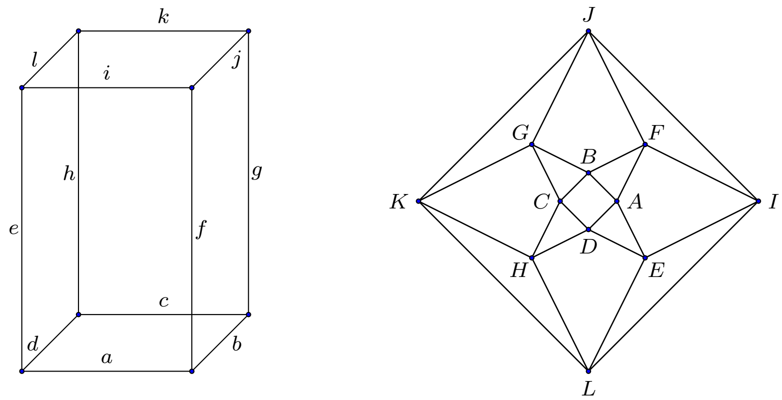


Figure 3: Four-sided prism and its line graph.

To determine a line graph of a given graph is quite straightforward, but the opposite task, to find a graph H such that a given graph G is the line graph of H, can be challenging. It is well-known that not all graphs are line graphs of other graphs. For more details we refer the reader to any textbook on graph theory.
It turns out that in some cases there is a simple relationship between the number of vertices and edges of a given graph G and the number of perfect matchings in its line graph $L(G)$. In particular, Dong, Yan and Zhang [3] proved the following result:

Theorem 1. *Let G be connected graph with n vertices and m edges, where m is even. If $\Delta(G) \leq 3$, then $\Phi(L(G)) = 2^{m-n+1}$.*

We will use this result to obtain the number of perfect matchings of graphs obtained from some configurations of Apollonian gaskets. We start from the case that is, in a way, canonical.

3 Tetrahedron - the canonical case

Let $\mathcal{A}_n$ denote an Apollonian gasket after n iterations and $|\mathcal{A}_n|$ the number of its circles. We denote the initial state of an Apollonian gasket $\mathcal{A}$ by $\mathcal{A}_1$.

Our canonical configuration consists of three starting circles having equal radii $2\sqrt{3}-3$ and one outside circle with radius 1. Figure 4 shows the initial state and the first two iterations of the corresponding Apollonian gasket. They have 4, 8 and 20 circles, respectively. Since the n^{th} iteration adds $4 \cdot 3^{n-2}$ new circles, for $n \geq 2$, the overall number of circles in $\mathcal{A}_n$ is given by

$$|\mathcal{A}_n| = 4 + 4\left(3^0 + \cdots + 3^{n-2}\right) = 2 \cdot \left(3^{n-1} + 1\right).$$

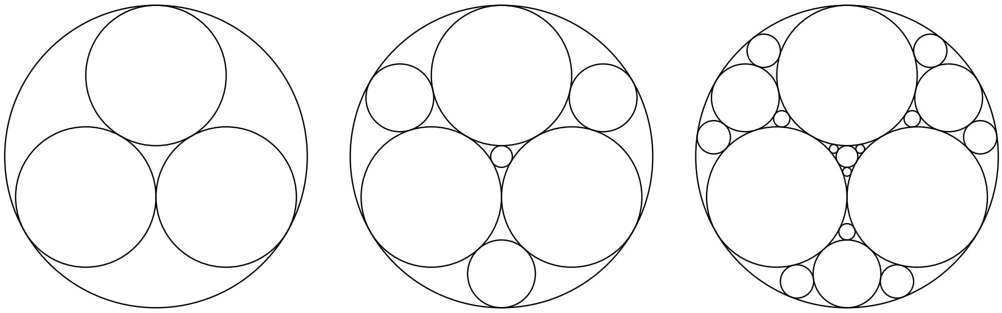


Figure 4: Apollonian gaskets $\mathcal{A}_n$ for $n = 1, 2, 3$.

Now we consider the graphs obtained from this Apollonian gasket. Let A_n denote a graph produced by the n^{th} level of $\mathcal{A}_n$. Then A_1 is a 4-regular graph with 6 vertices and 12 edges. In the second iteration there are 18 vertices, 36 edges, and the graph is again 4-regular. Since the n^{th} iteration adds $4 \cdot 3^{n-2}$ circles and every new circle is tangent to three previously added circles, the n^{th} iteration adds $4 \cdot 3^{n-1}$ new vertices. So,

$$|V(A_n)| = 6 + 4\sum_{i=1}^{n-1} 3^i = 2 \cdot 3^n.$$

Graphs A_n are shown in Figure 5 for $n = 1, 2, 3$. In is not hard to see that A_n is a 4-

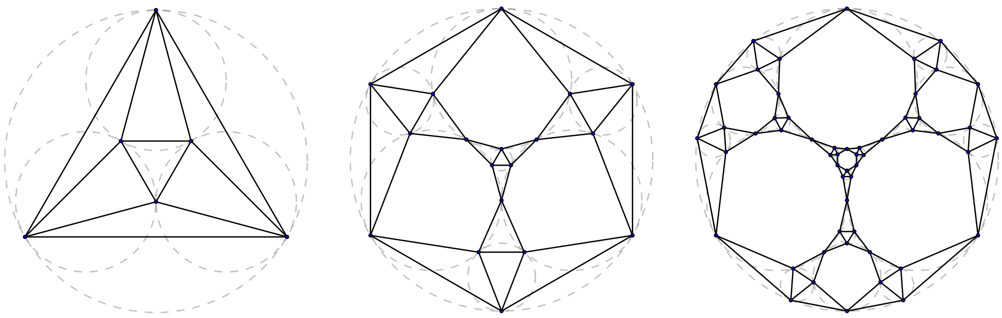


Figure 5: Graphs A_1, A_2 and A_3.

regular graph for every n. Addition of a new circle results in inscribing a new triangle in an already existing one. As shown in Figure 6, after inscribing new triangle shown in red, all new and old vertices have four neighbors. Thus, 4-regularity is preserved. It follows that the number of edges is $|E(A_n)| = 4 \cdot 3^n$.

One can see that A_1 is the octahedron, the line graph of tetrahedron, so we are motivated to ask whether other A_n also appear as line graphs of some related graphs.

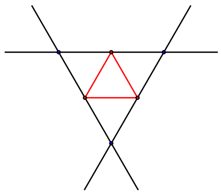


Figure 6: Inscribing a new triangle preserves 4-regularity.

We turn our attention to tetrahedron and start truncating it at its vertices. In every iteration we truncate every vertex of the graph constructed so far (Fig. 7). (Here we do not make distinction between a polyhedron and its graph, but there is no danger of confusion.) Let T_n denotes the graph obtained from the tetrahedron after $n-1$ iterations. Graph T_1 is non-truncated tetrahedron and has 4 vertices. To obtain T_2, we truncate every vertex and every vertex produces 3 vertices. Hence, $|V(T_n)| = 4 \cdot 3^{n-1}$. There are 6 edges in graph T_1 and n^{th} iteration brings $3|V(T_{n-1})|$ new edges. Hence,

$$\begin{aligned}|E(T_n)| =& 6 + 4 \cdot 3^1 + \cdots + 4 \cdot 3^{n-1} \\ =& 2 \cdot 3^n\end{aligned}$$

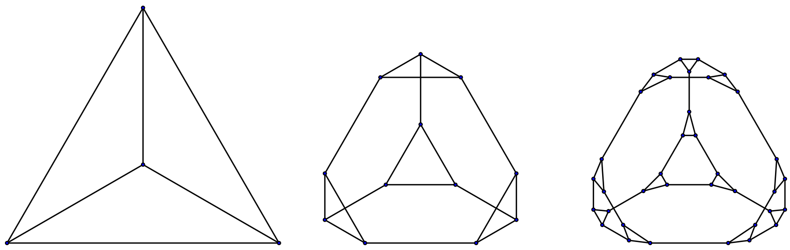


Figure 7: Graphs T_1, T_2 and T_3.

In the next lemma we establish the crucial connection between graphs A_n and T_n.

Lemma 1. $L(T_n) = A_n$.

Proof. The proof is by induction. The base of induction is true, as demonstrated in Figure 2. Graphs isomorphic to A_1, A_2 and A_3 are shown in Figure 8, where their fractal nature is more clear. As we stated, adding new circle results in a subdivision of graph A_n. More precisely, a new triangle is inscribed in an already existing one, thus producing four new triangles. The middle one of those four triangles is not further divided, but the other three are. The triangles that are to be divided are not hard to determine. The leftmost and the rightmost triangle with vertex at the top are divided in every iteration because the middle triangle with top vertex denotes the uppermost of the three original circles, and left and right of that circle is where we add new circles in every iteration. Since some triangles presents circles and other present blank spaces between them, no two divided triangles can share an edge. In that manner, all triangles that will be divided in next iteration are determined. The triangles that will be divided in the next iteration are the colored ones in Figure 8.

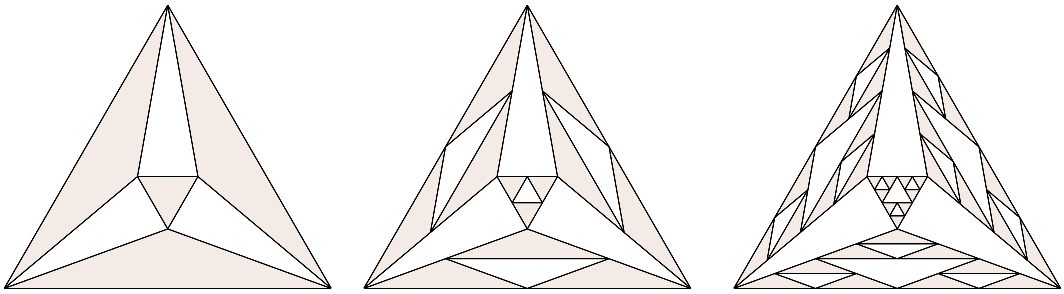


Figure 8: Graphs A_1, A_2 and A_3 with colored triangles that will be divided in the next iteration.

On the other hand, T_n is a 3-regular graph, and every triplet of edges that share a vertex forms a triangle in the corresponding line graph. We say that the vertex is represented by this triangle. Is is not hard to see that graphs shown in Figure 8 are also line graphs of T_1, T_2 and T_3, respectively, as truncating a common vertex inscribes a new triangle in the triangle that represents that vertex. The operation of truncating a vertex and inscribing a triangle in the line graph is shown in Figure 9. The left panel of Figure 9 shows newly-added edges, and the right part shows the corresponding vertices in the line graph, both marked red. We just need to show that triangles which represent vertices of T_n are precisely those colored triangles in Figure 8. First note that two triangles that share an edge in the line graph $L(T_n)$ cannot both represent a vertex. We consider three triangles in line graph that share top vertex. Since $|V(T_n)| = 4 \cdot 3^{n-1}$, there are at least $4 \cdot 3^{n-1}$ triangles in $L(T_n)$. The middle triangle can not represent a vertex, otherwise some vertices would have triangles with shared edge. Thus, alignment of triangles corresponds to Figure 8, and that concludes our proof.

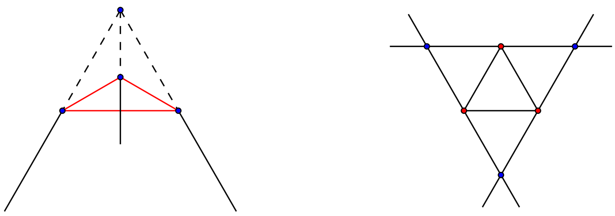


Figure 9: Truncation of a vertex in T_n and its effect in the line graph.

□

Figure 10 demonstrates the connection between graphs T_2 and A_2, more precisely, the fact that $L(T_2) = A_2$.
Now, by Lemma 1 we have $L(T_n) = A_n$ and $\Delta(T_n) = 3$, thus Theorem 1 is applicable to our graphs. Hence we can state the main result of this section.

Theorem 2.

$$\Phi(A_n) = 2 \cdot 4 \cdot 3^{n-1}.$$

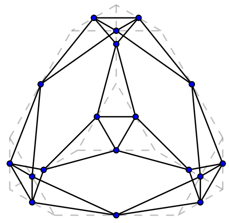


Figure 10: $L(T_2) = A_2$.

4 Other Apollonian gaskets

As we stated above, the initial number and the alignment of circles can vary. There are many different Apollonian packings yielding graphs with perfect matchings, and we here consider some possible simple alignments. Our first example starts with five circles, four with radius $\sqrt{2}-1$ and central one with radius $3-2\sqrt{2}$. As before, in each iteration we add circles in blank spaces between them. First three iterations of this gasket are shown in Figure 11. Similar as before, by B_n we denote a graph

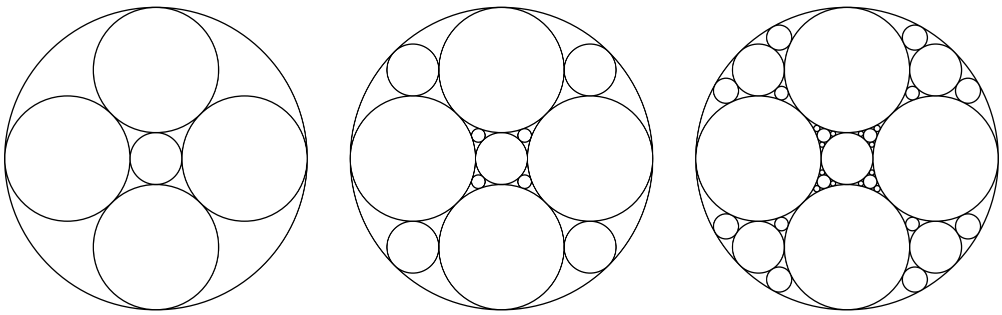


Figure 11: An Apollonian gasket starting with 5 circles.

obtained after $(n-1)^{\text{st}}$ iteration. Graph B_1 is 4-regular, it has 12 vertices and hence, 24 edges. There are $8 \cdot 3^{n-2}$ new circles in the n^{th} iteration that produce $8 \cdot 3^{n-1}$ new vertices. So graph B_n has

$$|V(B_n)| = 12 + 8 \cdot 3 + \cdots + 6 \cdot 3^{n-1} = 4 \cdot 3^n$$

vertices. Thus, it has $|E(B_n)| = 8 \cdot 3^n$ edges. Graphs isomorphic tor B_1, B_2 and B_3 are shown in Figure 12. Again, their fractal nature is clearly visible, because in each iteration, adding new circles in Apollonian gasket produces inscribed triangles in the corresponding graph. As before, B_n is 4-regular for every n.
As in the previous section, we want to identify a family of graphs P_n whose line graphs are graphs B_n. We start with a four-sided prism P_1, shown in Figure 3. Graph P_1 is 3-regular, it has 8 vertices and 12 edges. Furthermore, let P_n denote the starting prism which is truncated $n-1$ times. Every iteration has $8 \cdot 3^{n-1}$ vertices, and since truncation preserves 3-regularity, we have $|E(G)| = 4 \cdot 3^n$. Figure 3 shows the mapping $E(P_1) \to V(L(P_1))$ where edge a on the left corresponds to vertex A on the right, etc. So, $L(P_1) = B_1$. Since truncating a vertex produces same result as it did with tetrahedron, same arguments can be applied to prove that $L(P_n) = A_n$.

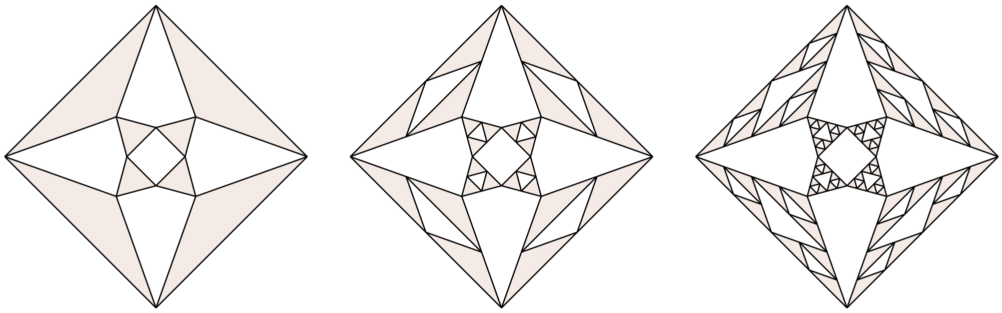


Figure 12: Graphs of initial levels of an Apollonian gasket starting with 5 circles.

Since graphs P_n satisfy assumptions of Theorem 1, we can compute the number of perfect matchings for this class of Apollonian gaskets:

Corollary 1.

$$\Phi(B_n) = 2 \cdot 16^{3^{n-1}}.$$

The Apollonian gasket we have just considered, the one defined by circle curvatures -1 and $1+\sqrt{2}$, can be generalized in a straightforward way. Instead of four circles whose centers are vertices of a square, we can start with m circles whose centers are vertices of regular polygon with m vertices, where $m \geq 4$ is an even integer. As before, let the radius of the outside circle be 1, and let R denote the radius of m circles. The central circle of radius r touches each of those m circles. Since $\sin\left(\frac{\pi}{m}\right) = \frac{R}{R+r}$ and $r + 2R = 1$, we have

$$R = \frac{\sin\left(\frac{\pi}{m}\right)}{1+\sin\left(\frac{\pi}{m}\right)}.$$

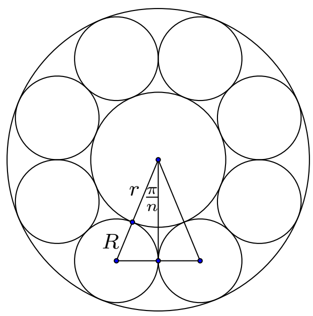


Figure 13: Graph of an Apollonian gasket starting with 9 circles.

We denote the n^{th} iteration of this Apollonian gasket configuration by $\mathcal{A}_n^m$ and the corresponding graph by A_n^m. The graph A_1^m has $3m$ vertices and since it is 4-regular, it has $6m$ edges. With every iteration we add $2m \cdot 3^{n-2}$ circles, hence, $2m \cdot 3^{n-1}$ new vertices. The overall number of vertices is now $V(A_n^m) = m3^n$ and the overall number of edges is $E(A_n^m) = 2m \cdot 3^n$. As we are interested in the number of perfect matchings of graph A_n^m, we only consider cases with m even, otherwise there are no perfect matchings.

By applying the same reasoning as in the previous cases, one can show that A_n^m is the line graph of $n-1$ times truncated m-sided prism, denoted by P_m^n. Again, the graph P_n^m is 3-regular and it has $2m \cdot 3^{n-1}$ vertices and $m \cdot 3^n$ edges. Since $\Delta(P_m^n) = 3$ and m is even, by Theorem 1, we have:

Corollary 2.

$$\Phi(A_n^m) = 2 \cdot 2^{m \cdot 3^{n-1}}.$$

We notice that our results apply also to $\mathcal{B} = \mathcal{A}^4$.

5 Universality of entropy

We conclude the paper with a few quick glances back to our original problem.
First, we notice that our canonical tetrahedral case could be brought within the framework of the gaskets generated by even-sided prisms by writing its growth rate $4^{3^{n-1}}$ as $2^{2 \cdot 3^{n-1}}$ and observing that it fits the pattern $2^{m \cdot 3^{n-1}}$ followed by their growth rates. Hence, the tetrahedron could be considered in this context as an "honorary" even-sided prism.
Our second glance back is toward entropies. The *entropy* of a lattice is defined as the limit of the ratio of the logarithm of the number of perfect matchings and the length of a perfect matching when the number of sites tends to infinity. Hence,

$$\mathcal{E}(\mathcal{G}) = \lim_{n \to \infty} \frac{2 \ln \Phi(G_n)}{|V(G_n)|},$$

where by G_n we denote the graph corresponding to the n^{th} level of a gasket $\mathcal{G}$. From our results it immediately follows that the entropy of $\mathcal{A}^m$ does not depend on m.

Corollary 3.

$$\mathcal{E}(\mathcal{A}^m) = \frac{\ln 4}{3}.$$

Hence, while the growth rates depend strongly on m, the corresponding entropy seems to be universal. This fact also implies the universality of the quantity known as per-dimer molecular freedom, defined as

$$W(\mathcal{G}) = e^{\mathcal{E}(\mathcal{G})}.$$

When the considered dimer configurations are not close-packed, i.e., when one deals with monomer-dimer configurations, all quantites of interest become much more difficult to determine; see, for example, a recent paper concerned with generalized Tower of Hanoi graphs [7]. It would be interesting to examine whether some progress could be made by using the approach of the present paper.
Our last remark is concerned with a recent paper by Li *et al.* [6] which extends the line of research of Dong *et al.* by looking at the dimer problem at the vertex-edge graphs of cubic graphs. It would be interesting to find classes of fractal lattices for which the results by Li *et al.* would provide compact explicit solutions of the type presented here.

Acknowledgements

Partial support of Slovenian ARRS (Grant no. J1-3002) is gratefully acknowledged by T. Došlić.

References

[1] R. F. S. Andrade, H. J. Herrmann, Magnetic models on Apollonian networks, *Phys. Rev. E* 71 (2005) 056131.

[2] J. A. Anema, K. Tsougkas, Counting spanning trees on fractal graphs and their asymptotic complexity, *J. Phys. A: Math. Theor.* 49 (2016) 355101.

[3] F. Dong, W. Yan, F. Zhang On the number of perfect matching of line graphs, *Discrete Appl. Math.* **161**, 794–801 (2013).

[4] M. E. Fisher, Statistical mechanics of dimers on a plane lattice, *Phys. Rev.* 124 (1961) 1664–1672.

[5] P. W. Kasteleyn, The statistics of dimers on a lattice, *Physica* 27 (1961) 1209–1225.

[6] S. Li, D. Li, W. Yan, On the dimer problem of the vertex-edge graph of a cubic graph, *Discrete Math.* 346 (2023) 113427.

[7] W.-B. Li, S.-C. Chang, Study of dimer–monomer on the generalized Hanoi graph, *Comput. Appl. Math.* 39 (2020) 77.

[8] Y. Liao, Y. Hou, X. Shen, Tutte polynomials of the Apollonian networks, *J. Stat. Mech.* (2014) P10043.

[9] L. Lovász, M. D. Plummer, *Matching Theory*, in: Ann. Discrete Math. vol. 29, North-Holland, New York, 1986.

[10] D. Mackenzie, A tisket, a tasket, an Apollonian gasket, *Amer. Sci.* 98 (2010) 10–14.

[11] M. Serva, U. L. Fulco, E. L. Albuquerque, Ising models on regularized Apollonian networks, *Phys. Rev. E* 88 (2013) 042823.

[12] H. N. V. Temperley, M. E. Fisher, Dimer problem in statistical mechanics – An exact result, *Phil. Mag.* 6 (1961) 1061–1063.

[13] F. Y. Wu, Dimers on two-dimensional lattices, *Int. J. Mod. Phys. B* 20 (2006) 5357–5371.

[14] Z. Zhang, B. Wu, F. Comellas, The number of spanning trees in Apollonian networks, *Discrete Appl. Math.* 169 (2014) 206–213.

Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.03

The Birkhoff polytope of the groups F_4 and H_4

Mathieu Dutour Sikirić

Abstract

We compute the set of facets of the polytope which is the convex hull of the Coxeter groups F_4 or H_4:

- For the group F_4 we found 2 orbits of facets which contradicts previous results published in [19].
- For the group H_4 we found 1063 orbits of facets which provides a counterexample to the conjecture of [19].

1 Introduction

Given a finite group G acting linearly on a real vector space $\mathbb{R}^n$, there is a strong interest in finding the orbits of facets of the convex hull Gx of a vector x.
For a coxeter group G with its natural action on $\mathbb{R}^n$, the structure is well known and given by the Wythoff construction (See [5]). Other representations were considered in [4] for the alternating group. Another very interesting case for a group having a n-dimensional representation is to consider the action of the group on itself. For the symmetric group S_n this gets us the Birkhoff polytope.
In [1, 19] the description was extended to other Coxeter groups by introducing the Birkhoff tensors B_G which are facets of conv G. In [18] the symmetry group of conv G are determined for all the finite Coxeter groups.
The authors proved the following result:

Theorem 1. *For $G = \mathsf{A}_n$, B_n, $\mathsf{I}_2(n)$ and H_3 all facets of* conv G *are Birkhoff tensors.*

They also proved that for D_4 the result does not hold and they claim in Theorem 8.1 that this implies that the result does not hold for D_n and E_n. Note that the authors also claimed that F_4 satisfy the theorem but we prove that this is not true. The authors conjectured [19, Problem 8.1] that for H_4 the result does hold. As it turns out, this is not true since while there is just one orbit of Birkhoff tensors, there is more than one orbit of facets:

(Mathieu Dutour Sikirić) Ruđer Bošković Institute, Zagreb, Croatia, mathieu.dutour@gmail.com

Theorem 2. *The polytope* conv H_4 *has* 188455824000 *facets in* 1063 *orbits.*

The proof of this result is computational with the algorithms presented in Section 3 and the results presented in Section 4.

2 Definitions

2.1 Convex geometry

A set $S \subset \mathbb{R}^n$ is called *convex* if for all $x, y \in S$ we have $[x, y] \subset S$. A convex set is called a *polytope* if there exist vectors $x_1 \dots, x_N$ such that

$$S = \left\{ x \in \mathbb{R}^n \text{ s.t. } x = \sum_{i=1}^{N} \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1 \right\}$$

and S is called the convex hull $Conv(\{x_1, \dots, x_N\})$ of $x_1, \dots, x_N$. A point $x \in S$ which cannot be expressed as the middle of two points $y_1, y_2 \in S$ is called a *vertex*. A polytope P has finitely many vertices and the vertices $v_1, \dots, v_N$ allow to write $P = Conv(\{v_1, \dots, v_N\})$.
For a polytope P there exist a number of linear forms f_i and constants C_i such that

$$P = \{x \in \mathbb{R}^n \text{ s.t. } f_i(x) \leq C_i \text{ for } 1 \leq i \leq M\}$$

The inequalities f_i which cannot be expressed as the sum of two other such inequalities are called facets. Passing from the description from vertices to the description by facets is called the *dual description problem*. See [21] for an introduction to the subject. This paper is just two special cases of dual descriptions problems being solved.

2.2 Coxeter group

A *finite Coxeter group* G is a finite group of isometries acting on a real space $\mathbb{R}^n$ which is generated by N reflections. A Coxeter group is called irreducible if there exist a non-trivial subspace preserved by G. Such groups were classified by Coxeter himself and we refer to [13] for related definitions and terminology. In dimension 4 there are the A_4, B_4, D_4, F_4, H_4.
Coxeter groups are tightly related to regular polytopes (see [3]). For example the isometry group of the 24-cell is the Coxeter group F_4. Also the isometry group of the 120-cell and 600-cell is the Coxeter group H_4.

3 Algorithms

The effective computation of dual description of polytopes is a classic problem in [11]. We designed software for using symmetries of polytope when computing their dual descriptions. The initial version of the code was written in `GAP` with some parts in `C/C++`. After several extensions, the code was completely ported into `C++`. As a side product of that, we are allowed to select the numeric type of the occurring matrix entries.

3.1 Dual description algorithms

In order to compute the facets of the polytope, we used the recursive adjacency decomposition technique. This method has been used for many different computations and is explained in [2, 6, 10, 20].
The idea of the adjacency decomposition technique is to compute one facet and from this facet to compute the adjacent facets. The obtained facets are then tested for equivalence. Computing the facets adjacent to a given facet is itself a dual description problem, therefore one may need to apply the method recursively hence the name recursive adjacency decomposition method.
Some early termination criterion are given in [6] and allow us to avoid having to compute the adjacencies of all the facets.
For a long time we used the code developed in [8] which is a package of [12]. For several reasons we have developed a new `C++` implementation (see [9]) that allow us to gain additional speed and functionality.

3.2 Fields

The commonly used numerical type is **mpq_class** from the GMP library which is a multiprecision rational type. It is supported in GAP as well as `C++`. In order to compute with polytopes related to the Coxeter groups H_3 and H_4 one needs to allow for the ring $\mathbb{Q}[\sqrt{5}]$. Implementing the arithmetic operations $(+, -, *, /)$ is relatively easy but the sign determinations require more care.
Testing if $a + b\sqrt{5}$ is positive can be done in the following way. If a and b are of the same sign it is easy to conclude. Otherwise, a and b are of opposite sign and we write

$$a + b\sqrt{5} = \frac{a - b\sqrt{5}}{a^2 - 5b^2}$$

The sign of $a - b\sqrt{5}$ can be decided and together with the sign of the rational number $a^2 - 5b^2$ we can conclude. The same strategy allows one to decide the sign in mixed cubic rings. For other real fields of algebraic numbers, different approaches would have to be used.
Also, for some subroutines like `lrs` one needs only to use ring operations. In that case one can reduce to the case of $\mathbb{Z}[\sqrt{5}]$. For the kind of computations we are doing here there is no need for algebraic closures or such kind of constructions.

3.3 Canonicalization strategies

In preceding works, when we had two orbits of facets in order to check isomorphism, we used the [12] implementation of the partition backtrack. See [16, 17] and [15] for accounts of this class of algorithms. That is we encode facets by the subset of their incident vertices and then use the partition backtrack for set equivalence.
However, in [14] an algorithm for finding a canonical representative of a subset for a permutation group action was found. This greatly simplifies the code since for N orbits instead of having to compute N equivalences, we simply have to do one canonicalization and one string comparison.

4 Results

4.1 Results for F_4

For the group F_4 we use the following two generators:

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

There are exactly two orbits of facets for conv(F_4):

1. Orbit 1 of Birkhoff tensor with incidence 288. Stabilizer has order 4608. One representative inequality is $Tr(XA) \leq 1$ with
$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$
2. Orbit 2 of facets with incidence 288. Stabilizer has order 48. One representative inequality is $Tr(XA) \leq 1$ with
$$A = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

The total number of facets is 55872.

4.2 Results for H_4

For the group H_4 we use the following two generators:

$$\frac{1}{4}\begin{pmatrix} 1 & -2 & -1 & 0 \\ 2 & 2 & -2 & 2 \\ 1 & -2 & 0 & 1 \\ 0 & -2 & -1 & 1 \end{pmatrix} + \frac{\sqrt{5}}{4}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We found 1063 orbits of facets of conv H_4. If we express such facets in the form $Tr(AX) \leq 1$ then one 1, 4, 130, 928 orbits of facets of rank 1, 2, 3 and 4.
Tables 1 and 2 give the statistics about the incidence of orbits and about the size of their stabilizers. The full list of orbits is presented in [7]. See below one matrix of incidence 120 with a stabilizer of size 120. This suffices to show that the conjecture is false.

$$A = \frac{1}{4}\begin{pmatrix} -6 & -11 & -7 & 4 \\ 0 & -2 & 3 & 1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{\sqrt{5}}{4}\begin{pmatrix} 2 & 5 & 3 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

p	Nr.	p	Nr.	p	Nr.	p	Nr.
16	376	17	282	18	116	19	85
20	48	21	30	22	12	23	5
24	44	25	3	26	31	28	10
30	7	32	5	36	4	38	1
48	1	100	1	120	1	480	1

Table 1: For each incidence p the number of orbits of incidence p is given.

s	Nr.	s	Nr.	s	Nr.	s	Nr.
1	800	2	189	4	50	6	1
8	8	12	4	16	3	24	1
36	1	40	1	48	2	120	1
576	1	2880	1				

Table 2: For each size s the number of orbits of orbits having a stabilizer of size s is given.

References

[1] J. Brandman, J. Fowler, B. Lins, I. Spitkovsky, and N. Zobin, *Convex hulls of Coxeter groups*, Function spaces, interpolation theory and related topics (Lund, 2000), de Gruyter, Berlin, 2002, pp. 213–240.

[2] D. Bremner, M. Dutour Sikirić, and A. Schürmann, *Polyhedral representation conversion up to symmetries*, Polyhedral computation, CRM Proc. Lecture Notes, vol. 48, Amer. Math. Soc., Providence, RI, 2009, pp. 45–71.

[3] H. S. M. Coxeter, *Regular polytopes*, third ed., Dover Publications, Inc., New York, 1973.

[4] J. Cruickshank and S. Kelly, *Rearrangement inequalities and the alternahedron*, Discrete Comput. Geom. **35** (2006), no. 2, 241–254.

[5] M. Deza, M. Dutour Sikirić, and S. Shpectorov, *Hypercube embeddings of Wythoffians*, Ars Math. Contemp. **1** (2008), no. 1, 99–111.

[6] M. Deza and M. D. Sikirić, *Enumeration of the facets of cut polytopes over some highly symmetric graphs*, Int. Trans. Oper. Res. **23** (2016), no. 5, 853–860.

[7] M. Dutour Sikirić, *Convex hull of Coxeter groups*, URL: https://mathieudutour.altervista.org/ConvCoxeter.

[8] M. Dutour Sikirić, `Polyhedral`, *a GAP package*, 2015, http://mathieudutour.altervista.org/Polyhedral/index.html.

[9] M. Dutour Sikirić, `Polyhedral, a C++ package`, 2020, https://github.com/MathieuDutSik/polyhedral_common.

[10] M. Dutour Sikirić, A. Schürmann, and F. Vallentin, *The contact polytope of the Leech lattice*, Discrete Comput. Geom. **44** (2010), no. 4, 904–911.

[11] J. E. Goodman, J. O'Rourke, and C. D. Tóth (eds.), *Handbook of discrete and computational geometry*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2018, Third edition of [MR1730156].

[12] T. G. group, *GAP — groups, algorithms, and permutations, version 4.12.2*, URL: http://www.gap-system.org.

[13] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[14] C. Jefferson, E. Jonauskyte, M. Pfeiffer, and R. Waldecker, *Minimal and canonical images*, J. Algebra **521** (2019), 481–506.

[15] C. Jefferson, M. Pfeiffer, W. A. Wilson, and R. Waldecker, *Permutation group algorithms based on directed graphs*, J. Algebra **585** (2021), 723–758.

[16] J. S. Leon, *Permutation group algorithms based on partitions. I. Theory and algorithms*, vol. 12, 1991, Computational group theory, Part 2, pp. 533–583.

[17] J. S. Leon, *Partitions, refinements, and permutation group computation*, Groups and computation, II (New Brunswick, NJ, 1995), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 28, Amer. Math. Soc., Providence, RI, 1997, pp. 123–158.

[18] C.-K. Li, I. Spitkovsky, and N. Zobin, *Finite reflection groups and linear preserver problems*, Rocky Mountain J. Math. **34** (2004), no. 1, 225–251.

[19] N. McCarthy, D. Ogilvie, I. Spitkovsky, and N. Zobin, *Birkhoff's theorem and convex hulls of Coxeter groups*, Linear Algebra Appl. **347** (2002), 219–231.

[20] M. D. Sikirić, A. Schürmann, and F. Vallentin, *Classification of eight-dimensional perfect forms*, Electron. Res. Announc. Amer. Math. Soc. **13** (2007), 21–32.

[21] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.

Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.04

Grids of equidistant walks

Biserka Kolarec

Abstract

We look at arithmetical progressions as equidistant walks, the difference being the steps' length. One can construct equidistant grids by putting equidistant walks to rows and columns. We introduce a notion of diagonal walks across grids. One specific diagonal walk is interesting because the sum of its elements gives polygonal numbers. We give one specific construction method for obtaining equidistant grids. It assumes alternating two equidistant walks placed in the first two columns, shifted up by the same shift whenever some of them repeat. We find conditions on the steps' lengths and the shift size that ensure that grids obtained in this way are equidistant. Also, we define a zig-zag pattern walks over a two-column grid. It is possible to identify them with horizontal walks in an infinite grid made of given columns by the mentioned construction method. Finally, we form equidistant walks of differences of consecutive products with overlapping odd and even factors and show that the distance between them is constant.

Keywords: *arithmetical progression, equidistant walks, equidistant grids, polygonal numbers, products with overlapping factors*

1 Introduction and preliminaries

An arithmetical progression or an arithmetical sequence is a sequence of numbers such that the difference between consecutive numbers is constant. If a_1 is the initial term and d is the common difference of consecutive numbers, then the n-th term a_n of the progression is given by: $a_n = a_1 + (n-1)d$. In general $a_n = a_m + (n-m)d$ for natural numbers n and m, $n > m$. We restrict here to sequences of positive integers, so $a_1, d \in \mathbb{N}^+$.

We consider arithmetical progressions as **equidistant walks** to infinity with a common difference as "the step length." Indeed, starting from the first term a_1 one obtains the next term from the previous by taking the step of the length d:

$$a_1, a_1 + d, (a_1 + d) + d, \ldots$$

(Biserka Kolarec) University of Zagreb, Faculty of Agriculture, Zagreb, Croatia, bkudelic@agr.hr

The arithmetical progression of odd numbers: $1, 3, 5, 7, 9, \ldots$ is an equidistant walk with the steps' length equal to 2, the same as the arithmetical progression of even numbers: $2, 4, 6, 8, 10, \ldots$. The only difference between those two equidistant walks is in the initial term.

Equidistant walks serve as building blocks for grids. To form a grid, we put one equidistant walk that starts with a_1 and has the steps' length d_c to the first column, and the other starting again with a_1 with the step length d_r to the first row. Then we add equidistant walks with step lengths equal to d_c (or d_r) to columns (or rows). So, equidistant grids are given by the first term a_1 and the pair of differences (d_r, d_c).

There are different kinds of number grids in literature. For example, the most well-known is the Pascal triangle based on the usual Euclidean square grid

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \vdots & \vdots & \vdots & \ddots & . \end{array}$$

and the hyperbolic Pascal triangles based on the hyperbolic square grids, see [1]. Here we consider the Euclidean grid and new number grids: equidistant grids with equidistant walks in rows and columns.

Example 1. *Having $a_1 = 1$ and $d_c = d_r = 2$, we obtain the next grid of odd numbers*

$$\begin{array}{ccccc} 1 & 3 & 5 & 7 & \cdots \\ 3 & 5 & 7 & 9 & \cdots \\ 5 & 7 & 9 & 11 & \cdots \\ 7 & 9 & 11 & 13 & \cdots \\ \vdots & \vdots & \vdots & \ddots & . \end{array} \tag{1}$$

Notice also the other possible construction of this grid: besides odd numbers in the first column, we add new columns by shifting the walk from the previous column up by one and omitting the first number.

Here, we define walks over equidistant grids and a grid characteristic. We will show that equidistant grids are means to generate polygonal numbers. There are known problems of finding arithmetical progressions in polygonal numbers tackled in [2] and [3]. Polygonal numbers may not contain long arithmetical sequences. However, they constitute them. It is not new that polygonal numbers are sums of numbers that form an arithmetical sequence. However, equidistant grids provide more than one way to do this and give all possible ways. More precisely, we shall prove that s-gonal numbers are sums of elements of a diagonal walk in the grid of the characteristic s. Adding numbers in different directions of a number grid is an old idea. For example, it is known that the shallow diagonal sums of Pascal's triangle give Fibonacci numbers:

Some equidistant grids, like the (1) from above, are obtained by a specific construction method. It is the method of alternating columns of two equidistant walks of

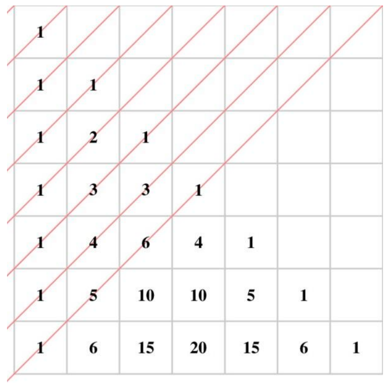


Figure 1: Sums in a Pascal triangle

the same steps' length, with each column shifted up by some fixed shift k in every new appearance. Some conditions on d_c, d_r, and k ensure the grid is equidistant.
Define an (R, D)**-walk** across the grid: for a pair of integers R and D, R gives a move to the right and D a move down that leads from one number of the walk to another. Here the negative value of R indicates a move to the left by $|R|$, and the negative value of D is a move up by $|D|$. Figure 1 illustrates directions of (R, D)-walks starting from the position A for different signs of R and D.

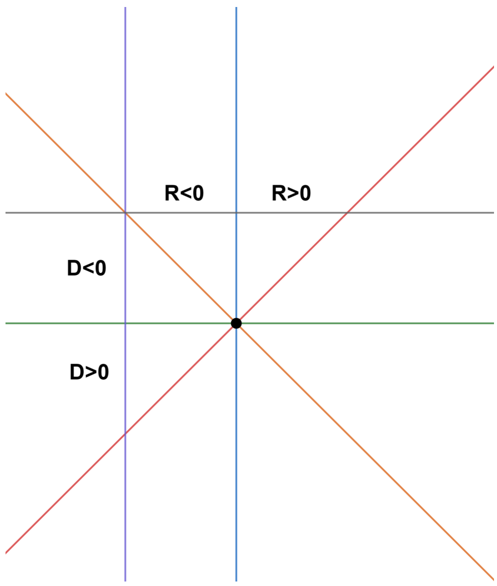


Figure 2: Directions of (R,D)-walks

Walks may start at any point in a grid. Observe that $(R, 0)$-walks are horizontal walks and $(0, D)$-walks are vertical walks. We call all other (R, D)-walks diagonal, not just the walk in the direction of the main diagonal for which $R = D$. Negative R and/or D results in a finite (R, D)-walk. Among infinite walks, we distinguish diagonal walks of the steep incline down for which $R < D$ and diagonal walks of the slight incline down if opposite.
Observe that each walk across the grid is an equidistant walk. We call such grid an **equidistant grid**. Being equidistant does not mean all walks through a grid

have the same step length. In the grid (1) any horizontal $(R,0)$-walk has the steps' length $2R$, for vertical $(0,D)$-walk it is $2D$, while a diagonal (R,D)-walk has the steps' length $2(R+D)$.
Notice that in a grid of odd numbers (1) any (R,D)-walk that starts with some number a_1 can be identified simply to an $(0,R+D)$-walk over the first column of the grid, i. e. a walk in which, starting from a_1, one steps on every $(R+D)$th number of the walk.
Observe that all the data necessary for the construction of the grid, (1) in particular, or any other equidistant grid in general, are given in the upper-left triangle of the grid, namely

$$\begin{array}{cc} 1 & 3 \\ 3 & \end{array} .$$

To an equidistant grid given by a_1, d_c and d_r **a grid characteristic** s is $s = 2a_1 + d_c + d_r$. This is the sum of elements $a_1 + d_c, a_1 + d_r$ of the first non-trivial $(1,-1)$-walk in an equidistant grid

$$\begin{array}{cc} a_1 & a_1 + d_r \\ a_1 + d_c & \end{array} .$$

Example 2. *Next grids have grid characteristics* 13 *and* 8*, respectively*

$$\begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 6 & 12 & 18 & 24 & \cdots \\ 11 & 17 & 23 & 29 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}, \quad \begin{array}{ccccc} 1 & 7 & 13 & 19 & \cdots \\ 1 & 7 & 13 & 19 & \cdots \\ 1 & 7 & 13 & 19 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} .$$

Notice that above, sums of elements of $(1,-1)$*-walks give tridecagonal and octagonal numbers, respectively:*

$$\begin{array}{rcl} 1 & = & 1 \\ 13 & = & 6+7 \\ 36 & = & 11+12+13 \\ \vdots & & \end{array}, \quad \begin{array}{rcl} 1 & = & 1 \\ 8 & = & 1+7 \\ 21 & = & 1+7+13 \\ \vdots & & \end{array} .$$

It happens by no chance: as we shall see, an equidistant grid with $a_1 = 1$ *and a characteristic* $s = 2 + d_c + d_r$ *generates* s*-gonal numbers as sums of members of* $(1,-1)$*-walks. In addition, observe that the first grid gives a non-standard decomposition of tridecagonal numbers; the standard one that describes the geometrical formation of* 13*-gonal numbers is:*

$$\begin{array}{rcl} 1 & = & 1 \\ 13 & = & 1+12 \\ 36 & = & 1+12+23 \\ \vdots & & \end{array} .$$

2 Equidistant grids and polygonal numbers

Polygonal numbers are numbers that, represented by dots, form a usual polygonal pattern. So, triangular numbers $1, 3, 6, \ldots$ form triangles, square numbers $1, 4, 9, \ldots$ form squares, and so.
Let s denote the number of polygon sides. There is a formula for the nth s-gonal number [2]

$$P_n{}^{(s)} = \frac{n}{2}\Big(2 + (n-1)(s-2)\Big).$$

Notice that the list of s-gonal numbers starts with 1 and then proceeds with s, since the second s-gonal number equals exactly s.

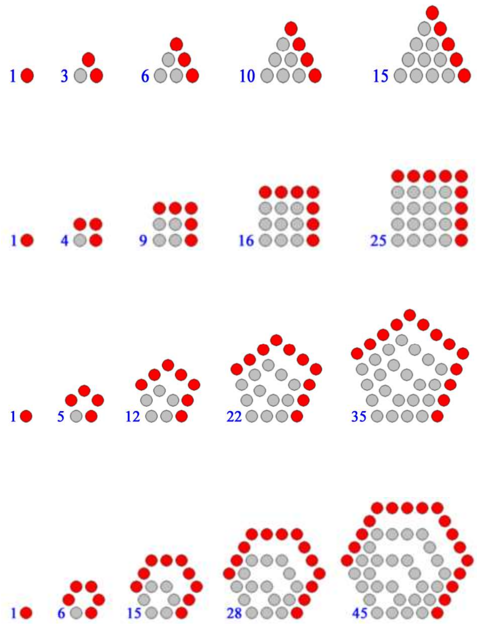


Figure 3: s-gonal numbers for $s = 3, 4, 5, 6$ (retrieved from Wikipedia.)

There are natural decompositions of nth s-gonal number to a sum of n members of an arithmetical sequence. These decompositions testify geometrical formation of s-gonal numbers. Namely, for triangular numbers, the list starts with 1, proceeds with $3 = 1 + 2$ because to one dot we must add two more dots to get the smallest non-trivial triangle, continues with $6 = 1 + 2 + 3$ because one has to add three more dots to get the third triangular number, and so on. In general, one enlarges the sides of an s-gon by the next member of an arithmetical sequence:

- for $s = 3$ addends are $\ldots(((1 + 2) + 3) + 4)\ldots$
- for $s = 4$ one has $\ldots(((1 + 3) + 5) + 7)\ldots$
- for $s = 5$, $\ldots(((1 + 4) + 7) + 10)\ldots$

These decompositions are justified geometrically: one can construct the $(n + 1)$st s-gonal number from the nth s-gonal number. Indeed, there is an initial point from which the s-gons begin to grow. Each of the two sides that meet in the initial point contains n points, so one adds $n-2$ points to each of the other $s-2$ sides to form the next s-gon. Since $(n-2)(s-2) = (n-1)(s-2) + (s-2)$ we see that the new s-gonal formation originates from the old one enlarged by $s-2$. Therefore $s-2$ is the difference of an arithmetical sequence whose consecutive sums give s-gonal numbers.

Besides the natural decomposition of s-gonal numbers as sums of members of equidistant walks, we can read others in the equidistant grids of a characteristic s. The total number of such decompositions is $\left\lfloor \frac{s}{2} \right\rfloor$. The following theorem holds.

Theorem 1. *For a given integer s, $s \geq 2$ there are exactly $\left\lfloor \frac{s}{2} \right\rfloor$ decompositions of s-gonal numbers as sums of members of $(1,-1)$-walks in equidistant grids of characteristic s.*

Proof. We claim that in every equidistant grid of the characteristics s sums of elements of $(1,-1)$-walks generate s-gonal numbers. Let $p \in \{1, 2, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\}$. Look at the equidistant grid of characteristics s:

$$\begin{array}{ll} 1 & s-p \\ p & \end{array}.$$

An equidistant walk in the first column starts with $a_1 = 1$ and has a step length equal to $p-1$. The one in the first row starts with $b_1 = 1$; its step length is $s-p-1$. So, $a_n = 1 + (n-1)(p-1)$ and $b_n = 1 + (n-1)(s-p-1)$. The sum of elements of a $(1,-1)$-walk is

$$s_n = \frac{n}{2}(a_n + b_n) = \frac{n}{2}\Big(2 + (n-1)(s-2)\Big)$$

and this is exactly the nth s-gonal number. □

Corollary 1. *For s even, every nth s-gonal number can be represented as the sum of n equal numbers.*

Proof. The decomposition follows from the equidistant grid of the characteristics s:

$$\begin{array}{ll} 1 & \frac{s}{2} \\ \frac{s}{2} & \end{array}.$$

□

Remark 1. Observe that s-gonal numbers are partial sums of elements of $(1,1)$-walk on the main diagonal of an equidistant grid of characteristic s. That is so because differences of consecutive s-gonal numbers form an equidistant walk with the step length $s-2$. Every $(1,1)$-walk has this step length, particularly the main diagonal walk. Thus, in the grid (2) from Example 3, nonagonal numbers are sums of elements of an $(1,-1)$-walk, but also partial sums of elements of an $(1,1)$-walk. Generally, the main diagonal is an invariant of all $\left\lfloor \frac{s}{2} \right\rfloor$ different equidistant grids of characteristic s. Indeed, for $p \in \{1, 2, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\}$ one has

$$\begin{array}{ll} 1 & s-p \\ p & s-1 \end{array}.$$

(The element $s-1$ in this grid is: $p + (s-p-1)$, in the horizontal direction, or $s-p+(p-1)$, in the vertical direction.) So, regardless of p, on the main diagonal is the walk $1, s-1, 2s-3, \ldots$ whose nth partial sum equals the nth s-gonal number.

3 Equidistant grids made by alternating and shifting columns

Example 3. *Look at the next equidistant grid in which $a_1 = 1, d_r = 5$ and $d_c = 2$*

$$\begin{array}{ccccc}
1 & 6 & 11 & 16 & \cdots \\
3 & 8 & 13 & 18 & \cdots \\
5 & 10 & 15 & 20 & \cdots \\
7 & 12 & 17 & 22 & \cdots \\
9 & 14 & 19 & 23 & \cdots \\
11 & 16 & 21 & 26 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array} . \tag{2}$$

The grid (2) is of a specific construction interest. Namely, we can start its construction with two equidistant walks having the same length of the step ($d_c = 2$) given in the first two columns. We add additional columns by alternating existing columns and shifting them up by five in each new appearance. Indeed, the shift by five, and only by five, of each column ensures that all horizontal walks are equidistant. (Five is the difference of the first elements in given columns or d_r).

We are interested in just-described equidistant grid construction. It starts by putting two walks of the same step length to the first two columns and then alternating columns shifted up by some fixed shift k. The grid (2) shows that not every shift size will result in an equidistant grid. The question is: how can one choose k to get it?

Assume there are equidistant walks with initial terms a_1 and $a_1 + d_r$ and the steps' lengths equal to d_c in the first two columns of a grid. Then add new columns in a way to alternate given walks shifted up by some k, $k > 0$ in each new appearance to obtain the grid

$$\begin{array}{ccccc}
a_1 & a_1 + d_r & a_1 + kd_c & a_1 + d_r + kd_c & \cdots \\
a_1 + d_c & a_1 + d_r + d_c & a_1 + (k+1)d_c & a_1 + d_r + (k+1)d_c & \cdots \\
a_1 + 2d_c & a_1 + d_r + 2d_c & a_1 + (k+2)d_c & a_1 + d_r + (k+2)d_c & \cdots \\
a_1 + 3d_c & a_1 + d_r + 3d_c & a_1 + (k+3)d_c & a_1 + d_r + (k+3)d_c & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array} .$$

Since d_r gives the steps' length of equidistant walks in rows, we have the requirement $a_1 + kd_c - (a_1 + d_r) = d_r$. That implies $d_c k = 2d_r$.

The condition $kd_c = 2d_r$ relates the shift k and the steps' lengths d_c and d_r of equidistant walks in columns and rows, respectively. Notice that for $d_c = 2$, the condition implies $k = d_r$ and therefore justifies that for $d_c = 2$ the only shift that gives an equidistant grid is the shift by d_r, the fact we already noticed in the grid (2).

Given d_r, each ordered pair (d_c, k) that satisfies $d_c k = 2d_r$ gives one equidistant grid. So, the number of all such grids equals the number of ordered pairs (d_c, k) such that $d_c k = 2d_r$. To investigate the total number of equidistant grids, we consider separate cases of prime and composite d_r.

In case of a prime d_r, $d_r \neq 2$ the condition $d_c k = 2d_r$ gives four possible choices of d_c and k: $d_c = 1, k = 2d_r$; $d_c = 2d_r, k = 1$; $d_c = 2, k = d_r$ and $d_c = d_r, k = 2$. For the first two, we have the following grids:

$$\begin{array}{cccc} a_1 & a_1 + d_r & a_1 + 2d_r & \cdots \\ a_1 + 1 & a_1 + d_r + 1 & a_1 + 2d_r + 1 & \cdots \\ a_1 + 2 & a_1 + d_r + 2 & a_1 + 2d_r + 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

and

$$\begin{array}{cccc} a_1 & a_1 + d_r & a_1 + 2d_r & \cdots \\ a_1 + 2d_r & a_1 + 3d_r & a_1 + 4d_r & \cdots \\ a_1 + 4d_r & a_1 + 5d_r & a_1 + 6d_r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} .$$

The case $d_c = 2, k = d_r$ gives already treated equidistant grids

$$\begin{array}{cccc} a_1 & a_1 + d_r & a_1 + 2d_r & \cdots \\ a_1 + 2 & a_1 + d_r + 2 & a_1 + 2d_r + 2 & \cdots \\ a_1 + 4 & a_1 + d_r + 4 & a_1 + 2d_r + 4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} .$$

Finally, the case $d_c = d_r, k = 2$ gives the equidistant grid of a specific (symmetric) layout

$$\begin{array}{cccc} a_1 & a_1 + d_r & a_1 + 2d_r & \cdots \\ a_1 + d_r & a_1 + 2d_r & a_1 + 3d_r & \cdots \\ a_1 + 2d_r & a_1 + 3d_r & a_1 + 4d_r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} .$$

If d_r is a composite number, besides the preceding four, there are other choices of d_c and k such that $d_c k = 2d_r$. We distinguish two cases: the case of odd and even d_r.
If d_r is odd, we can factorize it as $d_r = p_1^{n_1} p_2^{n_2} \ldots$. Here $p_1, p_2, \ldots$ are different (odd) prime numbers and $n_1, n_2, \ldots$ their respective multiplicities. The number $D(d_r)$ of all divisors of d_r equals $D(d_r) = (n_1 + 1)(n_2 + 1) \ldots$. Since $2d_r = 2p_1^{n_1} p_2^{n_2} \ldots$, the number of divisors of $2d_r$ is $D(2d_r) = 2D(d_r)$. That is exactly the number of all ordered pairs (d_c, k) such that $d_c k = 2d_r$ and also the number of all possible equidistant grids. Indeed, except for $d_r = 2$, the number of divisors of $2d_r$ is even. We can arrange them to an increasing sequence: $p_1, p_2, \ldots, p_{l-1}, p_l$ with $l = 2D(d_r)$, $p_1 = 1$ and $p_l = 2d_r$. Then $(p_1, p_l), (p_2, p_{l-1}), \ldots$ and $(p_l, p_1), (p_{l-1}, p_2), \ldots$ give all possible pairs of lengths of the step d_c of equidistant walks in columns and shifts k of columns that give equidistant grids.
For $d_r = 2$ there are three divisors of $2d_r = 4$, namely $1, 2$ and 4 as well as three possible ordered pairs (d_c, k): $(1, 4), (4, 1), (2, 2)$.
Even d_r has a prime factor decomposition $d_r = 2^{n_1} p_2^{n_2} p_3^{n_3} \ldots$, with different prime

factors $p_j \neq 2$. Then $2s = 2^{n_1+1}p_2^{n_2}p_3^{n_3}\dots$. Consequently,

$$\begin{aligned} D(2d_r) &= (n_1+2)(n_2+1)(n_3+1)\cdots \\ &= (n_1+1)(n_2+1)+\cdots+(n_2+1)(n_3+1)\cdots \\ &= D(d_r)+D\left(\frac{d_r}{2^{n_1}}\right). \end{aligned}$$

Similarly as in the case of odd d_r, the number of all possible equidistant grids for even d_r is $D(d_r)+D\left(\frac{d_r}{2^{n_1}}\right)$. To summarize: up to the choice of a_1 and d_r, the number of equidistant grids equals the number of all divisors of $2d_r$. The sequence $D(n)$ is in the On-Line Encyclopedia of Integer Sequences [4] with the label A000005.

Example 4. *Let us look at some concrete examples of equidistant grids.*
Case $d_r = 5$
There are four divisors of $2d_r = 10$*:* $1, 2, 5, 10$*. There are also four possible choices for* (d_c, k)*:* $(1, 10), (10, 1), (2, 5)$ *and* $(5, 2)$*. If we take* $a_1 = 1$*, there are four equidistant grids*

$$\begin{array}{ccccc|ccccc}
1 & 6 & 11 & 16 & \cdots & 1 & 6 & 11 & 16 & \cdots \\
2 & 7 & 12 & 17 & \cdots & 11 & 16 & 21 & 26 & \cdots \\
3 & 8 & 13 & 18 & \cdots & 21 & 26 & 31 & 36 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
1 & 6 & 11 & 16 & \cdots & 1 & 6 & 11 & 16 & \cdots \\
3 & 8 & 13 & 18 & \cdots & 6 & 11 & 16 & 21 & \cdots \\
5 & 10 & 15 & 20 & \cdots & 11 & 16 & 21 & 26 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

In given grids, we look at characteristics s *to see which* s*-gonal numbers they generate. We list here* s*-gonal numbers from the above grids together with their OEIS labels: octagonal numbers (*$s = 8$*, A000567), heptadecagonal numbers (*$s = 17$*, A051869), nonagonal numbers (*$s = 9$*, A001106) and dodecagonal numbers (*$s = 12$*, A051624).*
Case $d_r = 6$
There are six divisors of $2d_r = 12$*:* $1, 2, 3, 4, 6, 12$ *and six possible ordered pairs* (d_c, k)*:* $(1, 12), (12, 1), (2, 6), (6, 2), (3, 4)$ *and* $(4, 3)$*. For* $a_1 = 1$*, we have the following equidistant grids*

$$\begin{array}{ccccc|ccccc|ccccc}
1 & 7 & 13 & 19 & \cdots & 1 & 7 & 13 & 19 & \cdots & 1 & 7 & 13 & 19 & \cdots \\
2 & 8 & 14 & 20 & \cdots & 13 & 19 & 25 & 28 & \cdots & 3 & 9 & 15 & 21 & \cdots \\
3 & 9 & 15 & 21 & \cdots & 25 & 31 & 37 & 40 & \cdots & 5 & 11 & 17 & 23 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
1 & 7 & 13 & 19 & \cdots & 1 & 7 & 13 & 19 & \cdots & 1 & 7 & 13 & 19 & \cdots \\
7 & 13 & 19 & 25 & \cdots & 4 & 10 & 16 & 22 & \cdots & 5 & 11 & 17 & 23 & \cdots \\
13 & 19 & 25 & 31 & \cdots & 7 & 13 & 19 & 25 & \cdots & 9 & 15 & 21 & 27 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Given grids again generate s-gonal numbers: nonagonal ($s = 9$, A001106), icosagonal ($s = 20$, A051872), decagonal ($s = 10$, A001107), tetradecagonal ($s = 14$, A051866), hendecagonal ($s = 11$, A051682) and dodecagonal ($s = 12$, A051624).

3.1 Zig-zag equidistant walks and bi-equidistant walks

Now we introduce zig-zag walks and investigate conditions under which they are equidistant. Zig-zag walks involve two equidistant walks of the same step length d_c placed in a two-column grid (the members of the walk are bolded)

$$\begin{array}{cc} \mathbf{a_1} & a_1' \\ a_1 + d_c & a_1' + d_c \\ \vdots & \vdots \\ a_1 + rd_c & \mathbf{a_1' + rd_c} \\ \vdots & \vdots \\ \mathbf{a_1 + (r+l)d_c} & a_1' + (r+l)d_c \\ \vdots & \vdots \\ a_1 + (2r+l)d_c & \mathbf{a_1' + (2r+l)d_c} \\ \vdots & \vdots \end{array} .$$

The zig-zag walk starts with a_1, the first term of an equidistant walk from the first column, continues to $a_1' + rd_c$ from the second column r places lower than its first term, then proceeds to the first column $r + l$ places below the first term, and so on. So, in each right turn zig-zag walk steps to the element of the second column r places lower than the present one, and in each left turn to the one of the first column l places below the present one. Look at conditions that ensure that a zig-zag walk is equidistant. Let $d_r = a_1' - a_1$ and assume $d_r > 0$. From

$$a_1' + rd_c - a_1 = a_1 + (r + l)d_c - (a_1' + rd_c)$$

it follows that the zig-zag walk is equidistant if and only if $2d_r = (l - r)d_c$. The condition $d_r > 0$ together with $d_c > 0$ (as assumed so far) forces $r < l$. Further, $l - r = \frac{2d_r}{d_c}$ provides the condition that the steps' length d_c must be a divisor of $2d_r$. There is a finite number of possible step lengths, but an infinite number of choices of l and r satisfying $r < l$.
Zig-zag walks can become horizontal walks through an infinite grid of given walks. Like above, we can construct an equidistant grid by alternating shifted column walks. Namely, the one from the first column stays as it is, and we shift the second up by r. After this, we alternate columns shifted up by $l + r$ in each new appearance. The original zig-zag walk now appears in the first row

$$a_1 \quad a_1' + rd_c \quad a_1 + (r + l)d_c \quad a_1' + (2r + l)d_c \quad a_1 + (2r + 2l)d_c \quad \cdots$$

Example 5. *In a special case when $a_1 = a_1'$ (or $d_r = 0$) and $d_c = 1$, the previous horizontal walk turns into a walk we call a* **bi-equidistant walk**

$$a_1 \quad a_1 + r \quad a_1 + r + l \quad a_1 + 2r + l \quad a_1 + 2r + 2l \quad \cdots$$

It starts with a_1 with the first step length r and the second step length l and continues to infinity by alternating those two steps. There is also a bi-equidistant walk

$$a_1 \quad a_1+l \quad a_1+l+r \quad a_1+2l+r \quad a_1+2l+2r \quad \cdots$$

that starts with a_1, and then alternates steps of lengths l and r to infinity. Notice that odd members of given two bi-equidistant walks coincide, while the distance between even members is constant and equals $|r-l|$.

3.2 Equidistant walks over progressions of products with overlapping odd and even factors

The sum and the difference of corresponding members of two arithmetical progressions is again an arithmetical progression. The same does not hold for a Hadamard product of two arithmetical progressions unless one is constant. For given n let us look at equidistant walks over odd and even numbers

$$\begin{matrix} 2n-1 & 2n \\ 2n+1 & 2n+2 \\ 2n+3 & 2n+4 \\ \vdots & \vdots \end{matrix}.$$

By a progression of **products with overlapping factors** of odd numbers we mean the sequence $(2n-1)(2n+1), (2n+1)(2n+3), (2n+3)(2n+5), \ldots$.
Let us further observe the progressions of products with overlapping factors starting with the odd number $(2n-1)(2n+1)$ and even number $2n(2n+2)$, respectively

$$\begin{matrix} (2n-1)(2n+1) & 2n(2n+2) \\ (2n+1)(2n+3) & (2n+2)(2n+4) \\ (2n+3)(2n+5) & (2n+4)(2n+6) \\ \vdots & \vdots \end{matrix}.$$

Since differences of consecutive members are not constant, none of the progressions is arithmetical. However, differences in products of consecutive elements do form arithmetical progressions. Indeed, look at progressions of differences of products of consecutive numbers

$$\begin{matrix} (2n+1)(2n+3)-(2n-1)(2n+1) & (2n+2)(2n+4)-2n(2n+2) \\ (2n+3)(2n+5)-(2n+1)(2n+3) & (2n+4)(2n+6)-(2n+2)(2n+4) \\ \vdots & \vdots \end{matrix}.$$

They are both equidistant walks with a step length of 8

$$\begin{matrix} 8n+4 & 8n+8 \\ 8n+12 & 8n+16 \\ 8n+20 & 8n+24 \\ \vdots & \vdots \end{matrix}.$$

Further, the second walk is ahead of the first walk by 4 all the time. Namely, the difference of the corresponding numbers in sequences is constant and equals 4

$$\Big((2n+2)(2n+4) - 2n(2n+2)\Big) - \Big((2n+1)(2n+3) - (2n-1)(2n+1)\Big) = 4.$$

If we further shift the progression of even numbers up by one

$$\begin{matrix} & 2n \\ 2n-1 & 2n+2 \\ 2n+1 & 2n+4 \\ 2n+3 & 2n+6 \\ \vdots & \vdots \end{matrix}\,.$$

we may look at the progressions of respective products with overlapping factors as before

$$\begin{matrix} (2n-1)(2n+1) & (2n+2)(2n+4) \\ (2n+1)(2n+3) & (2n+4)(2n+6) \\ \vdots & \vdots \end{matrix}\,.$$

Distance between the corresponding differences is again constant; now it equals $12 = 2 \cdot 3!$.

$$\begin{aligned} &\Big((2n+4)(2n+6) - (2n+2)(2n+4)\Big) - \\ &- \Big((2n+1)(2n+3) - (2n-1)(2n+1)\Big) = 12. \end{aligned}$$

In general, we may fix a natural number n and the progression of products with overlapping odd factors. Further, we shift the progression of products with overlapping even factors by k. Here $k > 0$ and $k < 0$ denote the shift up or down by $|k|$, respectively. Because $k = 1 - n$ is the maximal possible shift down, there is a condition $k \geq 1 - n$. After we shift the second sequence by k, we get

$$\begin{matrix} \vdots & \vdots \\ (2n-1)(2n+1) & (2n+2k)(2n+2(k+1)) \\ (2n+1)(2n+3) & \Big(2n+2(k+1)\Big)\Big(2n+2(k+2)\Big) \\ \vdots & \vdots \end{matrix}\,.$$

The following holds for the distance of differences between two consecutive products of numbers in sequences.

Proposition 1. *For a natural number n and an integer $k \geq 1 - n$,*

$$\begin{aligned} &\Bigg(\Big(2n+2(k+1)\Big)\Big(2n+2(k+2)\Big)\Bigg) - (2n+2k)\Big(2n+2(k+1)\Big)\Bigg) - \\ &\qquad - \Bigg((2n+1)(2n+3) - (2n-1)(2n+1)\Bigg) = 8k+4. \end{aligned}$$

Remark 2. Products of overlapping factors can include more than two numbers that overlap in more than one factor. So we can speak of products of n consecutive odd or even numbers that overlap in m factors, $m < n$, i.e., of **m-overlapping n-products**. In this notation, the above overlapping products are 1-overlapping 2-products. As above, the distance between differences of two consecutive m-overlapping n-products of even numbers and odd numbers is constant.

References

[1] Belbachir H., Nemeth L., Szalay L.: *Hyperbolic Pascal triangles*, Applied Mathematics and Computation 273 (2016), 453-464.

[2] Brown K. A., Dunn S., Harrington J.: *Arithmetic progressions in polygonal numbers*, Integers 12 (2012), A43

[3] Jones L., Phillips T.: *Arithmetic progressions of polygonal numbers with common difference a polygonal number*, Integers 17 (2017), A43

[4] Sloane N. J. A.: The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org.

Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.05

On divisibility properties of some binomial sums connected with the Catalan and Fibonacci numbers

Jovan Mikić

Abstract

We show that an alternating binomial sum which is connected with the Catalan numbers is divisible by n. A natural generalization of this sum is connected with the generalized Catalan numbers and also divisible by n. A new class of binomial sum is used. In Appendix A, we consider a positive binomial sum connected with Fibonacci and Lucas numbers. In Appendix B, we consider an alternating binomial sum which is also connected with Catalan numbers and divisible by $(a+1)n+1$. Similar reasoning was already used by the author to reprove more simply Calkin's result for divisibility of the alternating sum of powers of binomials coefficients by the central binomial coefficient.

Key words: Catalan number, generalized Catalan number, Fibonacci number, Lucas number, M sum, alternating binomial sum.

AMS subject classification (2000): 11B37, 05C15

1 Introduction

Let us consider the following alternating binomial sum:

$$S_1(n,m) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \binom{2n+k}{2n+1}; \tag{1}$$

where n and m are natural numbers.
Let x and y be non-negative numbers. It is well-known [5, Eq. (10.15), p. 47] that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{y} = (-1)^n \binom{x}{y-n}. \tag{2}$$

(Jovan Mikić) University of Banja Luka, Faculty of Technology, Bosnia and Herzegovina, jovan.mikic@tf.unibl.org

The Vandermonde convolution formula [4, Eq. (5.24)] is equivalent to Eq. (2). Also there are, at least, two combinatorial proofs of Eq. (2) by using sign-reversing involutions [2].
Let C_n denote [7, Section 5, p. 103] the n-th Catalan number. For $m = 1$, by the Eq. (2), it follows that

$$S_1(n,1) = (-1)^n nC_n. \tag{3}$$

We assert that:

Theorem 1. *The sum $S_1(n,m)$ is always divisible by n for all natural numbers n and m.*

The sum $S_1(n,m)$ has a natural generalization:

$$S_1(n,m;a) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}^m \binom{an+k}{an+1}; \tag{4}$$

where a is a natural number.
Obviously, for $a = 2$, $S_1(n,m;2) = S_1(n,m)$.
For $m = 1$, by the Eq. (2), it can be shown that

$$S_1(n,1;a) = (-1)^n \frac{n}{(a-1)n+1}\binom{an}{n}. \tag{5}$$

Due to $gcd(n,(a-1)n+1) = 1$, by the Eq. (5), it follows that $S_1(n,1;a)$ is divisible by n. Note that the number $C(n,a) = \frac{1}{(a-1)n+1}\binom{an}{n}$ is known [7, Section 17, Eq. (17.1), p. 375] as generalized Catalan number or Fuss-Catalan number. See also [1, Eq. (2.2)]. For $a = 2$, $C(n,2) = C_n$.
We assert that:

Theorem 2. *Let a be a fixed natural number. The sum $S_1(n,m;a)$ is always divisible by n for all natural numbers n and m.*

Furthermore, let us consider the following alternating binomial sum:

$$S_2(n,m) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}^m \binom{2n+1+k}{2n}; \tag{6}$$

where n and m are natural numbers.
For $m = 1$, by the Eq. (2), it follows that

$$S_2(n,1) = (-1)^n(2n+1)C_n. \tag{7}$$

We assert that:

Theorem 3. *The sum $S_2(n,m)$ is always divisible by $2n+1$ for all natural numbers n and m.*

The sum $S_2(n,m)$ has a natural generalization:

$$S_2(n,m;a)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}^m\binom{an+1+k}{an}; \tag{8}$$

where a is a natural number.
For $m=1$, by the Eq. (2), it can be shown that

$$S_2(n,1;a)=(-1)^n\frac{an+1}{n+1}\binom{an}{n}. \tag{9}$$

Note that the integers $an+1$ and $n+1$ are not relatively prime in general.
We assert that:

Theorem 4. *Let a be a fixed natural number. The sum $S_2(n,m;a)$ is always divisible by $\frac{an+1}{\gcd(a-1,n+1)}$ for all natural numbers n and m.*

We prove Thms. (2) and (4) by using a new class of binomial sums. Theorem 1 is a special case of a Theorem 2 for $a=2$. Similarly, Theorem 3 is a special case of a Theorem 4 for $a=2$.
Let us consider the following sum:

$$S(n,m;a)=\sum_{k=0}^{n}\binom{n}{k}^m F(n,k,a); \tag{10}$$

where n, m and a are natural numbers, and $F(n,k,a)$ is an integer-valued function. Our goal is to investigate some divisibility properties of the sum $S(n,m;a)$. In order to do so, we use a new class of binomial sums which we called M sums.

Definition 1. *Let $S(n,m;a)$ be a sum from the Eq. (10) . Then*

$$M_S(n,j,t;a)=\binom{n-j}{j}\sum_{k=0}^{n-2j}\binom{n-2j}{k}\binom{n}{j+k}^t F(n,j+k,a); \tag{11}$$

where j and t are non-negative integers such that $j\le\lfloor\frac{n}{2}\rfloor$.

See [9, Def. 7, Eq. (28), p. 9].
Obviously, by setting $j=0$ in the Eq. (11), it follows that [9, Eq. (29), p. 9]

$$S(n,t+1;a)=M_S(n,0,t;a). \tag{12}$$

Due to Eq. (12), we can see $M_S(n,j,t;a)$ sum as a generalization of $S(n,m;a)$.
Furthermore, M sums satisfy [9, Thm. 8, p. 9] the following recurrence:

$$M_S(n,j,t+1;a)=\binom{n}{j}\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n-j}{u}M_S(n,j+u,t;a). \tag{13}$$

Eqns. (12) and (13) have a simple consequence which is important to us. Let us suppose that an integer $q(n,a)$ divides $M_S(n,j,t_0;a)$ for all $0\le j\le\lfloor\frac{n}{2}\rfloor$, where t_0 is a fixed non-negative integer. By using the Eq. (13) and the induction principle,

it can be shown that $q(n,a)$ must divide $M_S(n,j,t;a)$, for all integers t such that $t \geq t_0$. By the Eq. (12), it follows that $q(n,a)$ divides $S(n,t;a)$ for all integers t such that $t \geq t_0+1$.
By setting $t := 0$ in the Eq. (11), we obtain that

$$M_S(n,j,0;a) = \binom{n-j}{j} \sum_{k=0}^{n-2j} \binom{n-2j}{k} F(n,j+k,a). \tag{14}$$

The M sums give an elementary proof of Calkin result [3, Thm. 1]. See also [6, Thm. 1.2, Thm. 1.3, p. 2]. Note that there are also another applications [10, Section 2, p. 4], [8,9] of M sums.

2 The Main Lemmas

We present four lemmas.
We calculate M sums for the sum $S_1(n,m;a)$ for $t=0$ and $t=1$.

Lemma 1.

$$M_{S_1}(n,j,0;a) = \frac{(-1)^{n-j}(n-j)\binom{an+j}{j}\binom{an+1}{n-2j}}{an+1}. \tag{15}$$

Lemma 2.

$$M_{S_1}(n,j,1;a) = n \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \frac{(-1)^{n-j-u}\binom{n-1}{j}\binom{j+u}{u}\binom{an+j+u}{j+u}\binom{an+1}{n-2j-2u}}{an+1}. \tag{16}$$

Also we calculate M sums for the sum $S_2(n,m;a)$ for $t=0$ and $t=1$.

Lemma 3.

$$M_{S_2}(n,j,0;a) = \frac{(-1)^{n-j}(an+1)\binom{an+1+j}{j}\binom{an}{n-2j}}{n+1-j}. \tag{17}$$

Lemma 4.

$$M_{S_2}(n,j,1;a) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \frac{(-1)^{n-j-u}(an+1)\binom{n+1}{j+u}\binom{j+u}{j}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}}{n+1}. \tag{18}$$

3 A Proof of Lemma 1

Proof. Obviously, the sum $S_1(n,m;a)$ is an instance of the sum $S(n,m;a)$ from the Eq. (10), where $F_1(n,k,a) = (-1)^k\binom{an+k}{an+1}$.
By the Eq. (14), we have:

$$\begin{aligned} M_{S_1}(n,j,0;a) &= \binom{n-j}{j} \sum_{k=0}^{n-2j} \binom{n-2j}{k} F_1(n,j+k,a) \\ &= (-1)^j \binom{n-j}{j} \sum_{k=0}^{n-2j} (-1)^k \binom{n-2j}{k} \binom{an+j+k}{an+1} \end{aligned} \tag{19}$$

By the Eq. (2), it follows that

$$\sum_{k=0}^{n-2j}(-1)^k\binom{n-2j}{k}\binom{an+j+k}{an+1}=(-1)^{n-2j}\binom{an+j}{n-j-1}. \tag{20}$$

By the Eq. (20), the Eq. (19) becomes

$$M_{S_1}(n,j,0;a)=(-1)^{n-j}\binom{an+j}{n-j-1}\binom{n-j}{j}. \tag{21}$$

Due to well-known [7, Section 1, Eq. (1.3), p. 5] formula:

$$\binom{n}{k-1}=\frac{k}{n-k}\binom{n}{k},$$

we have that:

$$\binom{an+j}{n-j-1}=\frac{n-j}{(a-1)n+2j+1}\binom{an+j}{n-j}, \tag{22}$$

By the Eq. (22), the Eq. (21) becomes:

$$M_{S_1}(n,j,0;a)=(-1)^{n-j}\frac{n-j}{(a-1)n+2j+1}\binom{an+j}{n-j}\binom{n-j}{j}. \tag{23}$$

It is well-known [7, Section 1, Eq. (1.4), p. 5] that:

$$\binom{a}{b}\binom{b}{c}=\binom{a}{c}\binom{a-c}{b-c}; \tag{24}$$

where a, b, and c are non-negative integers such that $a\geq b\geq c$.
By the Eq. (24), it follows that

$$\binom{an+j}{n-j}\binom{n-j}{j}=\binom{an+j}{j}\binom{an}{n-2j}. \tag{25}$$

By the Eq. (25), the Eq. (23) becomes

$$M_{S_1}(n,j,0;a)=(-1)^{n-j}\frac{n-j}{(a-1)n+2j+1}\binom{an+j}{j}\binom{an}{n-2j}. \tag{26}$$

Due to another well-known [7, Section 1, Eq. (1.2), p. 5]formula:

$$\frac{1}{n+1-k}\binom{n}{k}=\frac{1}{n+1}\binom{n+1}{k},$$

it follows that:

$$\frac{1}{(a-1)n+2j+1}\binom{an}{n-2j}=\frac{1}{an+1}\binom{an+1}{n-2j}. \tag{27}$$

By using Eqns. (26) and (27), we obtain that

$$M_{S_1}(n,j,0;a)=(-1)^{n-j}\frac{n-j}{an+1}\binom{an+j}{j}\binom{an+1}{n-2j}. \tag{28}$$

The Eq. (28) completes the proof of Lemma 1. □

Note that that Eq. (21) leads directly to Eq. (28) by expressing binomial coefficients in terms of factorials and cancellation of equal factors.

4 A Proof of Lemma 2

Proof. We use the Eq. (13) and Lemma 1.
By setting $t := 0$ in the Eq. (13), we have that:

$$M_{S_1}(n, j, 1; a) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_{S_1}(n, j+u, 0; a). \tag{29}$$

By Lemma 1, it follows that:

$$M_{S_1}(n, j+u, 0; a) = (-1)^{n-j-u} \frac{n-j-u}{an+1} \binom{an+j+u}{j+u} \binom{an+1}{n-2j-2u}. \tag{30}$$

By the Eq. (24), it follows that

$$\binom{n}{j}\binom{n-j}{u} = \binom{n}{j+u}\binom{j+u}{u}. \tag{31}$$

By using Eqns. (30) and (31) in the Eq. (29), we obtain that $M_{S_1}(n, j, 1; a)$ is equal to the following sum:

$$\sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} (-1)^{n-j-u} \binom{n}{j+u}\binom{j+u}{u} \frac{n-j-u}{an+1} \binom{an+j+u}{j+u}\binom{an+1}{n-2j-2u}. \tag{32}$$

It is readily verified that:

$$(n-j-u)\binom{n}{j+u} = n\binom{n-1}{j+u}. \tag{33}$$

By using the Eq. (33) in the Eq. (32), we obtain that $M_{S_1}(n, j, 1; a)$ is equal to

$$(-1)^{n-j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} (-1)^u \frac{n\binom{n-1}{j+u}\binom{j+u}{u}\binom{an+j+u}{j+u}\binom{an+1}{n-2j-2u}}{an+1}. \tag{34}$$

This completes the proof of Lemma 2.

□

5 A Proof of Theorem 2

Proof. Let n and a be fixed natural numbers.
By the Eq. (5), we know that $S_1(n, m; a)$ is divisible by n for $m = 1$.
Due to fact that $F_1(n, m, a)$ is an integer-valued function, we know that $M_{S_1}(n, j, 0; a)$ is an integer. Also, see the Eq. (21).
The number $\binom{n}{j}\binom{n-j}{u} M_{S_1}(n, j+u, 0; a)$ is also an integer; and, by the Eq. (34), it is equal to the number $\frac{n\binom{n-1}{j+u}\binom{j+u}{u}\binom{an+j+u}{j+u}\binom{an+1}{n-2j-2u}}{an+1}$

By Lemma 2 and the fact that $\gcd(n, an+1) = 1$, it follows that the sum $M_{S_1}(n, j, 1; a)$ is divisible by n for all non-negative integers j such that $j \leq \lfloor \frac{n}{2} \rfloor$. Therefore, we can take $q_1(n, a) = n$. By the Eq. (13) and the induction principle on t, it follows that $M_{S_1}(n, j, t; a)$ is divisible by n for all natural numbers t. By the Eq. (12), it follows that $S_1(n, t+1; a)$ is divisible by n for all natural numbers t. Hence, $S_1(n, m; a)$ is divisible by n for all natural numbers m such that $m \geq 2$. This completes the proof of Theorem 2.

□

6 A Proof of Lemma 3

Proof. Obviously, the sum $S_2(n, m; a)$ is an instance of the sum $S(n, m; a)$ from the Eq. (10), where $F_2(n, k, a) = (-1)^k \binom{an+1+k}{an}$.
By the Eq. (14), we have:

$$\begin{aligned} M_{S_2}(n, j, 0; a) &= \binom{n-j}{j} \sum_{k=0}^{n-2j} \binom{n-2j}{k} F_2(n, j+k, a) \\ &= (-1)^j \binom{n-j}{j} \sum_{k=0}^{n-2j} (-1)^k \binom{n-2j}{k} \binom{an+1+j+k}{an} \end{aligned} \tag{35}$$

By the Eq. (2), it follows that

$$\sum_{k=0}^{n-2j} (-1)^k \binom{n-2j}{k} \binom{an+1+j+k}{an} = (-1)^{n-2j} \binom{an+1+j}{n-j+1}. \tag{36}$$

By the Eq. (36), the Eq. (35) becomes

$$M_{S_2}(n, j, 0; a) = (-1)^{n-j} \binom{an+1+j}{n-j+1} \binom{n-j}{j}. \tag{37}$$

It is readily verified that

$$\binom{an+1+j}{n-j+1} = \frac{(an+1+j}{n-j+1} \binom{an+j}{n-j}. \tag{38}$$

By the Eq. (38), the Eq. (37) becomes

$$M_{S_2}(n, j, 0; a) = (-1)^{n-j} \frac{an+1+j}{n-j+1} \binom{an+j}{n-j} \binom{n-j}{j}. \tag{39}$$

By the Eq. (25), it follows that

$$M_{S_2}(n, j, 0; a) = (-1)^{n-j} \frac{an+1+j}{n-j+1} \binom{an+j}{j} \binom{an}{n-2j}. \tag{40}$$

By the Eq. (24), it follows that

$$(an+1+j) \binom{an+j}{j} = \binom{an+1+j}{j} (an+1). \tag{41}$$

By the Eq. (41), the Eq. (40) becomes:

$$M_{S_2}(n,j,0;a) = (-1)^{n-j}\frac{an+1}{n-j+1}\binom{an+1+j}{j}\binom{an}{n-2j}. \tag{42}$$

The Eq. (42) completes the proof of Lemma 3.

□

7 A Proof of Lemma 4

Proof. We use the Eq. (13) and Lemma 3.
By setting $t := 0$ in the Eq. (13), we have that:

$$M_{S_2}(n,j,1;a) = \binom{n}{j}\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n-j}{u}M_{S_2}(n,j+u,0;a). \tag{43}$$

By Lemma 3, it follows that:

$$M_{S_2}(n,j+u,0;a) = (-1)^{n-j-u}\frac{an+1}{n-j-u+1}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}. \tag{44}$$

By using Eqns. (31) and (44) in the Eq. (43), we obtain that $M_{S_2}(n,j,1;a)$ is equal to the following sum:

$$\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}(-1)^{n-j-u}\binom{n}{j+u}\frac{an+1}{n-j-u+1}\binom{j+u}{u}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}. \tag{45}$$

It is readily verified that

$$\binom{n}{j+u}\frac{1}{n-j-u+1} = \frac{1}{n+1}\binom{n+1}{j+u}. \tag{46}$$

By using the Eq. (46) in the Eq. (45), it follows that $M_{S_2}(n,j,1;a)$ is equal to:

$$\frac{an+1}{n+1}\cdot\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}(-1)^{n-j-u}\binom{n+1}{j+u}\binom{j+u}{u}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}. \tag{47}$$

The Eq. (47) proves Lemma 4.

□

8 A Proof of Theorem 4

Proof. Let n and a be fixed natural numbers.
By the Eq. (9), we know that $S_2(n,m;a)$ is divisible by $\frac{an+1}{\gcd(an+1,n+1)}$ for $m = 1$.
Due to fact that $F_2(n,m,a)$ is an integer-valued function, we know that $M_{S_2}(n,j,0;a)$ is an integer. Also, see the Eq. (37).

The number $\binom{n}{j}\binom{n-j}{u}M_{S_2}(n,j+u,0;a)$ is an integer; and, by the Eq. (47), it is equal to the number $(-1)^{n-j-u}\cdot\frac{an+1}{n+1}\binom{n+1}{j+u}\binom{j+u}{u}\binom{an+1+j+u}{j+u}\binom{an}{n-2j-2u}$

By Lemma 4, it follows that the sum $M_{S_2}(n,j,1;a)$ is divisible by $\frac{an+1}{\gcd(an+1,n+1)}$ for all non-negative integers j such that $j\le\lfloor\frac{n}{2}\rfloor$.

Therefore, we can take $q_2(n,a)=\frac{an+1}{\gcd(an+1,n+1)}$. By the Eq. (13) and the induction principle on t, it follows that $M_{S_2}(n,j,t;a)$ is divisible by $q_2(n,a)$ for all natural numbers t. By the Eq. (12), it follows that $S_1(n,t+1;a)$ is divisible by $q_2(n,a)$ for all natural numbers t. Hence, $S_1(n,m;a)$ is divisible by $\frac{an+1}{\gcd(an+1,n+1)}$ for all natural numbers m such that $m\ge 2$.

Note that $\gcd(an+1,n+1)=\gcd(a-1,n+1)$. This completes the proof of Theorem 4.

□

Remark 1. By setting $a:=2$ in Theorem 4 and by using the fact $\gcd(2n+1,n+1)=1$, we obtain the proof of Theorem 3.

9 Concluding Remarks

Let us consider the following alternating sum:

$$S_1(n,m;a,b)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}^m\binom{an+k}{an+b};\tag{48}$$

where n, m, a, and b are natural numbers.

Obviously, $S_1(n,m;a,1)=S_1(n,m;a)$. Furthermore, by using M sums, it can be shown that:

$$M_{S_1}(n,j,0;a,b)=\frac{(-1)^{n-j}\binom{an+j}{j}\binom{n-j}{b}\binom{an+b}{n-2j}}{\binom{an+b}{b}},\tag{49}$$

$$M_{S_1}(n,j,1;a,b)=\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\frac{(-1)^{n-j-u}\binom{n}{b}\binom{n-b}{j+u}\binom{j+u}{j}\binom{an+j+u}{j+u}\binom{an+b}{n-2j-2u}}{\binom{an+b}{b}}.\tag{50}$$

The Eq. (50) suggests that $q_1(n,a,b)=\frac{\binom{n}{b}}{\gcd(\binom{n}{b},\binom{an+b}{b})}$. We assert that:

Theorem 5. *Let n, a, and b be fixed natural numbers. Then the sum $S_1(n,m;a,b)$ is always divisible by $\frac{\binom{n}{b}}{\gcd(\binom{n}{b},\binom{an+b}{b})}$ for all natural numbers m.*

By setting $b:=1$ in the Eq. (49), we obtain Lemma 1. Similarly, by setting $b:=1$ in the Eq. (50), we obtain Lemma 2. Finally, by setting $b:=1$ in Theorem 5, we obtain Theorem 2.

Let us now consider the following sum:

$$S_2(n,m;a,b)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}^m\binom{an+b+k}{an};\tag{51}$$

where n, m, a, and b are natural numbers.
Clearly, $S_2(n,m;a,1) = S_2(n,m;a)$. Furthermore, by using M sums, it can be shown that:

$$M_{S_2}(n,j,0;a,b) = \frac{(-1)^{n-j}\binom{an+b+j}{j}\binom{an}{n-2j}\binom{an+b}{b}}{\binom{n-j+b}{b}}, \tag{52}$$

$$M_{S_2}(n,j,1;a,b) = \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor} \frac{(-1)^{n-j-u}\binom{an+b}{b}\binom{n+b}{j+u}\binom{j+u}{j}\binom{an+b+j+u}{j+u}\binom{an}{n-2j-2u}}{\binom{n+b}{n}}. \tag{53}$$

The Eq. (53) suggests that $q_2(n,a,b) = \frac{\binom{an+b}{b}}{\gcd(\binom{an+b}{b},\binom{n+b}{n})}$. We assert that:

Theorem 6. *Let n, a, and b be fixed natural numbers. Then the sum $S_2(n,m;a,b)$ is always divisible by $\frac{\binom{an+b}{b}}{\gcd(\binom{an+b}{b},\binom{n+b}{n})}$ for all natural numbers m.*

By setting $b := 1$ in the Eq. (52), we obtain Lemma 3. Similarly, by setting $b := 1$ in the Eq. (53), we obtain Lemma 4. Finally, by setting $b := 1$ in Theorem 6, we obtain Theorem 4.
For the sake of brevity and clarity, we omit proofs of Thms. (5) and (6).

10 Appendix A

We give an example with the positive binomial sum with Fibonacci numbers.
Let F_n denote the n-th Fibonacci number, where n is a non-negative integer.
Let us consider the following binomial identity:

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\lfloor\frac{k}{2}\rfloor} = 2^n F_{n+1}. \tag{54}$$

It is well-known that the Binet formula for Fibonacci numbers states:

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}. \tag{55}$$

The Binet formula is equivalent with the following binomial identity:

$$\sum_{k=0}^{n} \binom{n}{2k+1} 5^k = 2^{n-1} F_n. \tag{56}$$

Let L_n denote the n-th Lucas number. Formula for Lucas numbers is also well-known:

$$L_n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n}. \tag{57}$$

The formula for Lucas numbers is equivalent with the following binomial identity:

$$\sum_{k=0}^{n} \binom{n}{2k} 5^k = 2^{n-1} L_n. \tag{58}$$

Note that Eq. (54) follows by Eqns. (56) and (58), and the fact that $L_n = F_{n+1} - F_{n-1}$.
Let us consider the following sum:

$$S_3(n,m) = \sum_{k=0}^{n} \binom{n}{k}^m 5^{\lfloor \frac{k}{2} \rfloor}. \tag{59}$$

Obviously the sum $S_3(n,m)$ is an instance of the sum S from the Eq. (10), where $F_3(n,k) = 5^{\lfloor \frac{k}{2} \rfloor}$.
By the Eq. (54), for $m = 1$, we have that

$$S_3(n,1) = 2^n F_{n+1}.$$

By using Eqns. (13), (56), and (58), it can be shown that:

$$M_{S_3}(n,j,0) = \begin{cases} \binom{n-j}{j} 5^{\frac{j}{2}} \cdot 2^{n-2j} \cdot F_{n-2j+1}, & \text{if } j \text{ is even;} \\ \binom{n-j}{j} 5^{\frac{j-1}{2}} \cdot 2^{n-2j} \cdot L_{n-2j+1}, & \text{if } j \text{ is odd.} \end{cases} \tag{60}$$

We see that the formula for the $M_{S_3}(n,j,0)$ sum appear both Fibonacci and Lucas numbers. Since the sum $M_{S_3}(n,j,0)$ is a slight generalization of the sum $S_3(n,1)$, this is expected, due to Eqns. (56) and (58).

Remark 2. By setting $t = 0$ in the Eq. (11), and by using Eqns. (12) and (60), we can calculate the sum $S_3(n,2)$. It can be shown that $S_3(n,2)$ is equal to:

$$\sum_{\substack{u=0 \\ u \text{ is even}}}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{u}\binom{n-u}{u} 5^{\frac{u}{2}} 2^{n-2u} F_{n-2u+1} + \sum_{\substack{u=0 \\ u \text{ is odd}}}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{u}\binom{n-u}{u} 5^{\frac{u-1}{2}} 2^{n-2u} L_{n-2u+1}. \tag{61}$$

11 Appendix B

Let $S_4(n,m;a)$ denote the following alternating binomial sum:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^m (\binom{an+k}{an} + 2\binom{an+k}{an+1}); \tag{62}$$

where n, m, and a are natural numbers.
Obviously the sum $S_4(n,m;a)$ is an instance of the sum S from the Eq. (10), where $F_4(n,k,a) = (-1)^k(\binom{an+k}{an} + 2\binom{an+k}{an+1})$.
By the Eq. (13) and the Eq. (2), it can be shown that:

$$M_{S_4}(n,j,0;a) = (-1)^{n-j} \binom{an+j}{j} \binom{an+1}{n-2j} \frac{(a+1)n+1}{an+1}. \tag{63}$$

Note that the integers $(a+1)n+1$ and $an+1$ are relatively prime. By the Eq. (63), it follows that the sum $M_{S_4}(n,j,0;a)$ is always divisible by $(a+1)n+1$. Hence, we can take $q_4(n,a) = (a+1)n+1$.
By using M sums, we can prove the following theorem:

Theorem 7. *Let n and a be fixed natural numbers. Then the sum $S_4(n,m;a)$ is always divisible by $(a+1)n+1$ for all natural integers m.*

Remark 3. By the Eq. (63), it follows that an integer $T(n,j,a)=\binom{an+j}{j}\binom{an+1}{n-2j}$ is always divisible by $an+1$ for all non-negative integers j such that $j \leq \lfloor\frac{n}{2}\rfloor$. These numbers also appear in Lemma 1 and Lemma 2. Note that the Eq. (15) can be also written as

$$M_{S_1}(n,j,0;a)=(-1)^{n-j}(n-j)T(n,j,a). \tag{64}$$

Similarly, the Eq. (16) can be written as

$$M_{S_1}(n,j,1;a)=n\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}(-1)^{n-j-u}\binom{n-1}{j}\binom{j+u}{u}T(n,j+u,a). \tag{65}$$

Obviously,

$$M_{S_4}(n,j,0;a)=(-1)^{n-j}((a+1)n+1)T(n,j,a). \tag{66}$$

It would be interesting to find a combinatorial interpretation for numbers $T(n,j,a)$.

Acknowledgments

I dedicate this article to my aunt Grozda Stanić who recently passed away. Also I am grateful to professor Tomislav Došlić for inviting me on the 4-th Croatian Combinatorial Days in Zagreb.

References

[1] Jean-Christophe Aval, Multivariate Fuss-Catalan Numbers, available at arXiv:0711.0906v1

[2] Arthur T. Benjamin and Jennifer J. Quinn, An Alternate Approach to Alternating Sums: A Method to DIE for.

[3] N. J. Calkin, Factors of sums of powers of binomial coefficients, *Acta Arith.* **86** (1998), 17–26

[4] R. L. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics*.

[5] H. W. Gould and J. Quaintance, Combinatorial identities: Table I: intermediate techniques for summing finite series, preprint, 2010. Available at `https://www.math.wvu.edu/~gould/Vol.4.PDF`,

[6] V. J. W. Guo, F. Jouhet, and J. Zeng, Factors of alternating sums of products of binomial and q-binomial coefficients, *Acta Arith.* **127** (2007), 17–31.

[7] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, 2009.

[8] J. Mikić, A method for examining divisibility properties of some binomial sums, *J. Integer Sequences* **21** (2018), Article 18.8.7.

[9] J. Mikić, New Class of Binomial Sums and Their Applications, Third Croatian Combinatorial Days, Zagreb, Septembar 21-22, 2020, p. 44–64 , DOI: htpps://doi.org/1.5592/CO/CCD.2020.05

[10] J. Mikić, Factors of Alternating Convolution of the Gessel numbers, ArXiv:2206.03808v1

Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.06

On a linear recurrence relation of the divide-and-conquer type

Daniele Parisse

Abstract

We study the six-parameter linear recurrence relation defined by

$$f(1) = \zeta,\ f(2n) = \alpha f(n) + \beta,\ f(2n+1) = \gamma f(n) + \delta f(n+1) + \varepsilon, \quad n \geq 1,$$

with $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{Z}$. We determine its ordinary generating function which shows that the sequence is of the divide-and-conquer type. Then we discuss some interesting special cases such as the Josephus problem, the number of 1's in the binary expression of $n \in \mathbb{N}$, the Gros sequence in the Tower of Hanoi with 3 pegs and n disks, $n \in \mathbb{N}$, the Prouhet-Thue-Morse sequence, Stern's diatomic sequence and others. We give its solution at first for the special case $\delta = 0$ and then for $\delta \neq 0$. Moreover, for $\delta \neq 0$, $\beta = 0 = \varepsilon$ and $\alpha \neq 0$, we conjecture that it satisfies a second-order recurrence relation. We prove this conjecture for three special cases and give the solution by means of continued fractions. Then, we generalize the recurrence relation by considering both β and ε as integer functions of n and discuss in detail the case $\delta = 0$ and a special case of this class. Finally, for $\delta \neq 0$, we only mention two known special cases.

Key words: Sequences of the divide-and-conquer type; Ordinary generating function; Stern's diatomic sequence; Prouhet-Thue-Morse sequence; Tower of Hanoi;

AMS subject classification (2010): Primary 05A10 ; Secondary 05A19.

1 Introduction

There are some problems such as the Tower of Hanoi puzzle, the Josephus problem, Stern's diatomic sequence, "the infinity sequence" of Per Nørgård, the Prouhet-Thue-Morse sequence and many others, which have apparently no connections among each other. However, they are all defined by the same six-parameter linear recurrence relation

$$f(1) = \zeta, \quad f(2n) = \alpha f(n) + \beta, \quad f(2n+1) = \gamma f(n) + \delta f(n+1) + \varepsilon, \quad n \geq 1 \quad (1)$$

(Daniele Parisse) Airbus Defence and Space GmbH, Manching, Germany, daniele.parisse@t-online.de

with $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \in \mathbb{Z}$. (Note that we begin the sequence by $n = 1$, although some of the sequences are also well-defined for all $n \in \mathbb{N}_0$. If possible, we shall define $f(0) := \eta \in \mathbb{Z}$ so that this value is consistent with (1), otherwise we set $f(0) := 0$.) Eq.(1) can also be written as follows:

$$f(1) = \zeta, \quad f(n) = a(n)f\Big(\lfloor n/2 \rfloor\Big) + b(n)f\Big(\lceil n/2 \rceil\Big) + c(n), \quad n \geq 2, \tag{2}$$

where

$$a(n) := \frac{\alpha + \gamma}{2} + (-1)^n \frac{\alpha - \gamma}{2}, \quad b(n) := \frac{1 - (-1)^n}{2}\delta, \quad c(n) := \frac{\beta + \varepsilon}{2} + (-1)^n \frac{\beta - \varepsilon}{2}.$$

Recurrence relations of this form are called *(binary) divide-and-conquer recurrences* and appear often in computer science, because algorithms based on the technique of *divide et impera* (divide and conquer) often reduce a problem of size n to the solution of two problems of approximately equal sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, where $n = \lfloor n/2 \rfloor + \lceil n/2 \rceil, n \in \mathbb{N}_0$. The solutions of the two subproblems are then used to solve the original problem.

In this paper we shall study the linear recurrence relation (1) and its ordinary generating function. We shall at first consider some known special cases of (1) and then we shall derive its general solution, first for $\delta = 0$ and then for $\delta \neq 0$. It will turn out that for $\delta = 0$ (1) is equivalent to a first-order linear recurrence relation with variable coefficients and that for $\delta \neq 0$, $\beta = 0 = \varepsilon$ and $\alpha \neq 0$ it seems to be equivalent to a second-order linear recurrence relation with variable coefficients. We prove this conjecture for three special cases. The solutions in both cases rely on the binary expansion of n. In the last chapter we study a generalization of (1) by considering instead of the constants β and ε the integer functions $g(n)$ and $h(n)$. Then, we give a solution for $\delta = 0$ and investigate in detail a special case. Finally, for $\delta \neq 0$, we only mention two known sequences.

2 Definitions and Examples

Definition 1. *A formal series $\varphi(s)$ is said to be of the* divide-and-conquer (DC) type *(see [31]), if it satisfies a functional equation of the form*

$$c_0(s)\varphi(s) + c_1(s)\varphi(s^2) + \cdots + c_n(s)\varphi(s^{2^n}) = b(s), \tag{3}$$

in which $b(s)$ is a formal series and $c_k(s), k \in [n+1]_0 := \{0, 1, \ldots, n\}$, are polynomials not all zero. If $b(s) = 0$, then the series $\varphi(s)$ is said to be Mahlerian*.*

A sequence is of the divide-and-conquer type*, if its ordinary generating function, defined by the power series $\sum_{n=0}^{\infty} f(n)s^n$, is of the divide-and-conquer type and* Mahlerian*, if its generating function is Mahlerian.*

We shall now prove that the generating function for the sequence $\Big(f(n)\Big)_{n \in \mathbb{N}_0}$ defined by (1) is of the DC type and, for some special values of the six parameters, even Mahlerian.

Proposition 1. *The generating function $\varphi(s)$ for the sequence $\big(f(n)\big)_{n\in\mathbb{N}_0}$ is of the DC type, since it satisfies the functional equation*

$$s\varphi(s)=(\delta+\alpha s+\gamma s^2)\varphi(s^2)-\delta\eta+\eta(1-\alpha)s+\Big(\zeta(1-\delta)-\eta\gamma\Big)s^2+\frac{(\beta+\varepsilon s)s^3}{1-s^2} \quad (4)$$

Proof. Let $\varphi(s):=\sum_{n=0}^{\infty}f(n)s^n$ and $f(0):=\eta$, then by (1)

$$\begin{aligned}\varphi(s)&=f(0)+\sum_{n=1}^{\infty}f(2n)s^{2n}+\sum_{n=0}^{\infty}f(2n+1)s^{2n+1}\\&=\eta+\alpha\sum_{n=1}^{\infty}f(n)s^{2n}+\beta\sum_{n=1}^{\infty}s^{2n}+f(1)s+\gamma\sum_{n=1}^{\infty}f(n)s^{2n+1}+\\&\quad+\delta\sum_{n=1}^{\infty}f(n+1)s^{2n+1}+\varepsilon\sum_{n=1}^{\infty}s^{2n+1}\\&=\eta+\alpha\Big(\varphi(s^2)-\eta\Big)+\beta\frac{s^2}{1-s^2}+\zeta s+\gamma s\Big(\varphi(s^2)-\eta\Big)+\\&\quad+\frac{\delta}{s}\left(\varphi(s^2)-\eta-\zeta s^2\right)+\varepsilon s\frac{s^2}{1-s^2},\end{aligned}$$

since $\sum_{n=1}^{\infty}f(n+1)s^{2n+1}=\frac{1}{s}\Big(\varphi(s^2)-\eta-\zeta s^2\Big)$ and $\sum_{n=1}^{\infty}s^{2n}=\Big(\frac{1}{1-s^2}-1\Big)=\frac{s^2}{1-s^2}$. Simplifying and multiplying both sides by s we obtain formula (4), which shows that $f(n)$ is of the DC type. □

We shall now discuss some examples (see also [2, 4]).

- **E1** The Josephus problem ([13, pp. 8–13]).
  Let $n\in\mathbb{N}$ people, numbered 1 to n, stand around a circle and eliminate every second remaining person until one survives. The problem consists in determining the survivor's number $J(n)$. (The original and more difficult problem (for $n=41$) was to eliminate every third remaining person.) It follows (see [13, p. 10])

  $$J(1)=1,\quad J(2n)=2J(n)-1,\quad J(2n+1)=2J(n)+1,\quad n\geq 1 \quad (5)$$

  This is the special case $\alpha=2=\gamma$, $\beta=-1$, $\delta=0$, $\varepsilon=1$, $\zeta=1$ and $\eta:=0$ of (1). The generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

  $$\varphi(s)=2(1+s)\varphi(s^2)+\frac{s}{1+s}$$

  and it is given by

  $$\varphi(s)=\frac{1}{1-s}\sum_{k=0}^{\infty}\frac{2^k\cdot s^{2^k}\left(1-s^{2^k}\right)}{1+s^{2^k}}, \quad (6)$$

since

$$\begin{aligned}
2(1+s)\varphi(s^2)+\frac{s}{1+s} &= \frac{2(1+s)}{1-s^2}\sum_{k=0}^{\infty}\frac{2^k\cdot s^{2^{k+1}}\left(1-s^{2^{k+1}}\right)}{1+s^{2^{k+1}}}+\frac{s}{1-s}\\
&= \frac{1}{1-s}\sum_{k=0}^{\infty}\frac{2^{k+1}\cdot s^{2^{k+1}}\left(1-s^{2^{k+1}}\right)}{1+s^{2^{k+1}}}+\frac{s}{1+s}\\
&= \frac{1}{1-s}\left(\sum_{k=0}^{\infty}\frac{2^{k+1}\cdot s^{2^{k+1}}\left(1-s^{2^{k+1}}\right)}{1+s^{2^{k+1}}}+\frac{s(1-s)}{1+s}\right)\\
&= \varphi(s).
\end{aligned}$$

The solution of **E1** can be given in several equivalent formulas in binary form.

1. Let $n=(b_m\dots b_0)_2 := \sum_{k=0}^{m} b_k\cdot 2^k$, $b_k\in\{0,1\}$, $k\in[m+1]_0$, $(b_m=1)$, be the binary expansion of $n\in\mathbb{N}$. Then (see [13, formula (1.10) at p. 11])

$$J\Big((b_m\dots b_0)_2\Big) = (b_{m-1}b_{m-2}\dots b_1b_0b_m)_2 \tag{7}$$

Since $(b_{m-1}b_{m-2}\dots b_1b_0b_m)_2=\sum_{k=1}^{m} b_{k-1}2^k+b_m$, we have

$$J\Big((b_m\dots b_0)_2\Big)=\sum_{k=1}^{m} b_{k-1}2^k+b_m=2\cdot\left(\sum_{k=0}^{m} b_k2^k-2^m\right)+1=2n-2^{m+1}+1,$$

where $m=\lfloor\log_2 n\rfloor$. Further, since $-2^{m+1}+1=-\sum_{k=0}^{m}2^k$ and $2b_k-1=(-1)^{1-b_k}$, we get

$$\begin{aligned}
J\Big((b_m\dots b_0)_2\Big) &= (b_{m-1}b_{m-2}\dots b_1b_0b_m)_2 = 2\sum_{k=0}^{m} b_k2^k-\sum_{k=0}^{m}2^k\\
&= \sum_{k=0}^{m}(2b_k-1)2^k=\sum_{k=0}^{m}(-1)^{1-b_k}\cdot 2^k.
\end{aligned} \tag{8}$$

2. Let $n=2^{a_0}+2^{a_1}+\dots+2^{a_\ell}$, $a_0>a_1>\dots>a_\ell\ge 0$, $\ell\ge 0$, then

$$J(2^{a_0}+2^{a_1}+\dots+2^{a_\ell})=1+\sum_{k=0}^{\ell-1}2^{a_{\ell-k}+1}=2^{a_1+1}+2^{a_2+1}+\dots+2^{a_\ell+1}+2^0 \tag{9}$$

The first few values of $J(n)$ (the sequence A006257 in the On-Line Encyclopedia of Integer Sequences (OEIS ®) [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	0	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Table 1: $J(n)$, $0\le n\le 16$

- **E2** Number of 1's in the binary expansion of $n\in\mathbb{N}$.
  For $n=(b_m\dots b_0)_2$ we define the sum of the bits in the binary expansion of

n to be $s_2(n) := \sum_{k=0}^{m} b_k$. Since $b_k \in \{0,1\}$, $k \in [m+1]_0$, this sum gives the number of 1's in the binary expansion of n and satisfies the recurrence relation

$$s_2(1) = 1, \quad s_2(2n) = s_2(n), \quad s_2(2n+1) = s_2(n) + 1, \quad n \geq 1 \tag{10}$$

This is the special case $\alpha = 1 = \gamma$, $\beta = 0$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta = 0$ of (1), since 0 has no 1's in the binary expansion. The generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = (1+s)\varphi(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi(s) = \frac{1}{1-s}\sum_{k=0}^{\infty}\frac{s^{2^k}}{1+s^{2^k}}, \tag{11}$$

since

$$\begin{aligned}(1+s)\varphi(s^2) + \frac{s}{1-s^2} &= \frac{1+s}{1-s^2}\sum_{k=0}^{\infty}\frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1-s^2} \\ &= \frac{1}{1-s}\Bigg(\sum_{k=0}^{\infty}\frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1+s}\Bigg) = \varphi(s).\end{aligned}$$

The solution of **E2** is by definition $s_2\Big((b_m \ldots b_0)_2\Big) = \sum_{k=0}^{m} b_k$ or, alternatively, for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$,

$$s_2(2^{a_0} + 2^{a_1} + \cdots + 2^{a_l}) = l + 1. \tag{12}$$

Closely related to $s_2(n)$ is the sequence **E2'**, denoted by $e_0(n)$, giving the number of nonleading 0's in the binary expansion of n. It satisfies the recurrence relation

$$e_0(1) = 0, \quad e_0(2n) = e_0(n) + 1, \quad e_0(2n+1) = e_0(n), \quad n \geq 1 \tag{13}$$

This is the special case $\alpha = 1 = \gamma$, $\beta = 1$, $\delta = 0$, $\varepsilon = 0$, $\zeta = 0$ and $\eta = 0$ of (1). The generating function $\varphi_0(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi_0(s) = (1+s)\varphi_0(s^2) + \frac{s^2}{1-s^2}$$

and it is given by

$$\varphi_0(s) = \frac{1}{1-s}\sum_{k=0}^{\infty}\frac{s^{2^{k+1}}}{1+s^{2^k}}, \tag{14}$$

since

$$\begin{aligned}(1+s)\varphi_0(s^2) + \frac{s^2}{1-s^2} &= \frac{1+s}{1-s^2}\cdot\sum_{k=0}^{\infty}\frac{s^{2^{k+2}}}{1+s^{2^{k+1}}} + \frac{s^2}{1-s^2} \\ &= \frac{1}{1-s}\Bigg(\sum_{k=0}^{\infty}\frac{s^{2^{k+2}}}{1+s^{2^{k+1}}} + \frac{s^2}{1+s}\Bigg) = \varphi_0(s).\end{aligned}$$

The solution of **E2'** for $n = (b_m \dots b_0)_2$ is given by

$$e_0(n) = m + 1 - \sum_{k=0}^{m} b_k = \lfloor \log_2 n \rfloor + 1 - s_2(n). \tag{15}$$

The first few values of $s_2(n)$ (the sequence A000120 in [24]) and $e_0(n)$ (the sequence A080791 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s_2(n)$	0	1	1	2	1	2	2	3	1	2	2	3	2	3	3	4	1
$e_0(n)$	0	0	1	0	2	1	1	0	3	2	2	1	2	1	1	0	4

Table 2: $s_2(n)$, $e_0(n)$, $0 \le n \le 16$

- **E3** The Tower of Hanoi with 3 pegs and n disks, $n \in \mathbb{N}$.
  This puzzle has been invented in 1883 by the French mathematician Édouard Lucas (1842-1891). It consists of n disks, $n \in \mathbb{N}$, of different sizes and three vertical pegs. At the beginning, all disks are stacked in decreasing order on one of the three pegs (this is called a *tower*). The objective is to transfer the entire tower to another peg using the minimum number of legal moves, where a *legal move* is to move one topmost disk at a time and never moving a larger one onto a smaller one (this is the *divine rule*). For more details on the Tower of Hanoi with 3 or more pegs and on some variations of this puzzle we refer to the comprehensive monograph [19]. It can be shown (see [17, p. 281] and [19, Theorem 2.1, pp. 79–83]) that the number $d_2(n)$ of the disk to be moved at the nth step of the optimal solution of the Tower of Hanoi puzzle satisfies the recurrence relation

$$d_2(1) = 1, \quad d_2(2n) = d_2(n) + 1, \quad d_2(2n+1) = 1, \quad n \ge 1 \tag{16}$$

  This is the special case $\alpha = 1$, $\beta = 1 = \varepsilon$, $\gamma = 0$, $\delta = 0$, $\zeta = 1$ and $\eta := 0$ of (1). This sequence is also known as the *Gros sequence* (see [19, pp. 79–83]). The generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = \varphi(s^2) + \frac{s}{1-s}$$

  and it is given by

$$\varphi(s) = \sum_{k=0}^{\infty} \frac{s^{2^k}}{1 - s^{2^k}}, \tag{17}$$

  since

$$\varphi(s^2) + \frac{s}{1-s} = \sum_{k=0}^{\infty} \frac{s^{2^{k+1}}}{1 - s^{2^{k+1}}} + \frac{s}{1-s} = \sum_{k=0}^{\infty} \frac{s^{2^k}}{1 - s^{2^k}} = \varphi(s).$$

  The solution of **E3** for $n = 2^r(2k+1)$, $r, k \in \mathbb{N}_0$, is given by

$$d_2(n) = r + 1, \tag{18}$$

or, for $n = (b_m \dots b_0)_2$ (cf. (74))

$$d_2\Big((b_m \dots b_0)_2\Big) = \sum_{k=0}^{m} \Bigg(\prod_{j=0}^{k-1} (1 - b_j) \Bigg). \tag{19}$$

Alternatively, for $n = 2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}$, $a_0 > a_1 > \dots > a_\ell \geq 0$, $\ell \geq 0$,

$$d_2(n) = a_\ell + 1. \tag{20}$$

Other interpretations of this sequence based on (18) and (20) are

1. $d_2(n)$ gives the exponent of highest power of 2 dividing $2n$.
2. $d_2(n)$ gives the position of the first 1 (counting from right to left and beginning with 1) of the binary representation of n, e.g., $12 = (1100)_2 \Rightarrow d_2(12) = 3$.
3. $d_2(n)$ gives the position of the bit (counting from right to left and beginning with 1) to be changed in the Gray-code (see Table 3).

Therefore, the sequence $\tilde{d_2}(n) := d_2(n) - 1$ for all $n \in \mathbb{N}$ (called the *binary carry sequence*) gives the exponent of highest power of 2 dividing n or, equivalently, the index of the right-most non-zero bit in the binary representation of n. It is the sequence A007814 in [24].

n decimal	n binary	Gray-code	$d_2(n)$
1	1	000**1**	1
2	10	00**1**1	2
3	11	001**0**	1
4	100	0**1**10	3
5	101	011**1**	1
6	110	01**0**1	2
7	111	010**0**	1
8	1000	**1**100	4
9	1001	110**1**	1
10	1010	11**1**1	2
11	1011	111**0**	1
12	1100	1**0**10	3
13	1101	101**1**	1
14	1110	10**0**1	2
15	1111	100**0**	1

Table 3: Position of the nth bit to be changed in the Gray-code, $1 \leq n \leq 15$

Remark 1. The sequence $p_f(n)$ of the changed bit in the Gray-code above is the so-called *regular paper-folding* (or *dragon curve sequence).* $p_f(n)$ is the one's complement of the bit to the left of the least significant "1" in the binary expansion of n, e.g., $n = 7 = 111_2$, that is $p_f(7) = 0$. This sequence is defined recursively by the recurrence relation

$$p_f(1) = 1, \quad p_f(4n) = 1, \quad p_f(4n+2) = 0, \quad p_f(2n+1) = p_f(n), \quad n \geq 1 \tag{21}$$

The first few values of $d_2(n)$ (the sequence A001511 in [24]) and $p_f(n)$ (the sequence A014577 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_2(n)$	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1	5
$p_f(n)$	1	1	0	1	1	0	0	1	1	1	0	0	1	0	0	1

Table 4: $d_2(n)$, $p_f(n)$, $1 \le n \le 16$

Eq.(17) can be written in the form

$$\varphi(s) = \sum_{n=1}^{\infty} \rho(n) \cdot \frac{s^n}{1 - s^n}, \tag{22}$$

where $\rho(n)$ is the characteristic function of the powers of 2, that is $\rho(n) = 1$, if n is a power of 2 and zero otherwise. It shows that $\varphi(s)$ is a Lambert series (cf. (114)). Moreover, since $d_2(n)$ is the number of 2's that divides $2n$, or, equivalently, the number of the positive powers of 2 which divides n, we have

$$d_2(n) = \sum_{t|n} \rho(t), \quad n \in \mathbb{N}. \tag{23}$$

There is a relation between $d_2(n)$ and $s_2(n)$.

Proposition 2. *For all $n \in \mathbb{N}$ it is*

$$d_2(n) = 2 - \Delta s_2(n) \tag{24}$$

and, conversely,

$$s_2(n) = 2n - D_2(n) \tag{25}$$

where $\Delta s_2(n) := s_2(n) - s_2(n-1)$ is the difference sequence of $s_2(n)$ and

$$D_2(n) := \sum_{k=1}^{n} d_2(k) \tag{26}$$

is the sequence of the partial sums of $d_2(n)$.

Proof. Let $\alpha(n) := 2-s_2(n)+s_2(n-1)$, then $\alpha(1) = 2-s_2(1)+s_2(0) = 2-1+0 = 1 = d_2(1)$. Further, by (16) and noting that $s_2(2n-1) = s_2(2(n-1)+1) = s_2(n-1)+1$ we have $\alpha(2n) = 2-s_2(2n)+s_2(2n-1) = 2-s_2(n)+s_2(n-1)+1 = \alpha(n)+1$ and $\alpha(2n+1) = 2-s_2(2n+1)+s_2(2n) = 2-(s_2(n)+1)+s_2(n) = 1$.

For all $n \ge 1$ the sequence $\alpha(n)$ satisfies the same recurrence relation as $d_2(n)$ and has the same initial value, therefore the two sequences are equal and this proves (24).

Conversely, by repeated iteration we obtain $s_2(n) = s_2(n-1) + 2 - d_2(n) = s_2(n-2)+2\cdot 2-(d_2(n-1)+d_2(n)) = \ldots = s_2(0)+2n-\sum_{k=1}^{n} d_2(k) = 2n-D_2(n)$ and this is (25), since $s_2(0) = 0$. □

Note that since $s_2(n) \ge 1$ for all $n \in \mathbb{N}$ it follows that $D_2(n) \le 2n$.

E4 Number of odd binomial coefficients in the nth row of Pascal's triangle. This sequence (also known as *Gould's sequence*) is given recursively by

$$G(1) = 2, \quad G(2n) = G(n), \quad G(2n+1) = 2G(n), \quad n \geq 1 \tag{27}$$

$G(n)$ gives the number of odd binomial coefficients in the nth row ($n \geq 0$) of Pascal's triangle. Since Pascal's triangle with 2^n rows modulo 2 is isomorphic to the graph of the Tower of Hanoi with 3 pegs and n disks [16], $G(n)$ gives also the number of regular states for which the distance to a given perfect state is equal to n. It is the special case $\alpha = 1$, $\beta = 0 = \varepsilon, \gamma = 2$, $\delta = 0$, $\zeta = 2$ and $\eta := 1$ of (1). The generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = (1+2s)\varphi(s^2)$$

which shows that $\varphi(s)$ is Mahlerian. It is given by the infinite product

$$\varphi(s) = \prod_{k=0}^{\infty}(1+2s^{2^k}), \tag{28}$$

since

$$(1+2s)\varphi(s^2) = (1+2s) \cdot \prod_{k=0}^{\infty}(1+2s^{2^{k+1}}) = \prod_{k=0}^{\infty}(1+2s^{2^k}) = \varphi(s).$$

The solution of **E4** for $n = (b_m \dots b_0)_2$ has been given by Glaisher [12, p. 10]

$$G((b_m \dots b_0)_2) = 2^{\sum_{k=0}^{m} b_k} = 2^{s_2(n)} \tag{29}$$

Alternatively, for $n = 2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}$, $a_0 > a_1 > \dots > a_\ell \geq 0$, $\ell \geq 0$,

$$G(2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}) = 2^{\ell+1}. \tag{30}$$

A slight modification of **E4** is the sequence **E4'** defined recursively by

$$G_1(1) = 1, \quad G_1(2n) = G_1(n), \quad G_1(2n+1) = 2G_1(n)+1, \quad n \geq 1 \tag{31}$$

$G_1(n)$ is the smallest number with the same number of 1's in its binary representation as n. It is the special case $\alpha = 1$, $\beta = 0, \gamma = 2$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta = 0$ of (1) and it is $G_1(n) = G(n) - 1$ for all $n \in \mathbb{N}_0$. Its generating function $\varphi_1(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi_1(s) = (1+2s)\varphi_1(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi_1(s) = \varphi(s) - \sum_{n=0}^{\infty} s^n = \prod_{k=0}^{\infty}(1+2s^{2^k}) - \frac{1}{1-s}, \tag{32}$$

since

$$\begin{aligned}(1+2s)\varphi_1(s^2) + \frac{s}{1-s^2} &= (1+2s)\Big(\prod_{k=0}^{\infty}(1+2s^{2^{k+1}}) - \frac{1}{1-s^2}\Big) + \frac{s}{1-s^2} \\ &= \prod_{k=0}^{\infty}(1+2s^{2^k}) - \frac{1+2s}{1-s^2} + \frac{s}{1-s^2} = \varphi_1(s).\end{aligned}$$

The solution of (31) for $n = (b_m \dots b_0)_2$ is given by (cf. (74))

$$G_1((b_m \dots b_0)_2) = 2^{\sum_{k=0}^{m} b_k} - 1 = 2^{s_2(n)} - 1 = \sum_{k=0}^{m} b_k \cdot 2^{\sum_{j=0}^{k-1} b_j} \tag{33}$$

or, for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$,

$$G_1(2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}) = 2^{\ell+1} - 1. \tag{34}$$

The first few values of $G(n)$ (the sequence A001316 in [24]) and $G_1(n)$ (the sequence A038573 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$G(n)$	1	2	2	4	2	4	4	8	2	4	4	8	4	8	8	16	2
$G_1(n)$	0	1	1	3	1	3	3	7	1	3	3	7	3	7	7	15	1

Table 5: $G(n)$, $G_1(n)$, $0 \leq n \leq 16$

E5 Base 3 representation of n contains no 2's.
This sequence is given recursively by

$$f_2(1) = 1, \quad f_2(2n) = 3f_2(n), \quad f_2(2n+1) = 3f_2(n) + 1, \quad n \geq 1 \tag{35}$$

$f_2(n)$ gives the nth number that can be written as a sum of distinct powers of 3. It is the special case $\alpha = 3 = \gamma$, $\beta = 0$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta = 0$ of (1). Its generating function $\varphi_2(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi_2(s) = 3(1+s)\varphi_2(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi_2(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{3^k \cdot s^{2^k}}{1+s^{2^k}}, \tag{36}$$

since

$$\begin{aligned} 3(1+s)\varphi_2(s^2) + \frac{s}{1-s^2} &= \frac{3(1+s)}{1-s^2} \sum_{k=0}^{\infty} \frac{3^k \cdot s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1-s^2} \\ &= \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{3^{k+1} \cdot s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1-s^2} \\ &= \frac{1}{1-s} \left(\sum_{k=0}^{\infty} \frac{3^{k+1} \cdot s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1+s} \right) = \varphi_2(s). \end{aligned}$$

The solution of **E5** for $n = (b_m \dots b_0)_2$ is given by

$$f_2((b_m \dots b_0)_2) = (b_m \dots b_0)_3 = \sum_{k=0}^{m} b_k \cdot 3^k. \tag{37}$$

Alternatively, for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$,

$$f_2(2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}) = 3^{a_0} + 3^{a_1} + \cdots + 3^{a_\ell}. \tag{38}$$

The first few values of $f_2(n)$ (the sequence A005836$(n+1)$ in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f_2(n)$	0	1	3	4	9	10	12	13	27	28	30	31	36	37	39	40	81

Table 6: $f_2(n)$, $0 \le n \le 16$

Remark 2. An interesting property of this sequence has been noted by D. Tseng in 2009, namely that $f_2(n) \equiv s_2(n) \pmod 2$, since $3^k \equiv 1 \pmod 2$ for all $k \in [m+1]_0$. By (37) it follows $f_2(n) \equiv \sum_{k=0}^{m} b_k = s_2(n) \pmod 2$. This is the Prouhet-Thue-Morse sequence $m(n)$ on $\{0,1\}$ which we shall now present as Example **E6**.

E6 The Prouhet-Thue-Morse sequence on $\{0,1\}$ (see [3, 14, 26]).
This famous sequence can be obtained in several ways:

1. consider the sequence of the natural numbers $(0,1,2,3,4,5,6,7,8,9,10\ldots)$ in binary form $(0,1,10,11,100,101,110,111,1000,1001,1010,\ldots)$ and build the sum of the digits of these numbers modulo 2, so we obtain the Prouhet-Thue-Morse sequence $(0,1,1,0,1,0,0,1,1,0,0,\ldots)$, that is $m(n) = s_2(n) \bmod 2$.
2. recursively, starting with $m(0) = 0$ and for all $k \in \mathbb{N}_0$, $m(0), m(1), \ldots, m(2^k - 1)$ is followed by its *one's complement*, i.e., $m(2^k) = \overline{m(0)}, \ldots, m(2^{k+1}) = \overline{m(2^k-1)}$. Hence, $0, 01, 0110, 01101001, 0110100110010110, \ldots$
3. by iteration according to the rules $0 \to 01$, $1 \to 10$,
that is $0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \cdots$.
4. by the logical function XOR (exclusive or) (see Table 7 for $n=2$)

x	y	$x\,XOR\,y$
0	0	0
0	1	1
1	0	1
1	1	0

Table 7: Truth table for XOR, x, y propositions, $1 = T(rue)$, $0 = F(alse)$

It satisfies the recurrence relation:

$$m(1) = 1, \quad m(2n) = m(n), \quad m(2n+1) = -m(n) + 1, \quad n \ge 1 \tag{39}$$

This is the special case $\alpha = 1$, $\beta = 0, \gamma = -1$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta := 0$ of (1). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = (1-s)\varphi(s^2) + \frac{s}{1-s^2}.$$

In order to get a simple expression for the generating function $\varphi(s)$ of $m(n)$ we use the transformation

$$m_1(n) := -2m(n) + 1 \tag{40}$$

which gives the Prouhet-Thue-Morse sequence on $\{-1, +1\}$ instead of $\{0, 1\}$. Note that by (40) we have $0 \to 1$ and $1 \to -1$, that is

$$m_1(n) = (-1)^{m(n)}. \tag{41}$$

This sequence **E6'** satisfies the recurrence relation:

$$m_1(1) = -1, \quad m_1(2n) = m_1(n), \quad m_1(2n+1) = -m_1(n), \quad n \geq 1 \tag{42}$$

This is the special case $\alpha = 1$, $\beta = 0 = \varepsilon, \gamma = -1$, $\delta = 0$, $\zeta = -1$ and $\eta = 1$ of (1). Its generating function $\psi(s)$ satisfies, by Proposition 1, the functional equation

$$\psi(s) = (1 - s)\psi(s^2).$$

$\psi(s)$ is Mahlerian and it is given by the infinite product

$$\psi(s) = \prod_{k=0}^{\infty} (1 - s^{2^k}) \tag{43}$$

known as *Euler's product*, since

$$(1 - s)\psi(s^2) = (1 - s) \prod_{k=0}^{\infty} (1 - s^{2^{k+1}}) = \prod_{k=0}^{\infty} (1 - s^{2^k}) = \psi(s).$$

By (41) and (43) we have

$$\sum_{n=0}^{\infty} (-1)^{m(n)} s^n = \prod_{k=0}^{\infty} (1 - s^{2^k}). \tag{44}$$

Using (40) and (43) we obtain the generating function $\varphi(s)$ of $m(n)$. Indeed, we have

$$\begin{aligned}
\varphi(s) &= \sum_{n=0}^{\infty} m(n) s^n = \sum_{n=0}^{\infty} \frac{1 - m_1(n)}{2} s^n \\
&= \frac{1}{2} \cdot \Big(\sum_{n=0}^{\infty} s^n - \sum_{n=0}^{\infty} m_1(n) s^n \Big) = \frac{1}{2} \cdot \Big(\frac{1}{1 - s} - \psi(s) \Big) \\
&= \frac{1}{2} \cdot \Big(\frac{1}{1 - s} - \prod_{k=0}^{\infty} (1 - s^{2^k}) \Big)
\end{aligned}$$

The solution of **E6'** for $n = (b_m \dots b_0)_2$ is given by

$$m_1(n) = (-1)^{\sum_{k=0}^{m} b_k} = (-1)^{s_2(n)} \tag{45}$$

or, for $n = 2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}$, $a_0 > a_1 > \dots > a_\ell \geq 0$, $\ell \geq 0$, by

$$m_1(2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}) = (-1)^{\ell + 1} \tag{46}$$

and that of **E6** is given by

$$m(n) = \frac{1}{2} \cdot \left(1 - (-1)^{s_2(n)}\right) \tag{47}$$

or, for $n = (b_m \dots b_0)_2$, by

$$m((b_m \dots b_0)_2) = \sum_{k=0}^{m} b_k (-1)^{\sum_{j=0}^{k-1} b_j} \tag{48}$$

or, for $n = 2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}$, $a_0 > a_1 > \dots > a_\ell \geq 0$, $\ell \geq 0$, by

$$m(2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}) = \sum_{k=0}^{\ell} (-1)^k = \frac{1}{2} \cdot \left(1 + (-1)^\ell\right) \tag{49}$$

Note that by (49) we obtain $|m(n)| = \left|\sum_{k=0}^{\ell} (-1)^k\right| \leq \sum_{k=0}^{\ell} |(-1)^k| = \sum_{k=0}^{\ell} 1 = \ell + 1 = s_2(n)$, for all $n \in \mathbb{N}$.
The first few values of $m(n)$ (the sequence A010060 in [24]) and $m_1(n)$ (the sequence A106400 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$m(n)$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1
$m_1(n)$	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	-1

Table 8: $m(n), m_1(n)$, $0 \leq n \leq 16$

E7 The *infinite sequence* of Per Nørgård (see [4, Example 23, pp. 21–22] and [5]).
This sequence is given recursively by

$$a(1) = 1, \quad a(2n) = -a(n), \quad a(2n+1) = a(n) + 1, \quad n \geq 1 \tag{50}$$

It is the special case $\alpha = -1$, $\beta = 0, \gamma = 1$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta := 0$ of (1). It is the *infinite sequence* of the Danish composer Per Nørgård (1932–), who invented it in an attempt to unify in a perfect way repetition and variation. Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = -(1-s)\varphi(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi(s) = \sum_{k=0}^{\infty} (-1)^k \frac{s^{2^k} \prod_{j=0}^{k-1} (1 - s^{2^j})}{1 - s^{2^{k+1}}}, \tag{51}$$

since

$$\begin{aligned} -(1-s)\varphi(s^2) + \frac{s}{1-s^2} &= -(1-s) \sum_{k=0}^{\infty} (-1)^k \frac{s^{2^{k+1}} \prod_{j=0}^{k-1} (1 - s^{2^{j+1}})}{1 - s^{2^{k+2}}} + \frac{s}{1-s^2} \\ &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{s^{2^{k+1}} \prod_{j=0}^{k} (1 - s^{2^j})}{1 - s^{2^{k+2}}} + \frac{s}{1-s^2} = \varphi(s). \end{aligned}$$

The solution of **E7** for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$, is given by

$$a(2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}) = \sum_{k=0}^{\ell} (-1)^k \cdot (-1)^{a_{\ell-k}}. \tag{52}$$

It is possible to generate this sequence by a procedure given by J. Mortensen: write n in binary form and read from left to right starting with 0 and interpreting 1 as "add 1" and 0 as "change sign". For example $23 = (10111)_2$ gives $0 \to 1 \to -1 \to 0 \to 1 \to 2$, so $a(23) = 2$. Note that by (52) we obtain $|a(n)| = \left|\sum_{k=0}^{\ell}(-1)^k \cdot (-1)^{a_{\ell-k}}\right| \leq \sum_{k=0}^{\ell} |(-1)^k| \cdot |(-1)^{a_{\ell-k}}| = \sum_{k=0}^{\ell} 1 = \ell + 1 = s_2(n)$, for all $n \in \mathbb{N}$.

A variation of **E7**, **E2** and **E6** is the sequence **E7'** defined by

$$a'(1) = 1, \quad a'(2n) = -a'(n), \quad a'(2n+1) = -a'(n) + 1, \quad n \geq 1 \tag{53}$$

It is the special case $\alpha = -1 = \gamma$, $\beta = 0$, $\delta = 0$, $\varepsilon = 1$, $\zeta = 1$ and $\eta := 0$ of (1). It can be defined as the alternating bit sum (adding from right to left and starting with a positive sign) of the binary expansion of n. For example: $n = 13 = (1101)_2$, hence $a'(13) = 1 - 0 + 1 - 1 = 1$. Its generating function $\varphi_1(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi_1(s) = -(1+s)\varphi_1(s^2) + \frac{s}{1-s^2}$$

and it is given by

$$\varphi_1(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} (-1)^k \frac{s^{2^k}}{1+s^{2^k}}, \tag{54}$$

since

$$\begin{aligned} -(1+s)\varphi_1(s^2) + \frac{s}{1-s^2} &= -\frac{1+s}{1-s^2} \sum_{k=0}^{\infty} (-1)^k \frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1-s^2} \\ &= \frac{1}{1-s} \left(\sum_{k=0}^{\infty} (-1)^{k+1} \frac{s^{2^{k+1}}}{1+s^{2^{k+1}}} + \frac{s}{1+s} \right) = \varphi_1(s). \end{aligned}$$

The solution of **E7'** for $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $\quad a_0 > a_1 > \cdots > a_\ell \geq 0$, $\quad \ell \geq 0$, is given by

$$a'(2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}) = \sum_{k=0}^{\ell} (-1)^{a_k} \tag{55}$$

This formula can be interpreted as follows: replace in the representation $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$ the terms 2^{a_k} by $(-1)^{a_k}$, $0 \leq k \leq \ell$. As in the case of $a(n)$ we have by (55) $|a'(n)| = \left|\sum_{k=0}^{\ell}(-1)^{a_k}\right| \leq \sum_{k=0}^{\ell} |(-1)^{a_k}| = \sum_{k=0}^{\ell} 1 = \ell + 1 = s_2(n)$, for all $n \in \mathbb{N}$.
The first few values of $a(n)$ (the sequence A004718 in [24]) and $a'(n)$ (the sequence A065359 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a(n)$	0	1	-1	2	1	0	-2	3	-1	2	0	1	2	-1	-3	4	1
$a'(n)$	0	1	-1	0	1	2	0	1	-1	0	-2	-1	0	1	-1	0	1

Table 9: $a(n), a'(n)$, $0 \le n \le 16$

Up to now all sequences were with $\delta = 0$. We shall now consider some example of (1) with $\delta \neq 0$. Perhaps the most simple sequence is

- **E8** Arithmetic progression
An arithmetic progression is defined recursively for all $n \in \mathbb{N}_0$ by

$$a(0) = a_0, \quad (\Rightarrow a(1) = a_0 + d), \quad a(n+1) = a(n) + d \tag{56}$$

The solution of (56) is given by

$$a(n) = a_0 + nd, \quad n \ge 0. \tag{57}$$

This solution satisfies also the recurrence relation

$$\begin{aligned} a(1) = a_0 + d, &\quad (\Rightarrow a(0) = a_0) \\ a(2n) = 2a(n) - a_0, &\quad a(2n+1) = a(n) + a(n+1) - a_0 \end{aligned} \tag{58}$$

It is the special case $\alpha = 2$, $\beta = -a_0, \gamma = 1$, $\delta = 1$, $\varepsilon = -a_0$, $\zeta = a_0 + d$ and $\eta = a_0$ of (1). Conversely, the difference between two consecutive terms of (58) is a constant, since for $d(n) := a(n+1) - a(n)$ we have

$$d(1) = d, \quad (\Rightarrow d(0) = d), \quad d(2n) = d(n), \quad d(2n+1) = d(n) \tag{59}$$

It is the special case $\alpha = 1 = \gamma$, $\beta = 0 = \varepsilon$, $\delta = 0$, $\zeta = d$ and $\eta = d$ of (1) with the solution $d(n) = d$, $n \in \mathbb{N}_0$ (the constant sequence).
The solution of (59) is given by (57) for $a_0 = -\beta$ and $d = \beta + \zeta$.

- **E9** Stern's diatomic sequence and two variations on it.
The first $2^n + 1$, $n \in \mathbb{N}_0$, terms of this sequence form the rth row of the so-called *Stern's diatomic array* $\big((0,1)_r\big)_{r \in \mathbb{N}_0}$ defined as follows: $(0,1)_0(0) := 0$, $(0,1)_0(1) := 1$ (0 and 1 are called the *atoms* or the *seeds* of the array) and the next rows are obtained by successively intercalating the sum of two neighbouring numbers. It satisfies the recurrence relation ($r, n \in \mathbb{N}_0$):

$$\begin{aligned} &(0,1)_0(0) = 0, \quad (0,1)_0(1) = 1, \\ &(0,1)_{r+1}(2n) = (0,1)_r(n), \quad (0,1)_{r+1}(2n+1) = (0,1)_r(n) + (0,1)_r(n+1) \end{aligned} \tag{60}$$

This array (originated by G. Eisenstein) has been studied in 1858 by M. A. Stern [32] and in 1860 (in a different form) by A. Brocot [7]. Later has been studied, among others, by É. Lucas [22], D. H. Lehmer [20] and D. A. Lind [21]. Stern's diatomic sequence can be defined by

$$s(n) := (0,1)_\infty(n), \quad n \in \mathbb{N}_0. \tag{61}$$

As a sequence it has been defined for the first time in 1947 by G. de Rham [29, p. 95] recursively as follows:

$$s(0) = 0, \quad s(1) = 1, \quad s(2n) = s(n), \quad s(2n+1) = s(n) + s(n+1), \quad n \geq 1 \tag{62}$$

It is the special case $\alpha = 1 = \gamma$, $\beta = 0 = \varepsilon$, $\delta = 1$, $\zeta = 1$ and $\eta = 0$ of (1). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$s\varphi(s) = (1 + s + s^2)\varphi(s^2)$$

which shows that $\varphi(s)$ is Mahlerian. It is given by the infinite product

$$\varphi(s) = s \cdot \prod_{k=0}^{\infty}(1 + s^{2^k} + s^{2^{k+1}}), \tag{63}$$

since

$$\begin{aligned}(1 + s + s^2)\varphi(s^2) &= (1 + s + s^2) \cdot s^2 \cdot \prod_{k=0}^{\infty}(1 + s^{2^{k+1}} + s^{2^{k+2}}) \\ &= s^2 \cdot \prod_{k=0}^{\infty}(1 + s^{2^k} + s^{2^{k+1}}) = s\varphi(s).\end{aligned}$$

This sequence has been studied by many mathematicians, see for example the recent publications by S. Northshield [23] and I. Urbiha [33], because it has remarkable combinatorial interpretations:

1. $s(n+1)$ is the third binary partition function $b(3; n)$, i.e., $s(n+1)$ is the number of representations of $n = \sum_{k=0}^{\infty} \xi \cdot 2^i$, $\xi \in \{0, 1, 2\}$ (see [28]).
2. in [9, 10] L. Carlitz defined the sequences $\Theta_0(n)$ as the number of odd Stirling numbers of second kind $\left\{ {n \atop m} \right\}$ with even k, $k \leq n$, and $\Theta_1(n)$ as the number of odd Stirling numbers of second kind $\left\{ {n \atop m} \right\}$ with odd k, $k \leq n$, and showed that $\Theta_1(n+1) = \Theta_0(n)$ and that the generating function of $\Theta_0(n)$ is given by $\prod_{k=0}^{\infty}(1 + s^{2^k} + s^{2^{k+1}})$ and consequently that of $\Theta_1(n)$ is given by (63). Moreover, he showed that $\Theta_0(n)$ satisfies the recurrence relation $\Theta_0(0) = 1$, $\Theta_0(1) = 1$, $\Theta_0(2n) = \Theta_0(n) + \Theta_0(n-1)$, $\Theta_0(2n+1) = \Theta_0(n)$, that is $\Theta_0(n) = s(n+1)$ and consequently $s(n) = \Theta_1(n)$.
3. $s(n)$ is the number of odd binomial coefficients in the nth subdiagonal (i.e. $\binom{\ell}{k}$ with $\ell + k = n$, $\ell \geq k \geq 0$) of Pascal's triangle.
4. $s(\mu) = z_n(2^n - \mu)$, $\mu = 0, 1, \ldots, 2^n$, where $z_n(\mu)$ (see [15, p. 305]) is the number of regular states in the graph of the Tower of Hanoi with 3 pegs and n disks, for which the difference of the distances to two distinct perfect states is equal to μ (see [18, Formula 2, p. 700]).

The first few values of $s(n)$ (the sequence A002487 in [24]) are given in Table 10.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s(n)$	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1

Table 10: $s(n)$, $0 \le n \le 16$

We consider now two variations on Stern's diatomic sequence. The first variation **E9'** has been introduced in [27] by B. Reznick and it is called *Reznick's sequence*. It is given recursively by

$$r(1) = 1, \quad r(2n) = r(n), \quad r(2n+1) = -r(n) + r(n+1), \quad n \ge 1 \tag{64}$$

It is the special case $\alpha = 1$, $\beta = 0 = \varepsilon, \gamma = -1$, $\delta = 1$, $\zeta = 1$ and $\eta := 0$ of (1). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$s\varphi(s) = (1 + s - s^2)\varphi(s^2)$$

which shows that $\varphi(s)$ is Mahlerian. It is given by the infinite product

$$\varphi(s) = s \prod_{k=0}^{\infty} (1 + s^{2^k} - s^{2^{k+1}}), \tag{65}$$

since

$$\begin{aligned}(1 + s - s^2)\varphi(s^2) &= (1 + s - s^2) \cdot s^2 \prod_{k=0}^{\infty} (1 + s^{2^{k+1}} - s^{2^{k+2}}) \\ &= s^2 \prod_{k=0}^{\infty} (1 + s^{2^k} - s^{2^{k+1}}) = s\varphi(s).\end{aligned}$$

The second variation **E9"** called *the twisted Stern sequence* has been introduced in [6] by R. Bacher and later studied by J. P. Allouche [1] and M. Coons [11]. It is given recursively by

$$t(1) = 1, \quad t(2n) = -t(n), \quad t(2n+1) = -t(n) - t(n+1), \quad n \ge 1 \tag{66}$$

It is the special case $\alpha = -1 = \gamma$, $\beta = 0 = \varepsilon$, $\delta = -1$, $\zeta = 1$ and $\eta := 0$ of (1). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$s\varphi(s) = -(1 + s + s^2)\varphi(s^2) + 2s^2$$

and it is given by

$$\varphi(s) = s \prod_{k=0}^{\infty} (-1)^k (1 + s^{2^k} + s^{2^{k+1}}) + 2s \sum_{n=0}^{\infty} (-1)^n \Big(\prod_{k=0}^{n-1} (1 + s^{2^k} + s^{2^{k+1}}) \Big), \tag{67}$$

since

$$-(1+s+s^2)\varphi(s^2)+2s^2=(1+s+s^2)\Bigg(s^2\prod_{k=0}^{\infty}(-1)^{k+1}(1+s^{2^{k+1}}+s^{2^{k+2}})+$$
$$+2s^2\sum_{n=0}^{\infty}(-1)^{n+1}\Bigg(\prod_{k=0}^{n-1}(1+s^{2^{k+1}}+s^{2^{k+2}})\Bigg)\Bigg)+2s^2$$
$$=s^2\prod_{k=0}^{\infty}(-1)^{k}(1+s^{2^{k}}+s^{2^{k+1}})+$$
$$+2s^2\Bigg(1+\sum_{n=0}^{\infty}(-1)^{n+1}\Bigg(\prod_{k=0}^{n}(1+s^{2^{k}}+s^{2^{k+1}})\Bigg)\Bigg)$$

and this is exactly $s\varphi(s)$, as the expression between the parentheses in the last term of the above equation is equal to $\sum_{n=0}^{\infty}(-1)^n\Big(\prod_{k=0}^{n-1}(1+s^{2^k}+s^{2^{k+1}})\Big)$.
The first few values of $r(n)$ (the sequence A005590 in [24]) and $t(n)$ (the sequence A213369 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$r(n)$	0	1	1	0	1	-1	0	1	1	-2	-1	1	0	1	1	0	1
$t(n)$	0	1	-1	0	1	1	0	-1	-1	-2	-1	-1	0	1	1	2	1

Table 11: $r(n)$, $t(n)$, $0 \le n \le 16$

- **E10** Tennis Tournament
  There are n participants to a (knock-out) tennis tournament. In the first round, all the losers are eliminated. The winners play in pairs in the second round, the losers are eliminated again, and so on. How many rounds $T(n)$ do we need to determine the winner?
  This sequence is given recursively for all $n \in \mathbb{N}$ by

$$T(1)=0,\quad T(2n)=T(n)+1,\quad T(2n+1)=T(n+1)+1 \tag{68}$$

It is the special case $\alpha=1$, $\beta=1=\varepsilon, \gamma=0$, $\delta=1$, $\zeta=0$ and $\eta:=0$ of (1). Equivalently, **E10** can be written as

$$T(1)=0,\quad T(n)=T\Big(\lceil n/2\rceil\Big)+1,\quad n\ge 2 \tag{69}$$

Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$s\varphi(s)=(1+s)\varphi(s^2)+\frac{s^3}{1-s}$$

and it is given by

$$\varphi(s)=\frac{s}{1-s}\sum_{k=0}^{\infty}s^{2^k}, \tag{70}$$

since

$$\begin{aligned}(1+s)\varphi(s^2)+\frac{s^3}{1-s} &= (1+s)\cdot\frac{s^2}{1-s^2}\cdot\sum_{k=0}^{\infty}s^{2^{k+1}}+\frac{s^3}{1-s}\\ &= \frac{s^2}{1-s}\Big(\sum_{k=0}^{\infty}s^{2^{k+1}}+s\Big)=s\varphi(s)\end{aligned}$$

The solution of **E10** for $n=2^{a_0}+2^{a_1}+\cdots+2^{a_\ell}$, $a_0>a_1>\cdots>a_\ell\geq 0,\quad \ell\geq 0$, is given by

$$T(2^{a_0})=a_0,\quad T(2^{a_0}+2^{a_1}+\cdots+2^{a_\ell})=a_0+1,\quad \ell\geq 1, \tag{71}$$

and since $a_0=\lfloor\log_2 n\rfloor$ we can also write

$$T(n)=\lceil\log_2 n\rceil,\quad n\geq 1. \tag{72}$$

The first few values of $T(n)$ (the sequence A029837 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$T(n)$	0	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4

Table 12: $T(n)$, $1\leq n\leq 16$

3 Solution of the special case $\delta=0$

We shall now study the special case $\delta=0$ of (1), that is

$$f(1)=\zeta,\quad f(2n)=\alpha f(n)+\beta,\quad f(2n+1)=\gamma f(n)+\varepsilon,\quad n\geq 1 \tag{73}$$

with $\alpha,\beta,\gamma,\varepsilon,\zeta\in\mathbb{Z}$, since the general case $\delta\neq 0$ can be reduced to (73).

Theorem 1. *The solution of (73) for $n=(b_m\dots b_0)_2$ is given by*

$$f(n)=\zeta\prod_{k=0}^{m-1}\Big(\gamma b_k+\alpha(1-b_k)\Big)+\sum_{k=0}^{m-1}\Big(\prod_{j=0}^{k-1}\Big(\gamma b_j+\alpha(1-b_j)\Big)\Big)\cdot\Big(\varepsilon b_k+\beta(1-b_k)\Big) \tag{74}$$

Proof. Let $n=(b_m\dots b_0)_2$. Then by (73):
if $b_0=0$ we have $f\Big((b_m\dots b_0)_2\Big)=\alpha f\Big((b_m\dots b_1)_2\Big)+\beta$ and
if $b_0=1$ we have $f\Big((b_m\dots b_0)_2\Big)=\gamma f\Big((b_m\dots b_1)_2\Big)+\varepsilon$.
Hence,

$$\begin{aligned}f\Big((b_m\dots b_0)_2\Big) &= b_0\cdot\Big(\gamma f\Big((b_m\dots b_1)_2\Big)+\varepsilon\Big)+(1-b_0)\cdot\Big(\alpha f\Big((b_m\dots b_1)_2\Big)+\beta\Big)\\ &= \Big(\gamma b_0+\alpha(1-b_0)\Big)\cdot f\Big((b_m\dots b_1)_2\Big)+\Big(\varepsilon b_0+\beta(1-b_0)\Big).\end{aligned}$$

Similarly, $f\Big((b_m\dots b_1)_2\Big) = \Big(\gamma b_1+\alpha(1-b_1)\Big)\cdot f\Big((b_m\dots b_2)_2\Big) + \Big(\varepsilon b_1+\beta(1-b_1)\Big)$. Substituting this expression into the above equation gives

$$\begin{aligned} f\Big((b_m\dots b_0)_2\Big) &= \Big(\gamma b_0+\alpha(1-b_0)\Big)\cdot\Big(\gamma b_1+\alpha(1-b_1)\Big)\cdot f\Big((b_m\dots b_2)_2\Big)+\\ &+\Big(\gamma b_0+\alpha(1-b_0)\Big)\cdot\Big(\varepsilon b_1+\beta(1-b_1)\Big)+\Big(\varepsilon b_0+\beta(1-b_0)\Big)\end{aligned}$$

By iteration and noting that $f\Big((b_m)_2\Big) = f(1) = \zeta$ one obtains formula (74) and this proves the theorem. $\square$

The next corollary gives the exact values of $f(n)$ for some special numbers.

Corollary 1. *We have*

$$f(2^m) = f\Big((10\dots0)_2\Big) = \zeta\alpha^m+\beta\sum_{k=0}^{m-1}\alpha^k,\quad m\ge 0 \tag{75}$$

$$f(2^m+1) = f\Big((10\dots01)_2\Big) = \zeta\gamma\alpha^{m-1}+\beta\gamma\sum_{k=0}^{m-2}\alpha^k+\varepsilon,\quad m\ge 1 \tag{76}$$

$$f(2^{m+1}-1) = f\Big((11\dots1)_2\Big) = \zeta\gamma^m+\varepsilon\sum_{k=0}^{m-1}\gamma^k,\quad m\ge 0 \tag{77}$$

In the next theorem we present an alternative expression of the solution of (73).

Theorem 2. *Let $n = 2^{a_0}+2^{a_1}+\dots+2^{a_\ell}$, $a_0>a_1>\dots>a_\ell\ge 0$, $\ell\ge 0$, a strictly decreasing sequence of positive integers. Then the solution of (73) for $\alpha\ne 0$ is given by*

$$f(n) = \Big(\frac{\gamma}{\alpha}\Big)^\ell\cdot\Big(\zeta\alpha^{a_0}+\beta\sum_{k=0}^{a_0-1}\alpha^k\Big)+\sum_{k=0}^{\ell-1}\Big(\frac{\gamma}{\alpha}\Big)^k\cdot\Big((\varepsilon-\beta)\alpha^{a_{\ell-k}}+\beta\Big(1-\frac{\gamma}{\alpha}\Big)\sum_{j=0}^{a_{\ell-k}}\alpha^j\Big) \tag{78}$$

and for $\alpha = 0$ by

$$f(n) = \beta+\gamma^\ell(\zeta-\beta)\prod_{k=0}^{\ell}[a_{\ell-k}=k]+(\beta\gamma+\varepsilon-\beta)\sum_{k=0}^{\ell-1}\gamma^k\Big(\prod_{j=0}^{k}[a_{\ell-j}=j]\Big), \tag{79}$$

where $[A]$ is Iverson's convention meaning 1 if the assertion A is true and 0 if A is false.

Proof. 1) Let $\alpha\ne 0$ and $A_i := f(2^{a_0}+\dots+2^{a_i})$, $i\in[\ell+1]_0$, $a_0>a_1>\dots>a_\ell\ge 0$, a strictly decreasing sequence of positive integers. Then $f(2^{a_0+1}) = \alpha\cdot f(2^{a_0})+\beta$ with the initial value $f(2^0) = f(1) = \zeta$. This is an arithmetic progression with the solution

$$f(2^{a_0}) = f(1)\cdot\alpha^{a_0}+\beta\sum_{k=0}^{a_0-1}\alpha^k = \zeta\alpha^{a_0}+\beta\sum_{k=0}^{a_0-1}\alpha^k.$$

Now let $i\ge 0$, then by repeated use of (1)

$$\begin{aligned} A_{i+1} &= f(2^{a_0}+\dots+2^{a_{i+1}}) = f\Big(2^{a_{i+1}}\cdot(2^{a_0-a_{i+1}}+\dots+2^{a_i-a_{i+1}}+1)\Big)\\ &= \alpha^{a_{i+1}}\cdot f(2^{a_0-a_{i+1}}+\dots+2^{a_i-a_{i+1}}+1)+\beta\sum_{k=0}^{a_{i+1}-1}\alpha^k\\ &= \alpha^{a_{i+1}}\cdot\Big(\gamma\cdot f(2^{a_0-a_{i+1}-1}+\dots+2^{a_i-a_{i+1}-1})+\varepsilon\Big)+\beta\sum_{k=0}^{a_{i+1}-1}\alpha^k,\end{aligned}$$

that is

$$A_{i+1}=\gamma\alpha^{a_{i+1}}\cdot f(2^{a_0-a_{i+1}-1}+\cdots+2^{a_i-a_{i+1}-1})+\varepsilon\alpha^{a_{i+1}}+\beta\sum_{k=0}^{a_{i+1}-1}\alpha^k. \tag{80}$$

We have

$$\begin{aligned}f(2^{a_0}+\cdots+2^{a_i})&=f\Big(2^{a_{i+1}+1}\cdot(2^{a_0-a_{i+1}-1}+\cdots+2^{a_i-a_{i+1}-1})\Big)\\&=\alpha^{a_{i+1}+1}\cdot f(2^{a_0-a_{i+1}-1}+\cdots+2^{a_i-a_{i+1}-1})+\beta\sum_{k=0}^{a_{i+1}}\alpha^k\end{aligned}$$

or solving for $f(2^{a_0-a_{i+1}-1}+\cdots+2^{a_i-a_{i+1}-1})$ and since $\alpha\neq 0$ it follows $f(2^{a_0-a_{i+1}-1}+\cdots+2^{a_i-a_{i+1}-1})=\Big(f(2^{a_0}+\cdots+2^{a_i})-\beta\sum_{k=0}^{a_{i+1}}\alpha^k\Big)/\alpha^{a_{i+1}+1}$. Inserting this formula into (80) yields

$$A_{i+1}=\frac{\gamma}{\alpha}A_i-\frac{\beta\gamma}{\alpha}\sum_{k=0}^{a_{i+1}}\alpha^k+\varepsilon\alpha^{a_{i+1}}+\beta\sum_{k=0}^{a_{i+1}}\alpha^k-\beta\alpha^{a_{i+1}}=\frac{\gamma}{\alpha}A_i-g(i+1),$$

where $g(i+1):=\beta\Big(1-\frac{\gamma}{\alpha}\Big)\sum_{k=0}^{a_{i+1}}\alpha^k+(\varepsilon-\beta)\alpha^{a_{i+1}}$.

The solution of this arithmetic progression is given by

$$\begin{aligned}A_\ell&=A_0\cdot\left(\frac{\gamma}{\alpha}\right)^\ell+\sum_{k=0}^{\ell-1}\left(\frac{\gamma}{\alpha}\right)^k\cdot g(\ell-k)\\&=\left(\frac{\gamma}{\alpha}\right)^\ell\cdot\Big(\zeta\alpha^{a_0}+\beta\sum_{k=0}^{a_0-1}\alpha^k\Big)+\sum_{k=0}^{\ell-1}\left(\frac{\gamma}{\alpha}\right)^k\cdot\Big((\varepsilon-\beta)\alpha^{a_{\ell-k}}+\beta\Big(1-\frac{\gamma}{\alpha}\Big)\sum_{j=0}^{a_{\ell-k}}\alpha^j\Big)\end{aligned}$$

and this is formula (78).
2) Let $\alpha=0$, then we shall prove (79) by induction on $\ell\in\mathbb{N}_0$.
Base step: For $\ell=0$ we have by (73) : if $a_0=0:\ f(2^0)=\zeta$ and if $a_0\neq 0:\ f(2^{a_0})=\beta$ or $f(2^{a_0})=(a_0=0)\cdot\zeta+(1-(a_0=0))\cdot\beta=\beta+(\zeta-\beta)\cdot(a_0=0)$, and this is (79) for $\ell=0$.
Induction step: By (73) we have: $f(2^{a_0}+\cdots+2^{a_{\ell+1}})=\gamma f(2^{a_0-1}+\cdots+2^{a_\ell-1})+\varepsilon$, if $a_{l+1}=0$ and $f(2^{a_0}+\cdots+2^{a_{\ell+1}})=\beta$, if $a_{l+1}\neq 0$ or

$$\begin{aligned}f(2^{a_0}+\cdots+2^{a_\ell})&=(a_{\ell+1}=0)\cdot\Big(\gamma f(2^{a_0-1}+\cdots+2^{a_\ell-1})+\varepsilon\Big)+(1-(a_{\ell+1}=0))\cdot\beta\\&=\beta+\Big(\gamma f(2^{a_0-1}+\cdots+2^{a_\ell-1})+\varepsilon\Big)(a_{\ell+1}=0)\end{aligned}\tag{81}$$

By induction assumption (with $a_0-1,\dots,a_\ell-1$ instead of $a_0,\dots,a_\ell$) we have

$$f(2^{a_0-1}+\cdots+2^{a_\ell-1})=\beta+\gamma^\ell(\zeta-\beta)\prod_{k=0}^{\ell}[a_{\ell-k}=k+1]+(\beta\gamma+\varepsilon-\beta)\sum_{k=0}^{\ell-1}\gamma^k\Big(\prod_{j=0}^{k}[a_{\ell-j}-1=j]\Big).$$

Hence, inserting this equation into (81) we obtain

$$\begin{aligned}
f(2^{a_0}+\cdots+2^{a_{\ell+1}}) = \beta + \Bigg(\gamma\cdot\Bigg(&\beta+\gamma^\ell(\zeta-\beta)\prod_{k=0}^{\ell}[a_{\ell-k}=k]+\\
&+(\beta\gamma+\varepsilon-\beta)\sum_{k=0}^{\ell-1}\gamma^k\Big(\prod_{j=0}^{k}[a_{\ell-j}=j]\Big)\Bigg)+\varepsilon-\beta\Bigg)\cdot[a_{\ell+1}=0]\\
=&\ \beta+\gamma^{\ell+1}(\zeta-\beta)\cdot[a_{\ell+1}=0]\cdot\prod_{k=0}^{\ell}[a_{\ell-k}=k+1]+\\
&+(\beta\gamma+\varepsilon-\beta)\cdot[a_{\ell+1}=0]\cdot\Bigg(1+\sum_{k=0}^{\ell-1}\gamma^{k+1}\Big(\prod_{j=0}^{k}[a_{\ell-j}=j+1]\Big)\Bigg)
\end{aligned}$$

and this is (79) for $\ell+1$ instead of ℓ, since

$$\begin{aligned}
[a_{l+1}=0]\prod_{k=0}^{\ell}[a_{\ell-k}=k+1] &= [a_{l+1}=0]\cdot[a_l=1]\cdot[a_{l-1}=2]\cdots[a_0=\ell+1]\\
&=\prod_{k=0}^{\ell+1}[a_{\ell+1-k}=k]
\end{aligned}$$

and

$$\begin{aligned}
&[a_{l+1}=0]\cdot\Bigg(1+\sum_{k=0}^{\ell-1}\gamma^{k+1}\Big(\prod_{j=0}^{k}[a_{\ell-j}=j+1]\Big)\Bigg)=\\
&=[a_{l+1}=0]\big(1+\gamma[a_l=1]+\gamma^2[a_l=1][a_{l-1}=2]+\cdots+\gamma^\ell[a_l=1][a_{l-1}=2]\cdots[a_1=\ell]\big)\\
&=\gamma^0[a_{l+1}=0]+\gamma[a_{l+1}=0][a_l=1]+\gamma^2[a_{l+1}=0][a_l=1][a_{l-1}=2]+\cdots+\\
&\qquad+\gamma^\ell[a_{l+1}=0][a_l=1][a_{l-1}=2]\cdots[a_1=\ell]=\sum_{k=0}^{\ell}\gamma^k\Big(\prod_{j=0}^{k}[a_{\ell+1-j}=j]\Big).
\end{aligned}$$

By induction (79) is true and this proves Theorem 2. $\square$

All sequences **E1** ..... **E8** considered in Chapter 2 are special cases of (73). We shall now mention some other interesting special cases of (73).

1. Choosing $\alpha=0$, $\beta=1=\varepsilon$, $\gamma=1$, $\zeta=1$ we obtain the recurrence relation

$$d_c(1)=1,\quad d_c(2n)=1,\quad d_c(2n+1)=d_c(n)+1,\quad n\ge 1 \tag{82}$$

with the solution

$$d_c\big((b_m\dots b_0)_2\big)=\sum_{k=0}^{m}\Big(\prod_{j=0}^{k-1}b_j\Big) \tag{83}$$

Note that by (19) and (83) we have $d_c(n)=d_2(c(n))$, $n\ge 1$, where $c(n)$ is the one's complement of $n=(b_m\dots b_0)_2$.
The first few values of $d_c(n)$ (the sequence A091090 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_c(n)$	1	1	2	1	2	1	3	1	2	1	3	1	2	1	4	1

Table 13: $d_c(n)$, $1 \leq n \leq 16$

2. Choosing $\alpha = 1$, $\beta = 1 = \varepsilon$, $\gamma = 2$, $\zeta = 1$ we obtain the recurrence relation

$$f(1) = 1, \quad f(2n) = f(n) + 1, \quad f(2n+1) = 2f(n) + 1, \quad n \geq 1 \tag{84}$$

with the solution given by (74)

$$f\Big((b_m \dots b_0)_2\Big) = \sum_{k=0}^{m} \left(\prod_{j=0}^{k-1} (b_j + 1)\right) = \sum_{k=0}^{m} 2^{\sum_{k=0}^{k-1} b_j}, \tag{85}$$

since $b_j + 1 = 2^{b_j}$, $b_j \in \{0, 1\}$.
Alternatively, by (78)

$$f\Big(2^{a_0} + 2^{a_1} + \dots + 2^{a_\ell}\Big) = 2^\ell a_0 + 1 - \sum_{k=0}^{\ell-1} 2^k a_{\ell-k}. \tag{86}$$

The first few values of $f(n)$ (the sequence A135533 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	1	2	3	3	5	4	7	4	7	6	11	5	11	8	15	5

Table 14: $f(n)$, $1 \leq n \leq 16$

3. The special case $\alpha = \gamma$ leads by (74) to the solution $(n = (b_m \dots b_0)_2)$

$$f(n) = \zeta\alpha^m + \sum_{k=0}^{m-1} \alpha^k \Big(\varepsilon b_k + \beta(1 - b_k)\Big) = \zeta\alpha^m + (\varepsilon - \beta) \sum_{k=0}^{m-1} \alpha^k b_k + \beta \sum_{k=0}^{m-1} \alpha^k \tag{87}$$

This formula shows that the solution of (73) for $\alpha = \gamma$ and $\beta = \varepsilon$ depends only on $m = \lfloor \log_2 n \rfloor$. Perhaps, one of the most important of these cases is for $\alpha = 1 = \gamma$, $\beta = 1 = \varepsilon$ and $\zeta = 1$, that is the recurrence relation

$$B(1) = 1, \quad B(2n) = B(n) + 1, \quad B(2n+1) = B(n) + 1, \quad n \geq 1 \tag{88}$$

or, equivalently,

$$B(1) = 1, \quad B(n) = B\Big(\lfloor n/2 \rfloor\Big) + 1, \quad n \geq 2 \tag{89}$$

with the solution

$$B(n) = 1 + \lfloor \log_2 n \rfloor. \tag{90}$$

This sequence is, for $n \geq 1$, equal to the number of bits in the binary representation of n, that is $B(n) = s_2(n) + e_0(n)$, $n \in \mathbb{N}_0$, (see Examples **E2** and **E2'**). It gives also the number of comparisons in the worst case with *binary search* in an (ordered) list of size n (see [30, pp. 64–65]). Its generating function $\varphi(s)$ satisfies, by Proposition 1, the functional equation

$$\varphi(s) = (1+s)\varphi(s^2) + \frac{s}{1-s}$$

and it is given by

$$\varphi(s) = \frac{1}{1-s} \cdot \sum_{k=0}^{\infty} s^{2^k}, \tag{91}$$

since

$$\begin{aligned}(1+s)\varphi(s^2) + \frac{s}{1-s} &= \frac{1+s}{1-s^2} \cdot \sum_{k=0}^{\infty} s^{2^{k+1}} + \frac{s}{1-s} \\ &= \frac{1}{1-s}\Big(\sum_{k=0}^{\infty} s^{2^{k+1}} + s\Big) = \varphi(s).\end{aligned}$$

The generating function (91) is the Cauchy product of two power series, namely

$$\frac{1}{1-s} \cdot \sum_{k=0}^{\infty} s^{2^k} = \Big(\sum_{k=0}^{\infty} s^n\Big) \cdot \Big(\sum_{k=1}^{\infty} \rho(n) s^n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=1}^{n} \rho(k)\Big) s^n,$$

that is $B(n) = \sum_{k=1}^{n} \rho(k)$ is the sequence of the partial sums of $\rho(n)$, where $\rho(n)$ is the characteristic function of the powers of 2 (cf. Example **E3** at p. 8). Note also that $B(n) = T(n+1)$, $n \geq 0$, where $T(n)$ is the sequence **E10**.
The first few values of $\rho(n)$ (the sequence A209229 in [24]) and $B(n)$ (the sequence A029837 in [24]) are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\rho(n)$	-	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1
$B(n)$	0	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4	5

Table 15: $\rho(n), B(n)$, $0 \leq n \leq 16$

4 Solution of the general case $\delta \neq 0$

Let $\vec{f}(n) := \begin{pmatrix} f(n) \\ f(n+1) \end{pmatrix}$, then $\vec{f}(1) = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix} = \begin{pmatrix} \zeta \\ \alpha\zeta+\beta \end{pmatrix}$, $\vec{f}(2n) = \begin{pmatrix} f(2n) \\ f(2n+1) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \vec{f}(n) + \begin{pmatrix} \beta \\ \varepsilon \end{pmatrix}$ and $\vec{f}(2n+1) = \begin{pmatrix} f(2n+1) \\ f(2n+2) \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ 0 & \alpha \end{pmatrix} \vec{f}(n) + \begin{pmatrix} \varepsilon \\ \beta \end{pmatrix}$.
In this way Eq.(1) is equivalent to

$$\vec{f}(1) = \begin{pmatrix} \zeta \\ \alpha\zeta+\beta \end{pmatrix}, \quad \vec{f}(2n) = A \cdot \vec{f}(n) + \vec{b}, \quad \vec{f}(2n+1) = B \cdot \vec{f}(n) + C\vec{b} \tag{92}$$

where $A := \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$, $B := \begin{pmatrix} \gamma & \delta \\ 0 & \alpha \end{pmatrix}$, $C := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{b} := \begin{pmatrix} \beta \\ \varepsilon \end{pmatrix}$ and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{Z}$.
Before giving the general solution of (92) we shall determine the values of this sequence for the numbers $n = 2^m$, $2^m + 1$, $2^{m+1} - 1$.

Lemma 1. *Let $m \geq 0$, then*

$$f(2^m) = \zeta\alpha^m + \beta \sum_{k=0}^{m-1} \alpha^k \tag{93}$$

$$f(2^m+1) = \delta^m(\alpha\zeta+\beta) + \sum_{k=0}^{m-1} \delta^k \Big(\gamma\zeta\alpha^{m-1-k} + \beta\gamma \sum_{j=0}^{m-2-k} \alpha^j + \varepsilon\Big) \tag{94}$$

$$f(2^{m+1}-1) = \zeta\gamma^m + \sum_{k=0}^{m-1} \gamma^k \Big(\delta\zeta\alpha^{m-k} + \beta\delta \sum_{j=0}^{m-1-k} \alpha^j + \varepsilon\Big) \tag{95}$$

Proof. 1) Let $A(m) := f(2^m)$, $A(0) = f(1) = \zeta$, then by (1):
$A(m) = \alpha f(2^{m-1}) + \beta = \alpha A(m-1) + \beta$. This is an arithmetic progression with constant coefficients. Its solution is given by $A(m) = A(0) \cdot \alpha^m + \sum_{k=0}^{m-1} \alpha^k \beta$ and this proves (93).

2) Let $B(m) := f(2^m+1)$, $B(0) = f(2) = \alpha\zeta + \beta$, then by (1):
$B(m) = \gamma f(2^{m-1}) + \delta f(2^{m-1}+1) + \varepsilon = \gamma A(m-1) + \delta B(m-1) + \varepsilon$.
Inserting (93) into this equation we obtain the arithmetic progression
$B(m) = \delta B(m-1) + g(m-1)$, where $g(m-1) := \gamma \cdot \Big(\zeta\alpha^{m-1} + \beta \sum_{k=0}^{m-2} \alpha^k\Big) + \varepsilon$ with the solution $B(m) = B(0) \cdot \delta^m + \sum_{k=0}^{m-1} \delta^k g(m-1-k)$ and this proves (94).

3) Let $C(m) := f(2^{m+1}-1) = f(2^m + \cdots + 2^1 + 2^0), C(0) = f(1) = \zeta$, then by (1):
$C(m) = \gamma f(2^{m-1}+\cdots+2^0) + \delta f(2^{m-1}+\cdots+2^0+1) + \varepsilon = \gamma C(m-1) + \delta f(2^m) + \varepsilon = \gamma C(m-1) + \delta A(m) + \varepsilon$.
Inserting (93) into this equation we obtain the arithmetic progression
$C(m) = \gamma C(m-1) + h(m)$, where $h(m) := \delta \cdot \Big(\zeta\alpha^m + \beta \sum_{j=0}^{m-1} \alpha^j\Big) + \varepsilon$ with the solution $C(m) = C(0) \cdot \gamma^m + \sum_{k=0}^{m-1} \gamma^k g(m-k)$ and this proves (95). $\square$

We shall now determine the general solution of (92).

Theorem 3. *Let $n = (b_m \dots b_0)_2$, then the solution of (92) is given by*

$$\begin{aligned} \vec{f}\Big((b_m \dots b_0)_2\Big) &= M(b_0) \cdot M(b_1) \cdots M(b_{m-1}) \cdot \begin{pmatrix} \zeta \\ \alpha\zeta+\beta \end{pmatrix} + \\ &+ \Big(M(b_0) \cdots M(b_{m-2}) \cdot C^{b_{m-1}} + \cdots + M(b_0) \cdot C^{b_1} + C^{b_0}\Big) \cdot \begin{pmatrix} \beta \\ \varepsilon \end{pmatrix}, \end{aligned} \tag{96}$$

where $M(b_k) := A^{1-b_k} \cdot B^{b_k} = (1-b_k)A + b_k B$, $k \in [m]_0$, i.e., $M(0) = A$, $M(1) = B$ and $C^0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 unit matrix E_2.

Proof. Let $n = (b_m \dots b_0)_2$. Then by (92):
if $b_0 = 0$ we have $\vec{f}\Big((b_m \dots b_0)_2\Big) = A \cdot \vec{f}\Big((b_m \dots b_1)_2\Big) + \vec{b}$ and
if $b_0 = 1$ we have $\vec{f}\Big((b_m \dots b_0)_2\Big) = B \cdot \vec{f}\Big((b_m \dots b_1)_2\Big) + C \cdot \vec{b}$.
Hence,

$$\begin{aligned}\vec{f}\Big((b_m \dots b_0)_2\Big) &= (1-b_0)\cdot\Big(A\cdot\vec{f}\Big((b_m \dots b_1)_2\Big)+\vec{b}\Big)+b_0\cdot\Big(B\cdot\vec{f}\Big((b_m \dots b_1)_2\Big)+C\cdot\vec{b}\Big)\\ &= \Big((1-b_0)A+b_0B\Big)\cdot\vec{f}\Big((b_m \dots b_1)_2\Big)+\Big((1-b_0)\vec{b}+b_0\cdot C\vec{b}\Big)\\ &= M(b_0)\cdot\vec{f}\Big((b_m \dots b_1)_2\Big)+C^{b_0}\cdot\vec{b}.\end{aligned}$$

Similarly, $\vec{f}\Big((b_m \dots b_1)_2\Big) = M(b_1)\cdot\vec{f}\Big((b_m \dots b_2)_2\Big)+C^{b_1}\cdot\vec{b}$.
Substituting this expression into the above equation gives

$$\vec{f}\Big((b_m \dots b_0)_2\Big) = M(b_0)\cdot M(b_1)\cdot\vec{f}\Big((b_m \dots b_2)_2\Big)+\Big(M(b_0)\cdot C^{b_1}+C^{b_0}\Big)\cdot\vec{b}$$

By iteration and noting that $\vec{f}\Big((b_m)_2\Big)=\vec{f}(1)=\begin{pmatrix}\zeta\\ \alpha\zeta+\beta\end{pmatrix}$ one obtains formula (96) and this proves the theorem. □

We now apply formula (96) to the sequences **E9**, **E9'** and **E9"**.

1. For **E9** we have $\alpha=1=\gamma$, $\beta=0=\varepsilon$, $\delta=1$, $\zeta=1$. Hence $A=\begin{pmatrix}1&0\\1&1\end{pmatrix}$, $B=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $\vec{b}:=\begin{pmatrix}0\\0\end{pmatrix}$, $\vec{s}(1)=\begin{pmatrix}1\\1\end{pmatrix}$ and $M(b_k)=\begin{pmatrix}1&b_k\\1-b_k&1\end{pmatrix}$, $k\in[m]_0$. The solution is given by

$$\vec{s}\Big((b_m \dots b_0)_2\Big)=\begin{pmatrix}1&b_0\\1-b_0&1\end{pmatrix}\cdot\begin{pmatrix}1&b_1\\1-b_1&1\end{pmatrix}\cdots\begin{pmatrix}1&b_{m-1}\\1-b_{m-1}&1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} \quad (97)$$

For example, if $n=13=(1101)_2$, then

$$\vec{s}(13)=\begin{pmatrix}s(13)\\s(14)\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2&3\\1&2\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$$

that is $s(13)=5$ and $s(14)=3$.

Remark 3. We have $B=CAC$ and since $C^2=E_2$ it follows $BC=CA=\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}F_2&F_1\\F_1&F_0\end{pmatrix}=:Q$, where Q is the Q-matrix that generates the Fibonacci numbers F_n, since $Q^n=\begin{pmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{pmatrix}$.

2. For **E9'** we have $\alpha=1$, $\beta=0=\varepsilon,\gamma=-1$, $\delta=1$, $\zeta=1$. Hence $A=\begin{pmatrix}1&0\\-1&1\end{pmatrix}$, $B=\begin{pmatrix}-1&1\\0&1\end{pmatrix}$, $\vec{b}:=\begin{pmatrix}0\\0\end{pmatrix}$, $\vec{r}(1)=\begin{pmatrix}1\\1\end{pmatrix}$ and $M(b_k)=\begin{pmatrix}1-2b_k&b_k\\b_k-1&1\end{pmatrix}$, $k\in[m]_0$. The solution is given by

$$\vec{r}\Big((b_m \dots b_0)_2\Big)=\begin{pmatrix}1-2b_0&b_0\\b_0-1&1\end{pmatrix}\cdot\begin{pmatrix}1-2b_1&b_1\\b_1-1&1\end{pmatrix}\cdots\begin{pmatrix}1-2b_{m-1}&b_{m-1}\\b_{m-1}-1&1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} \quad (98)$$

For example, if $n = 13 = (1101)_2$, then

$$\begin{aligned}\vec{r}(13) = \begin{pmatrix} r(13) \\ r(14) \end{pmatrix} &= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}\end{aligned}$$

that is $r(13) = 1$ and $r(14) = 1$.

3. For **E9”** we have $\alpha = -1 = \gamma$, $\beta = 0 = \varepsilon$, $\delta = -1$, $\zeta = 1$. Hence $A = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$, $\vec{b} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\vec{t}(1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $M(b_k) = \begin{pmatrix} -1 & -b_k \\ b_k - 1 & -1 \end{pmatrix}$, $k \in [m]_0$. The solution is given by

$$\vec{t}\big((b_m \dots b_0)_2\big) = \begin{pmatrix} -1 & -b_0 \\ b_0 - 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & -b_1 \\ b_1 - 1 & -1 \end{pmatrix} \cdots \begin{pmatrix} -1 & -b_{m-1} \\ b_{m-1} - 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{99}$$

For example, if $n = 13 = (1101)_2$, then

$$\begin{aligned}\vec{t}(13) = \begin{pmatrix} t(13) \\ t(14) \end{pmatrix} &= \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -3 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\end{aligned}$$

that is $t(13) = 1$ and $t(14) = 1$.

For $\beta = 0 = \varepsilon$ and $\alpha \neq 0$ the sequence (92) seems to satisfy a second-order linear recurrence relation with variable coefficients. We conjecture the following statement.

Conjecture 1. *Let $\beta = 0 = \varepsilon$, $\alpha \neq 0$ in (92) and $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$, be a strictly decreasing sequence of positive integers. Then*

$$f(2^{a_0}) = \zeta \cdot \alpha^{a_0}, \quad f(2^{a_0} + 2^{a_1}) = \zeta \cdot \alpha^{a_0} \left(\alpha \Big(\frac{\delta}{\alpha}\Big)^{a_0 - a_1} + \frac{\gamma}{\alpha} \sum_{k=0}^{a_0 - a_1 - 1} \Big(\frac{\delta}{\alpha}\Big)^k \right)$$
$$f(2^{a_0} + \cdots + 2^{a_\ell}) = A \cdot f(2^{a_0} + \cdots + 2^{a_{\ell-1}}) + B \cdot f(2^{a_0} + \cdots + 2^{a_{\ell-2}}), \quad \ell \geq 2 \tag{100}$$

where $A := \left(\frac{\delta}{\alpha}\right)^{a_{\ell-1} - a_\ell} + \frac{\gamma}{\alpha} \cdot \sum_{k=0}^{a_{\ell-1} - a_\ell - 1} \left(\frac{\delta}{\alpha}\right)^k$ and $B := (\alpha - \delta - \frac{\gamma}{\alpha}) \cdot \left(\frac{\delta}{\alpha}\right)^{a_{\ell-2} - a_\ell - 1}$.

Indeed, for the cases discussed in Section 2, namely **E9** (Stern's diatomic sequence $s(n)$), **E9'** (Reznick sequence $r(n)$), and **E9”** (twisted Stern sequence $t(n)$) we can prove three special cases of this Conjecture.

Theorem 4. *Let $\beta = 0 = \varepsilon$ in (92) and $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_\ell}$, $a_0 > a_1 > \cdots > a_\ell \geq 0$, $\ell \geq 0$, be a strictly decreasing sequence of positive integers. Then*

1. *For $\alpha = 1$, $\gamma = 1$, $\delta = 1$ we have for $\ell \geq 2$*

$$\begin{gathered} s(2^{a_0}) = 1, \quad s(2^{a_0} + 2^{a_1}) = a_0 - a_1 + 1, \\ s(2^{a_0} + \cdots + 2^{a_\ell}) = (a_{\ell-1} - a_\ell + 1) \cdot s(2^{a_0} + \cdots + 2^{a_{\ell-1}}) - s(2^{a_0} + \cdots + 2^{a_{\ell-2}}) \end{gathered} \tag{101}$$

2. *For $\alpha = 1$, $\gamma = -1$, $\delta = 1$ we have for $\ell \geq 2$*

$$\begin{gathered} r(2^{a_0}) = 1, \quad r(2^{a_0} + 2^{a_1}) = -a_0 + a_1 + 1, \\ r(2^{a_0} + \cdots + 2^{a_\ell}) = (-a_{\ell-1} + a_\ell + 1) \cdot r(2^{a_0} + \cdots + 2^{a_{\ell-1}}) + r(2^{a_0} + \cdots + 2^{a_{\ell-2}}) \end{gathered} \tag{102}$$

3. *For $\alpha = -1$, $\gamma = -1$, $\delta = -1$ we have for $\ell \geq 2$*

$$\begin{gathered} t(2^{a_0}) = (-1)^{a_0}, \quad t(2^{a_0} + 2^{a_1}) = (-1)^{a_0}(a_0 - a_1 - 1), \\ t(2^{a_0} + \cdots + 2^{a_\ell}) = (a_{\ell-1} - a_\ell + 1) \cdot t(2^{a_0} + \cdots + 2^{a_{\ell-1}}) - t(2^{a_0} + \cdots + 2^{a_{\ell-2}}) \end{gathered} \tag{103}$$

Proof. By induction on $\ell \in \mathbb{N}_0$.
1) Case $\ell = 0$: the first initial value of (101) follows from (93) for $m = a_0$ and $\zeta = 1$.
2) Case $\ell = 1$: Since $s(2^{a_0} + 2^{a_1}) = s\Big(2^{a_1} \cdot (2^{a_0 - a_1} + 1)\Big) = s(2^{a_0 - a_1} + 1)$, we obtain from (94) for $m = a_0 - a_1$ and $\zeta = 1$ the second initial value of (101).
Let now $\ell \geq 1$ and $A_{\ell+1} := s(2^{a_0} + \cdots + 2^{a_{\ell+1}})$, then by definition
$A_{\ell+1} = s\Big(2^{a_{\ell+1}} \cdot (2^{a_0 - a_{\ell+1}} + \cdots + 2^{a_\ell - a_{\ell+1}} + 1)\Big) = s(2^{a_0 - a_{\ell+1}} + \cdots + 2^{a_\ell - a_{\ell+1}} + 1)$.
Let $b_i := a_i - a_{\ell+1}$, $i \in [\ell + 2]_0$, then $b_0 > b_1 > \cdots > b_\ell > b_{\ell+1} = 0$. Hence, by definition

$$\begin{aligned} A_{\ell+1} &= s(2^{b_0} + \cdots + 2^{b_\ell} + 1) = s(2 \cdot (2^{b_0 - 1} + \cdots + 2^{b_\ell - 1}) + 1) \\ &= s(2^{b_0 - 1} + \cdots + 2^{b_\ell - 1}) + s(2^{b_0 - 1} + \cdots + 2^{b_\ell - 1} + 1) \\ &= s(2^{b_0} + \cdots + 2^{b_\ell}) + s(2^{b_0 - 1} + \cdots + 2^{b_\ell - 1} + 1). \end{aligned}$$

Repeating this procedure b_ℓ times we obtain by induction

$$\begin{aligned} A_{\ell+1} &= b_\ell \cdot s(2^{b_0} + \cdots + 2^{b_\ell}) + s(2^{b_0 - b_\ell} + \cdots + 2^{b_{\ell-1} - b_\ell} + 2^{b_\ell - b_\ell} + 1) \\ &= b_\ell \cdot s(2^{b_0} + \cdots + 2^{b_\ell}) + s(2^{b_0 - b_\ell - 1} + \cdots + 2^{b_{\ell-1} - b_\ell - 1} + 1) \\ &= (a_\ell - a_{\ell-1}) \cdot s(2^{a_0 - a_{\ell+1}} + \cdots + 2^{a_\ell - a_{\ell+1}}) + s(2^{a_0 - a_\ell - 1} + \cdots + 2^{a_{\ell-1} - a_\ell - 1} + 1) \end{aligned}$$

and this is by definition and since $b_i - b_\ell - 1 = a_i - a_{\ell+1} - (a_\ell - a_{\ell+1}) - 1 = a_i - a_\ell - 1$ equal to

$$A_{\ell+1} = (a_\ell - a_{\ell-1}) \cdot A_\ell + s(2^{a_0 - a_\ell - 1} + \cdots + 2^{a_{\ell-1} - a_\ell - 1} + 1) \tag{104}$$

We distinguish now two cases: 1.) $a_{\ell-1} - a_\ell - 1 > 0$ and 2.) $a_{\ell-1} - a_\ell - 1 = 0$.
First case: $a_{\ell-1} - a_\ell - 1 > 0$.
Applying by assumption the assertion to the sequence of $\ell + 1$ integers $a_0 - a_\ell - 1 >$

$a_1-a_\ell-1 > \ldots > a_{\ell-1}-a_\ell-1 > 0$ and by definition we obtain for $c_i := a_i-a_\ell-1$, $i \in [\ell]_0$, $c_\ell := 0$,

$$\begin{aligned} s(2^{c_0}+\cdots+2^{c_{\ell-1}}+1) &= (c_{\ell-1}-c_\ell+1)\cdot s(2^{c_0}+\cdots+2^{c_{\ell-1}}) - s(2^{c_0}+\cdots+2^{c_{\ell-2}}) \\ &= (a_{\ell-1}-a_\ell)\cdot s(2^{a_0}+\cdots+2^{a_{\ell-1}}) - s(2^{a_0}+\cdots+2^{a_{\ell-2}}) \\ &= (a_{\ell-1}-a_\ell)\cdot A_{\ell-1} - A_{\ell-2}, \end{aligned}$$

since $s(2^{c_0}+\cdots+2^{c_{\ell-1}}) = s(2^{a_0-a_\ell-1}+\cdots+2^{a_{\ell-1}-a_\ell-1}) = s(2^{a_0}+\cdots+2^{a_{\ell-1}})$. Inserting the above expression into (104) we obtain

$$\begin{aligned} A_{\ell+1} &= (a_\ell-a_{\ell-1})\cdot A_\ell + (a_{\ell-1}-a_\ell)\cdot A_{\ell-1} - A_{\ell-2} \\ &= (a_\ell-a_{\ell-1}+1-1)\cdot A_\ell + (a_{\ell-1}-a_\ell+1-1)\cdot A_{\ell-1} - A_{\ell-2} \\ &= (a_\ell-a_{\ell-1}+1)\cdot A_\ell - A_\ell + (a_{\ell-1}-a_\ell+1)\cdot A_{\ell-1} - A_{\ell-1} - A_{\ell-2} \\ &= (a_\ell-a_{\ell-1}+1)\cdot A_\ell - A_{\ell-1} \end{aligned}$$

and this is the assertion for $\ell+1$ instead of ℓ, since by assumption $-A_\ell + (a_{\ell-1} - a_\ell+1)\cdot A_{\ell-1} - A_{\ell-2} = 0$.
Second case: $a_{\ell-1}-a_\ell-1 = 0$.
In this case the sequence $c_i := a_i-a_\ell-1$, $i \in [\ell]_0$, $c_\ell := 0$ is no longer strictly decreasing, since $c_{\ell-1}-c_\ell = a_{\ell-1}-a_\ell-1 = 0$. We define $n := 2^{c_0}+\cdots+2^{c_{\ell-1}}$ and using the rule $s(n+1) = s(2n+1)-s(n)$ we obtain $s(2^{c_0}+\cdots+2^{c_{\ell-1}}+1) = s(2^{c_0+1}+\cdots+2^{c_{\ell-1}+1}+1) - s(2^{c_0}+\cdots+2^{c_{\ell-1}})$ or

$$\begin{aligned} s(2^{c_0}+\cdots+2^{c_{\ell-1}}+1) &= s(2^{a_0-a_\ell}+\cdots+2^{a_{\ell-1}-a_\ell}+1) - s(2^{a_0-a_\ell-1}+\cdots+2^{a_{\ell-1}-a_\ell-1}) \\ &= s(2^{a_0}+\cdots+2^{a_\ell}) - s(2^{a_0}+\cdots+2^{a_{\ell-1}}) \\ &= A_\ell - A_{\ell-1}. \end{aligned}$$

Inserting this expression into (104) we get

$$\begin{aligned} A_{\ell+1} &= (a_\ell-a_{\ell-1})\cdot s(2^{a_0}+\cdots+2^{a_\ell}) + s(2^{a_0}+\cdots+2^{a_\ell}) - s(2^{a_0}+\cdots+2^{a_{\ell-1}}) \\ &= (a_\ell-a_{\ell-1}+1)\cdot A_\ell - A_{\ell-1} \end{aligned}$$

and this is again the assertion for $\ell+1$ instead of ℓ. This proves (101).
The proofs of (102) and (103) are similar and will be omitted. $\square$
By means of Theorem 4 (and using the Euler-Wallis relations) the solution of Stern's diatomic sequence can be given by the numerator of the finite continued fraction

$$\frac{A_\ell}{B_\ell} = a_0-a_1+1 - \frac{1|}{|a_1-a_2+1} - \cdots - \frac{1|}{|a_{\ell-1}-a_\ell+1}, \quad \text{for} \quad \ell > 0$$

and $s(2^{a_0}) = 1$ for $\ell = 0$, that is $s(n) = A_\ell$ for $\ell > 0$ and $s(2^{a_0}) = 1$ for $\ell = 0$.

We recall that $b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_{n-1}|}{|b_{n-1}} + \frac{a_n|}{|b_n}$ is the notation of Pringsheim of a finite generalized continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots \cfrac{a_{n-1}}{b_{n-1} + \cfrac{a_n}{b_n}}}}$$

If some or all a_k, $k \in [n]$, are provided with minus signs, then these are also written in front of the a_k instead of the plus signs; thus instead of $b_0 + \frac{-a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{-a_n|}{|b_n}$ we will write $b_0 - \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots - \frac{a_n|}{|b_n}$.

If $a_k = 1$ and $b_k \in \mathbb{N}$ for all $k \in [n]$, then the continued fraction is said to be *simple*.

Remark 4. Other equivalent solutions of Stern's diatomic sequence have been given in [29, pp. 95–96], in [13, Exercise 6.50 at p. 300 and its solution at p. 531] and in [20] by different representations of n.

Similarly, the solution of Reznick's sequence can be given by the numerator of the finite simple continued fraction

$$\frac{A_\ell}{B_\ell} = -a_0 + a_1 + 1 + \frac{1|}{|-a_1 + a_2 + 1} + \cdots + \frac{1|}{|-a_{\ell-1} + a_\ell + 1}, \quad \text{for} \quad \ell > 0$$

and $r(2^{a_0}) = 1$ for $\ell = 0$, that is $r(n) = A_\ell$ for $\ell > 0$ and $r(2^{a_0}) = 1$ for $\ell = 0$.

Finally, the solution of the twisted Stern sequence can be given by the numerator of the finite continued fraction

$$\frac{A_\ell}{B_\ell} = (-1)^{a_0} \cdot \left(a_0 - a_1 - 1 - \frac{1|}{|a_1 - a_2 + 1} - \cdots - \frac{1|}{|a_{\ell-1} - a_\ell + 1} \right), \quad \text{for} \quad \ell > 0$$

and $t(2^{a_0}) = (-1)^{a_0}$ for $\ell = 0$, that is $t(n) = A_\ell$ for $\ell > 0$ and $t(2^{a_0}) = (-1)^{a_0}$ for $\ell = 0$.

5 Generalization

An obvious generalization of (1) is given by

$$f(1) = \zeta, \quad f(2n) = \alpha f(n) + g(n), \quad f(2n+1) = \gamma f(n) + \delta f(n+1) + h(n), \quad n \geq 2, \tag{105}$$

with $\alpha, \gamma, \delta, \zeta \in \mathbb{Z}$ and $g, h : \mathbb{N} \longrightarrow \mathbb{Z}$ two arbitrary integer functions.

5.1 The case $\delta = 0$

At first we shall give the general solution of (105) for the case $\delta = 0$.

Theorem 5. *The solution of (105) for $\delta = 0$ and for $n = (b_m \ldots b_0)_2$ is given by*

$$\begin{aligned} f(n) = \zeta \cdot \prod_{k=0}^{m-1} \Big(\gamma b_k + \alpha(1 - b_k)\Big) + \\ + \sum_{k=0}^{m-1} \left(\prod_{j=0}^{k-1} \Big(\gamma b_j + \alpha(1 - b_j)\Big) \right) \cdot \Big(b_k \cdot h\big((b_m \ldots b_{k+1})_2\big) + (1 - b_k) \cdot g\big((b_m \ldots b_{k+1})_2\big) \Big) \end{aligned} \tag{106}$$

Proof. Let $n = (b_m \dots b_0)_2$. Then by (105):
if $b_0 = 0$ we have $f\Big((b_m \dots b_0)_2\Big) = \alpha f\Big((b_m \dots b_1)_2\Big) + g\Big((b_m \dots b_1)_2\Big)$ and
if $b_0 = 1$ we have $f\Big((b_m \dots b_0)_2\Big) = \gamma f\Big((b_m \dots b_1)_2\Big) + h\Big((b_m \dots b_1)_2\Big)$.
Hence,

$$\begin{aligned} f\Big((b_m \dots b_0)_2\Big) &= b_0 \cdot \Big(\gamma f\Big((b_m \dots b_1)_2\Big) + h\Big((b_m \dots b_1)_2\Big)\Big) + \\ &+ (1-b_0) \cdot \Big(\alpha f\Big((b_m \dots b_1)_2\Big) + g\Big((b_m \dots b_1)_2\Big)\Big) \\ &= \Big(\gamma b_0 + \alpha(1-b_0)\Big) \cdot f\Big((b_m \dots b_1)_2\Big) + \Big(b_0 \cdot h\Big((b_m \dots b_1)_2\Big) + \\ &+ (1-b_0) \cdot g\Big((b_m \dots b_1)_2\Big)\Big). \end{aligned}$$

Similarly, $f\Big((b_m \dots b_1)_2\Big) = \Big(\gamma b_1 + \alpha(1-b_1)\Big) \cdot f\Big((b_m \dots b_2)_2\Big) + \Big(b_1 \cdot h\Big((b_m \dots b_2)_2\Big) + (1-b_1) \cdot g\Big((b_m \dots b_2)_2\Big)\Big)$.
Substituting this expression into the above equation gives

$$\begin{aligned} f\Big((b_m \dots b_0)_2\Big) &= \Big(\gamma b_0 + \alpha(1-b_0)\Big) \cdot \Big(\gamma b_1 + \alpha(1-b_1)\Big) \cdot f\Big((b_m \dots b_2)_2\Big) + \\ &+ \Big(\gamma b_0 + \alpha(1-b_0)\Big) \cdot \Big(b_1 \cdot h\Big((b_m \dots b_2)_2\Big) + (1-b_1) \cdot g\Big((b_m \dots b_2)_2\Big)\Big) + \\ &+ \Big(b_0 \cdot h\Big((b_m \dots b_1)_2\Big) + (1-b_0) \cdot g\Big((b_m \dots b_1)_2\Big)\Big) \end{aligned}$$

By iteration and noting that $f(b_m) = f(1) = \zeta$ one obtains formula (106) and this proves the theorem. ☐

An interesting special case of (105) is for $\gamma = \alpha$ and $g(n) := 2n$, $h(n) := 2n+1$, that is the recurrence relation

$$f(1) = \zeta, \quad f(2n) = \alpha f(n) + 2n, \quad f(2n+1) = \alpha f(n) + 2n + 1, \quad n \geq 1 \tag{107}$$

or, equivalently,

$$f(1) = \zeta, \quad f(n) = \alpha f\Big(\lfloor n/2 \rfloor\Big) + n, \quad n \geq 2 \tag{108}$$

Proposition 3. *The solution of (107) for $n = (b_m \dots b_0)_2$ is given by*

$$f\Big((b_m \dots b_0)_2\Big) = \begin{cases} (\zeta - 1) \cdot 2^m + \sum_{k=1}^m k \cdot b_k \cdot 2^k + (b_m \dots b_0)_2, & \text{if } \alpha = 2 \\ (\zeta - 1) \cdot \alpha^m + \frac{\alpha}{\alpha-2}(b_m \dots b_0)_\alpha - \frac{2}{\alpha-2}(b_m \dots b_0)_2, & \text{if } \alpha \neq 2 \end{cases} \tag{109}$$

where $(b_m \dots b_0)_\alpha := \sum_{k=0}^m b_k \cdot \alpha^k$.

Proof. By (106) with

$$h\Big((b_m \dots b_{k+1})_2\Big) = 2 \cdot (b_m \dots b_{k+1})_2 + 1 = (b_m \dots b_{k+1}1)_2,$$

$$g\Big((b_m \dots b_{k+1})_2\Big) = 2 \cdot (b_m \dots b_{k+1})_2 = (b_m \dots b_{k+1}0)_2$$

and

$$b_k \cdot (b_m \dots b_{k+1})_2 + (1-b_k) \cdot (b_m \dots b_{k+1})_2 = (b_m \dots b_{k+1} b_k)_2$$

we obtain

$$f\Big((b_m \dots b_0)_2\Big) = \zeta \cdot \alpha^m + \sum_{k=0}^{m-1} \alpha^k \cdot (b_m \dots b_{k+1} b_k)_2.$$

Since $(b_m \dots b_{k-1} b_k)_2 = \sum_{j=k}^{m} b_j 2^{j-k}$ it follows

$$\begin{aligned}
\sum_{k=0}^{m-1} \alpha^k \cdot (b_m \dots b_{k+1} b_k)_2 &= \sum_{k=0}^{m-1} \alpha^k \Big(\sum_{j=k}^{m} b_j 2^{j-k} \Big) \\
&= \alpha^0 \cdot (b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{m-1} 2^{m-1} + b_m 2^m) + \\
&+ \quad \alpha^1 \cdot (\qquad b_1 2^0 + b_2 2^1 + \dots + b_{m-1} 2^{m-2} + b_m 2^{m-1}) + \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
&+ \quad \alpha^{m-1} \cdot (\qquad\qquad\qquad\qquad b_{m-1} 2^0 + b_m 2^1) \\
&= b_0(\alpha^0 2^0) + b_1(\alpha^0 2^1 + \alpha^1 2^0) + b_2(\alpha^0 2^1 + \alpha^1 2^1 + \alpha^2 2^0) + \\
&+ \dots + b_m(\alpha^0 2^m + \alpha^1 2^{m-1} + \dots + \alpha^{m-1} 2^1 + \alpha^m 2^0 - \alpha^m 2^0) \\
&= \sum_{k=0}^{m} b_k \cdot 2^k \cdot \Big(\sum_{j=0}^{k} (\alpha/2)^j \Big) - b_m \alpha^m
\end{aligned}$$

Substituting the value of the finite geometric series

$$\sum_{j=0}^{k} (\alpha/2)^j = \begin{cases} k+1, & \text{if } \alpha = 2 \\ \frac{1}{2^k} \frac{\alpha^{k+1} - 2^{k+1}}{\alpha - 2}, & \text{if } \alpha \neq 2 \end{cases}$$

into the last formula and simplifying one obtains the formula (109) and this proves the proposition. □

An important example of (107) is for $\alpha = 1, \zeta = 1$. Its solution $f(n) = 2n - \sum_{k=0}^{n} b_k = 2n - s_2(n)$ shows, that this is the sequence $\big(D_2(n)\big)_{n \in \mathbb{N}}$ defined in (26). Besides, it is not difficult to show directly that by (16) this sequence satisfies the recurrence relation

$$D_2(1) = 1, \quad D_2(2n) = D_2(n) + 2n, \quad D_2(2n+1) = D_2(n) + 2n + 1, \quad n \geq 1 \tag{110}$$

We now derive some properties of this sequence.

Proposition 4. *1. The solution of (110) is given by*

$$D_2(n) = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \lfloor n/2^k \rfloor, \tag{111}$$

that is, $D_2(n)$ is the number of positive powers of 2 which divide numbers $\leq n$.

2. The generating function $\varphi(s) := \sum_{n=1}^{\infty} D_2(n) s^n$ satisfies the functional equation

$$\varphi(s) = (1+s)\varphi(s^2) + \frac{s}{(1-s)^2}, \tag{112}$$

which shows that $D_2(n)$ is of the DC type.

3. The solution of (112) is given either by

$$\varphi(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^k}} \tag{113}$$

or by the Lambert series

$$\varphi(s) = \sum_{n=1}^{\infty} a_n \frac{s^n}{1-s^n} \tag{114}$$

where $a_n := 2\phi(n) - \sum_{t|n} \mu(n/t) s_2(t)$*,* $\mu(n)$ *is the Möbius function and* $\phi(n)$ *is Euler's totient function.*

Proof. By (23) we have $d_2(n) = \sum_{t|n} \rho(t)$, where $\rho(n) = 1$, if n is a power of 2 and zero otherwise. Hence,

$$D_2(n) = \sum_{k=1}^{n} d_2(k) = \sum_{k=1}^{n} \Big(\sum_{t|k} \rho(t) \Big) \tag{115}$$

We collect terms with equal values of $\rho(t)$ in the above double sum. For every $j \le n$, $\rho(j)$ appears in $\sum_{t|k} \rho(t)$ if and only if $j \mid k$. Since each integer has itself as a divisor, the right-hand side of (115) includes $\rho(j)$ at least once. Furthermore, there are exactly $\lfloor n/j \rfloor$ integers among $1, 2, \ldots, n$ which are divisible by j, namely: $j, 2j, 3j, \ldots, \lfloor n/j \rfloor j$. Hence, $\sum_{k=1}^{n} \Big(\sum_{t|k} \rho(t) \Big) = \sum_{j=1}^{n} \rho(j) \cdot \lfloor n/j \rfloor = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \lfloor n/2^k \rfloor$ and this proves (111) (see [8, Theorem 6-11, pp. 119–120]).

Let $\varphi(s) := \sum_{n=1}^{\infty} D_2(n) s^n$, then by (110)

$$\begin{aligned}
\varphi(s) &= \sum_{n=1}^{\infty} D_2(2n) s^{2n} + \sum_{n=0}^{\infty} D_2(2n+1) s^{2n+1} \\
&= \sum_{n=1}^{\infty} D_2(n) s^{2n} + \sum_{n=1}^{\infty} (2n) s^{2n} + s \sum_{n=1}^{\infty} D_2(n) s^{2n} + \sum_{n=0}^{\infty} (2n+1) s^{2n+1} \\
&= (1+s)\varphi(s^2) + \sum_{n=1}^{\infty} (2n) s^{2n} + \sum_{n=0}^{\infty} (2n+1) s^{2n+1}
\end{aligned}$$

and this is (112), since $\sum_{n=1}^{\infty} (2n) s^{2n} + \sum_{n=0}^{\infty} (2n+1) s^{2n+1} = \sum_{n=1}^{\infty} n s^n = \frac{s}{(1-s)^2}$.

It is well-known that the generating function for the sequence of the partial sums of a sequence $a(n)$ is equal to the generating function of $a(n)$ divided by $1-s$. Hence,

$$\sum_{n=1}^{\infty} D_2(n) s^n = \frac{1}{1-s} \sum_{n=1}^{\infty} d_2(n) s^n \tag{116}$$

Substituting (17) into this equation gives immediately (113).
By (22) the generating function for $d_2(n)$ is given by a Lambert series. Therefore, it is reasonable to assume that the generating function of $D_2(n)$ is given by a Lambert series, too. Let $\varphi(s) = \sum_{n=1}^{\infty} a_n \frac{s^n}{1-s^n}$, where (a_n) is an unknown sequence to be

determined. Since $\frac{s^n}{1-s^n} = s^n(1 + s^n + s^{2n} + \cdots) = s^n + s^{2n} + s^{3n} + \cdots = \sum_{m=1}^{\infty} s^{m\cdot n}$ we obtain $\sum_{n=1}^{\infty} a_n \frac{s^n}{1-s^n} = \sum_{n=1}^{\infty} a_n \Big(\sum_{m=1}^{\infty} s^{m\cdot n} \Big)$. Summing by rows we obtain

$$\sum_{n=1}^{\infty} a_n \frac{s^n}{1-s^n} = \sum_{n=1}^{\infty} \Big(\sum_{t|n} a_t \Big) s^n,$$

that is $\sum_{t|n} a_t = D_2(n)$. By the Möbius Inversion Formula (see [8, Theorem 6-7, pp. 113–114]) it follows $a_n = \sum_{t|n} \mu(n/t) \cdot D_2(t)$, where

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = \prod_{j=1}^{k} p_j, \quad p_1, \ldots, p_k \text{ distinct primes} \\ 0, & \text{otherwise} \end{cases}$$

is the Möbius function. Since by (25) $D_2(t) = 2t - s_2(t)$ we obtain

$$a_n = 2 \sum_{t|n} \mu(n/t) t - \sum_{t|n} \mu(n/t) s_2(t) = 2\phi(n) - \sum_{t|n} \mu(n/t) s_2(t)$$

(see [8, Theorem 7-8, pp. 138–139]), where $\phi(n)$ is Euler's totient function giving the number of positive integers not exceeding n that are relative prime to n, and this proves (114). □

The first few values of $D_2(n)$ (the sequence A005187 in [24]) and a_n (the sequence A035532 in [24]) are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$D_2(n)$	1	3	4	7	8	10	11	15	16	18	19	22	23	25	26	31
a_n	1	2	3	4	7	4	10	8	12	8	18	8	22	12	15	16

Table 16: $D_2(n)$, a_n, $1 \le n \le 16$

Note that for all primes p we have

$$D_2(p) - a_p = 1, \tag{117}$$

since $a_p = 2\phi(p) - \sum_{t|p} \mu(p/t) s_2(t) = 2(p-1) - \mu(p) s_2(1) - \mu(1) s_2(p) = 2p - 2 - (-1) \cdot 1 - 1 \cdot s_2(p) = D_2(p) - 1$.

The next proposition lists some more properties of the sequences $(d_2(n))$, $(s_2(n))$ and $(D_2(n))$.

Proposition 5. *Let $n \in \mathbb{N}$, then*

1. *(Highest power of 2 that divides $n!$)*

$$D_2(n) - n = n - s_2(n) = \sum_{k=1}^{\lfloor \log_2 n \rfloor} \lfloor n/2^k \rfloor \tag{118}$$

is the highest power of 2 dividing $n!$, that is

$$n! = 2^{n - s_2(n)} \cdot (2k+1), \text{ for a } k \in \mathbb{N}_0 \tag{119}$$

2. (Asymptotic behavior)

$$\lim_{n\to\infty} \frac{D_2(n)}{n} = 2, \tag{120}$$

consequently,

$$\lim_{n\to\infty} \frac{s_2(n)}{n} = 0 \tag{121}$$

3. (Another representation of the generating function of $s_2(n)$)

$$\sum_{n=1}^{\infty} s_2(n)s^n = \frac{2s}{(1-s)^2} - \frac{1}{1-s}\sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^k}} \tag{122}$$

4. (Another representation of the generating function of $d_2(n)$)

$$\sum_{n=1}^{\infty} d_2(n)s^n = \frac{2s}{1-s} - \sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}} \tag{123}$$

5. (Another representation of the generating function of $D_2(n)$)

$$\sum_{n=1}^{\infty} D_2(n)s^n = \frac{2s}{(1-s)^2} - \frac{1}{1-s}\sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}} \tag{124}$$

Proof. By (2) and (111) one obtains the formula (118). The right-hand side of this (double) equation gives the exponent of the highest power of 2 that divides $n!$, since there are $\lfloor n/2 \rfloor$ numbers (namely, $2, 2\cdot 2, 3\cdot 2, \ldots, \lfloor n/2\rfloor \cdot 2$) which are divisible by 2, $\lfloor n/2^2 \rfloor$ numbers which are divisible by 2^2, etc. This proves (119).
By definition it holds that $x - 1 < \lfloor x \rfloor \leq x$ for all $x \in \mathbb{R}$. Choosing $x := n/2^k$ we obtain $n/2^k - 1 < \lfloor n/2^k \rfloor \leq 2/n^k$. Summing up for $k = 0, 1, \ldots, \lfloor \log_2 n \rfloor$ it follows

$$\sum_{k=0}^{\lfloor \log_2 n \rfloor} \left(n/2^k - 1\right) < \sum_{k=0}^{\lfloor \log_2 n \rfloor} \lfloor n/2^k \rfloor \leq \sum_{k=0}^{\lfloor \log_2 n \rfloor} n/2^k$$

or by (111) and using the formula for finite geometric series we obtain

$$2n\left(1 - 1/2^{\lfloor \log_2 n \rfloor + 1}\right) - (\lfloor \log_2 n \rfloor + 1) < D_2(n) \leq 2n\left(1 - 1/2^{\lfloor \log_2 n \rfloor + 1}\right)$$

or dividing by n

$$2\left(1 - \frac{1}{2^{\lfloor \log_2 n \rfloor + 1}}\right) - \frac{\lfloor \log_2 n \rfloor + 1}{n} < \frac{D_2(n)}{n} \leq 2\left(1 - \frac{1}{2^{\lfloor \log_2 n \rfloor + 1}}\right)$$

Assertion (120) follows by noting that $1/2^{\lfloor \log_2 n \rfloor + 1} \to 0$ and $(\lfloor \log_2 n \rfloor + 1)/n \to 0$ for $n \to \infty$.
Since $s_2(n) = 2n - D_2(n)$ we conclude by (120) that $\frac{s_2(n)}{n} = 2 - \frac{D_2(n)}{n} \to 0$ for $n \to \infty$ and this proves (121).
By (25) we have

$$\sum_{n=1}^{\infty} s_2(n)s^n = \sum_{n=1}^{\infty} (2n - D_2(n))s^n = 2\sum_{n=1}^{\infty} ns^n - \sum_{n=1}^{\infty} D_2(n)s^n$$

and this is formula (122) by (118) and noting that $\sum_{n=1}^{\infty} ns^n = \frac{s}{(1-s)^2}$.
By (24) and noting that the generating function for a difference sequence $\Delta a(n)$ is equal to $1-s$ times the generating function for the sequence $a(n)$ we have

$$\begin{aligned}\sum_{n=1}^{\infty} d_2(n)s^n &= \sum_{n=1}^{\infty} \Big(2 - \Delta s_2(n)\Big)s^n \\ &= 2\sum_{n=1}^{\infty} s^n - \sum_{n=1}^{\infty} \Delta s_2(n)s^n \\ &= \frac{2s}{1-s} - (1-s)\sum_{n=1}^{\infty} s_2(n)s^n\end{aligned}$$

and this is formula (123) by (11) and noting that $\sum_{n=1}^{\infty} s^n = \frac{s}{1-s}$.
Substituting (123) into (116) gives immediately the formula (124). □
Note that formula (118) has been obtained for the first time in 1808 by A. M. Legendre in the form (cf. Example **E3** at p. 7)

$$\sum_{k=1}^{n} \tilde{d}_2(k) = \sum_{k=1}^{n} \Big(d_2(k) - 1\Big) = D_2(n) - n = n - s_2(n).$$

By equating the two representations (11) and (122) of the generating function for $s_2(n)$ and simplifying one obtains the identities

Corollary 2.

$$\sum_{k=0}^{\infty} \frac{s^{2^k}}{1+s^{2^k}} = \frac{2s}{1-s} - \sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^k}} \tag{125}$$

$$\sum_{k=0}^{\infty} \frac{s^{2^k}}{1-s^{2^{k+1}}} = \frac{s}{1-s}. \tag{126}$$

5.2 The case $\delta \neq 0$

In this subsection we just mention two known sequences, namely

1. The Bodlaender sequence, that is the special case $\alpha = 1$, $\gamma = 0$, $\delta = 1$, $\zeta = -1$, $g(n) = 0$, $h(n) = n$ of (105)

$$f(1) = -1, \quad f(2n) = f(n), \quad f(2n+1) = f(n+1) + n, \quad n \geq 1 \tag{127}$$

   already studied in [25] with the solution

$$f\Big((b_m \dots b_0)_2 + 1\Big) = -1 + \sum_{k=0}^{m-1} (1-b_k)(b_m \dots b_{k+1})_2 \tag{128}$$

2. Merge sort (see [13, p. 79, Exercise 34 at p. 98])
   The total number of comparisons for merge sort of n records is at most $C(n)$, where

$$C(1) = 0, \quad C(n) = C\Big(\lfloor n/2 \rfloor\Big) + C\Big(\lceil n/2 \rceil\Big) + n - 1, \quad n \geq 2, \tag{129}$$

or, equivalently, for $n \geq 1$

$$C(1) = 0, \quad C(2n) = 2C(n)+2n-1, \quad C(2n+1) = C(n)+C(n+1)+2n \quad (130)$$

This is the special case $\alpha = 2$, $\gamma = 1$, $\delta = 1$, $\zeta = 0$, $g(n) = 2n - 1$, $h(n) = 2n$ of (105) with the solution (see [13, p. 496])

$$C(n) = \sum_{k=1}^{n} \lceil \log_2 k \rceil = n \cdot \lceil \log_2 n \rceil - 2^{\lceil \log_2 n \rceil} + 1. \quad (131)$$

Acknowledgments

I would like to thank Andreas M. Hinz (Munich) for some useful comments.

References

[1] J. P. Allouche, On the Stern sequence and its twisted version, *Integers* **12** (2012) #A58, available electronically at http://www.integers-ejcnt.org/vol12.html.

[2] J. P. Allouche, J. O. Shallit, The ring of k-regular sequences, *Theoret. Comput. Sci.* **98** (1992) 163–197.

[3] J. P. Allouche, J. O. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in: C. Ding, T. Helleseth, H. Niederreiter (eds.), *Sequences and their Applications*, Springer, London, 1999, 1–16.

[4] J. P. Allouche, J. O. Shallit, The ring of k-regular sequences, II, *Theoret. Comput. Sci.* **307** (2003) 3–29.

[5] Y. H. Au, C. Drexler-Lemire, J. O. Shallit, Notes and note pairs in Nørgård's infinity series, *Journal of Mathematics and Music* Volume **11**, Issue 1, (2017) 1–19.

[6] R. Bacher, Twisting the Stern sequence, arXiv preprint (2010), available electronically at http://arxiv.org/abs/1005.5627.

[7] A. Brocot, Calcul de rouages par approximation, *Revue Chronométrique* **3** (1859-61) 186–194. (Reprinted in: R. Chavigny, *Les Brocot, une dynastie d'horlogers*, Antoine Simonin, Neuchâtel, 1991, 187–197.)

[8] D. M. Burton, *Elementary Number Theory*, 3rd ed., McGraw-Hill, New York, 1997.

[9] L. Carlitz, Single variable Bell polynomials, *Collect. Math.* **14** (1962) 13–25.

[10] L. Carlitz, A problem in partitions related to the Stirling numbers, *Bull. Amer. Math. Soc.* **70** (1964) 275–278.

[11] M. Coons, On some conjectures concerning Stern's sequence and its twist, *Integers* **11** (2011) #A35, available electronically at http://www.integers-ejcnt.org/vol11.html.

[12] H. L. Glaisher, On the residue of a binomial coefficient theorem with respect to a prime modulus, *Quarterly J. Pure and Appl. Math.* **30** (1899) 150–156.

[13] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading MA, 1989.

[14] W. Hilberg, Mehrdimensionale Morse-Thue-Folgen, *Physik in unserer Zeit 22 Jahrg.* Nr. 1 (1991) 24–28.

[15] A. M. Hinz, The Tower of Hanoi, *L'Enseignement Mathématique* (2) **35** (1989) 289–321.

[16] A. M. Hinz, Pascal's Triangle and the Tower of Hanoi, *Amer. Math. Monthly* **99** (1992) 538–544.

[17] A. M. Hinz, The Tower of Hanoi, in: *Algebras and Combinatorics (An International Congress, ICAC '99, Hong Kong (Kar-Ping Shum, Earl J. Taft, and Zhe-Xian Wan, eds.))*, Springer, Singapore (1999) 277–289.

[18] A. M. Hinz, S. Klavžar, U. Milutinović, D. Parisse, C. Petr, Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence, *European J. Combin.* **26** (2005) 693–708.

[19] A. M. Hinz, S. Klavžar, C. Petr, *The Tower of Hanoi – Myths and Maths*, 2nd edition, Birkhäuser, Cham, 2018.

[20] D. H. Lehmer, On Stern's diatomic series, *Amer. Math. Monthly* **36** (1929) 59–67.

[21] D. A. Lind, An extension of Stern's diatomic series, *Duke Math. J.* **36** (1969) 55–60.

[22] É. Lucas, Sur le suites de Farey, *Bulletin de la Société mathématique de France* **6** (1878) 118–119.

[23] S. Northshield, Stern's diatomic sequence $0, 1, 1, 2, 1, 3, 2, 3, 1, 4, \ldots$, *Amer. Math. Monthly* **117** (2010) 581–598.

[24] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/`

[25] D. Parisse, On the Bodlaender Sequence, *Proceedings of the 2nd Croatian Combinatorial Days, Zagreb, Sept. 27-28, 2018* (Eds. T. Došlić and I. Martinjak), Fac. Civil Eng., Univ. Zagreb, (2018) 105–121.

[26] J. C. Puchta, J. Spilker, Die Thue-Morse-Folge, *Elem. Math.* **55** No. 3 (2000) 110–122.

[27] B. Reznick, A new sequence with many properties, *Abs. Amer. Math. Soc.* **5** (1984) 16.

[28] B. Reznick, Some binary partition functions, in: B. C. Berndt et al. (Eds.), *Analytic Number Theory (Proceedings of a Conference in Honor of Paul T. Bateman),* Birkhäuser, Boston (1990) 451-477.

[29] G. de Rham, Un peu de mathématiques à propos d'une courbe plane, *Elem. der Math.* Band II, **4**, **5** (1947) 73–88, 89–97.

[30] R. Sedgewick, P. Flajolet, *An Introduction to the Analysis of Algorithms,* Addison-Wesley, 1996.

[31] R. Stephan, Divide-and-conquer generating functions. I. Elementary sequences, available electronically at arXiv:math/0307027 [math.CO], 2003.

[32] M. A. Stern, Über eine zahlentheoretische Funktion, *J. Reine Angew. Math.* **55** (1858) 193-220.

[33] I. Urbiha, Some properties of a function studied by De Rham, Carlitz and Dijkstra and its relation to the (Eisenstein-)Stern's diatomic sequence, *Math. Commun.* **6** (2001) 181–198.

Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.07

Complexity function for a variant of Flory model on a ladder

Mate Puljiz, Stjepan Šebek and Josip Žubrinić

Abstract

In this article, we compare the famous Flory model with its variant that was recently introduced by the authors. Instead of looking at these models on a one-dimensional lattice, we consider a two-row ladder. For both models we compute the complexity function, and we analyze the differences of static and dynamic versions of these models.

Keywords: dynamic lattice systems, equilibrium lattice systems, complexity function, configurational entropy, jammed configuration.

AMS subject classification (2020): 82B20, 82C20, 05B40, 05A15, 05A16.

1 Introduction

Random sequential adsorption (RSA) refers to a process where particles are sequentially introduced in a system at random. Placement (adsorption) of a new particle is accepted if the particle does not fall into the exclusion region of another particle that has already been adsorbed. The process continues until a *jammed configuration* is reached, i.e. until no more new particles can be adsorbed. These jammed configurations for various models have been extensively studied in the literature (see [18, §7] for a comprehensive overview). The main question related to jammed configurations concerns the expected density of the adsorbed particles. There are two natural ways to interpret this question. One is through the explained RSA approach where particles are sequentially introduced in a system until a jammed configuration is reached. This is usually referred to as the *dynamic* model. The limit to which the expected density of particles converges in this case is called *jamming limit*. The other way to look at this problem is to consider the set of all the possible jammed configurations

(Mate Puljiz) University of Zagreb, Faculty of Electrical Engineering and Computing, Zagreb, Croatia, mate.puljiz@fer.hr

(Stjepan Šebek) University of Zagreb, Faculty of Electrical Engineering and Computing, Zagreb, Croatia, stjepan.sebek@fer.hr

(Josip Žubrinić) University of Zagreb, Faculty of Electrical Engineering and Computing, Zagreb, Croatia, josip.zubrinic@fer.hr

and to sample one such configuration at random. This is referred to as the *static* (or *equilibrium*) model. The static model is usually described with the so-called *complexity function* (also known as *configurational entropy*), and the expected density of particles in a jammed configuration converges to the argument of the maximum of the complexity function. The precise definitions of the jamming limit and the complexity function are given later. It is not clear, at first, that the two models are different. The assumption that both approaches lead to the same expected density of large jammed configurations is referred to as the *Edwards hypothesis*, see [2] for a recent review. In the models studied in the present paper, as is the case for most irreversible deposition models, Edwards hypothesis is violated.

The dynamic model (RSA) has been studied extensively because of its wide applicability to diverse aspects of physics, chemistry and biology. It was first studied in one dimension. In his pioneering paper [12], Paul Flory studied the attachment of pendant groups in a polymer chain. Closely related to his work is the Page-Rényi car-parking problem (see [24]), which is a discrete version of the original car-parking model introduced by Alfred Rényi in [27]. To determine the value of the jamming limit one can run experiments (see [22]), computer simulations (see [23, 30, 31]), or use mathematical analysis (see [1, 3, 10, 12, 13, 18, 27]). Even though various analytic solutions have been found, for both continuum and lattice versions of RSA, for most of the models studied in the literature, jamming limit is only known approximately and is obtained with the aid of computer simulations.

The static (equilibrium) model has also received attention in the literature (see [4, 6, 7, 15, 18–20]). The configurational entropy of jammed configurations is usually determined either by means of direct combinatorial reasoning [4, 7, 19], or by using the transfer-matrix approach [6, 9, 20]. Recently, a new method for determining complexity function has been developed in [17], inspired by the theory of renewal processes.

Most of the results in the literature (in both the static and the dynamic case) have been obtained for one-dimensional lattices. The problem gets more involved in higher dimensions. In this paper, we consider a slightly more general structure than the one-dimensional lattice, namely a two-row ladder (see Figure 1). We employ the new approach from [17] to develop the complexity function for both the original Flory model, and the modified version of Flory model that was introduced in [25] and subsequently studied in [8, 16, 26]. This modified version of the original Flory model was introduced in terms of a combinatorial settlement model where the impact of the architect would be as small as possible, and people would have a lot of freedom in the process of building the settlement. This minimal intervention from the side of the architect is given through the condition that houses are not allowed to be blocked from the sunlight, and that the tracts of land on which the settlements are built are of rectangular shapes. A house is blocked from the sunlight if there are other houses on the neighboring sites to the right (east), to the left (west), and below (to the south) of that house. Additionally, it is assumed that houses receive sunlight from the eastern, southern and western boundary of the tract of land on which the settlement is being built. Here, we consider the two-row ladder as the rectangular tract of land on which the houses are built. The described condition imposes that each particle (house) in the upper (northern) row of the two-row ladder needs to

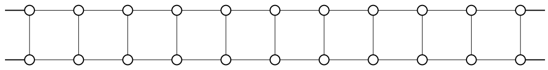


Figure 1: A two-row ladder graph.

have at least one of its neighboring sites free, while no restrictions are placed upon particles (houses) in the lower (southern) row as they are guaranteed to receive sunlight from the southern border. Compare this to the original Flory model, where each deposited particle enforces all of its neighboring sites to be vacant.
Flory model on a two-row ladder has already been studied in the literature in the dynamic case (see [1, 3, 10, 13]). The static case was considered in one dimension (see [18, §7.2]). On the other hand, the model introduced in [25] was originally studied on a two-dimensional lattice, and some observations have been made both in the static and the dynamic case (see [25,26]), but the complexity was addressed only in the one-dimensional version of that model, the so-called Riviera model (see [8,16]).

Notation. The length of the two-row ladder graph that we are considering is denoted by L. Notice that such a graph of length L has $2L$ sites that can be occupied by deposited particles (atoms, houses), or vacant. Each configuration of length L can be represented with two binary 0/1 sequences (both of length L, one representing the upper row, and the other one representing the lower row of a two-row ladder graph) where 1 is interpreted as an occupied site, and 0 as vacant site.
The total number of different jammed configurations on the two-row ladder of length L is denoted by J_L, and the total number of jammed configurations on the two-row ladder of length L that have precisely N deposited particles is denoted by $J_{N,L}$. The *density (saturation, coverage)* of any such configuration of length L with N particles is defined as N/L. Notice that, using the convention that the length of the two-row ladder graph with $2L$ sites is denoted by L, the density will always attain values in $[0, 2]$.

2 Flory model on a ladder

Flory model on a one-dimensional lattice is the most famous model within the theory of RSA. Each site on the lattice can be occupied by an atom, or vacant. Each atom blocks its nearest neighbor sites. This model (and the related Page-Rényi car-parking process) has been extensively studied in the literature, in both dynamic and static setting (see [12, 14, 24] for the dynamic version and [16, 18] for both versions). As announced in the introduction, we are dealing with Flory model on a two-row ladder graph. Most of the analytic treatments of such models are restricted to one-dimensional lattice. It would be very interesting to see how all these results translate to the truly two-dimensional lattice. This, however, seems to be an extremely technical task (if one wants to obtain some analytical result, and not just run simulations), so we provide here a very modest entrée into two-dimensional space.
To get a better feeling about the Flory model on a two-row ladder graph, and about the structure of jammed configurations of this model, we display in Figure 2 all the

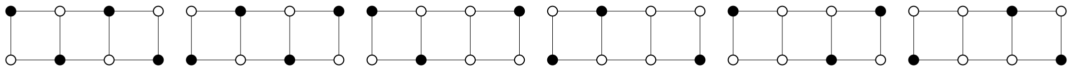


Figure 2: All the jammed configurations of Flory model on a two-row ladder graph of length 4.

possible jammed configurations on the two-row ladder graph of length 4.
Based on Figure 2, we can easily compare static and dynamic model in the case of the two-row ladder graph of length 4. Notice that $J_4^{\text{Flory}} = 6$. More precisely, $J_{3,4}^{\text{Flory}} = 4$, and $J_{4,4}^{\text{Flory}} = 2$. Denoting the random variable counting the number of occupied sites in the static model on a two-row ladder graph of length L by $X_s^{\text{Flory}}(L)$, and random variable counting the number of occupied sites in the dynamic model by $X_d^{\text{Flory}}(L)$, we read from Figure 2 that

$$X_s^{\text{Flory}}(4) \sim \begin{pmatrix} 3 & 4 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Inspecting in which order particles have to be introduced in the system to end up in each of the jammed configurations shown in Figure 2, one easily obtains

$$X_d^{\text{Flory}}(4) \sim \begin{pmatrix} 3 & 4 \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}.$$

Using the sequence of random variables $(X_d(L))_L$, that count the number of occupied sites in a dynamic model of length L, we can give a precise definition of the jamming limit of a certain model.

Definition 1. *For a fixed length $L \in \mathbb{N}$, let $X_d(L)$ denote the random variable which models the number of occupied sites in a jammed configuration of length L, reached through the RSA procedure. The* jamming limit *is then defined as*

$$\rho_\infty = \lim_{L\to\infty} \frac{\mathbb{E}[X_d(L)]}{L}, \tag{1}$$

if this limit exists, where $\mathbb{E}[X]$ denotes the expected value of a random variable X.

Remark 1. The convergence of expected values in (1) is the most common notion of convergence considered in these kind of problems. However, in some instances it was possible to prove convergence in probability for these random variables, in addition to convergence of expectations (see e.g. [14, 24] for discrete Rényi car-parking problem). Note that here, due to boundedness of random variables, the convergence in probability is strictly stronger than the convergence of expected values.

As mentioned in the introduction, the dynamic version of the Flory model on a two-row ladder has already been studied in the literature. It was independently shown in [10] and [1] that the jamming limit of this model is

$$\rho_\infty^{\text{Flory}} = 1 - \frac{1}{2e}, \tag{2}$$

and this result was later reconstructed in [3, 13].

Remark 2. Note that in [1, 3, 10, 13] the density is defined as $\frac{N}{2L}$, so the constant above is divided by two.

We now turn to the static model and to determining the complexity function. We first recall the definition of complexity function of a certain model.

Definition 2. *For a fixed density $\rho \geq 0$, take $((N_i, L_i))_i$ to be any sequence of pairs of non-negative integers such that $\lim_{i\to\infty} L_i = +\infty$ and $\lim_{i\to\infty} \frac{N_i}{L_i} = \rho$. Note that J_{N_i,L_i} is the number of configurations of length L_i with density $\frac{N_i}{L_i}$. We are interested in the quantity*

$$\limsup_{i\to\infty} \frac{\ln J_{N_i,L_i}}{L_i}$$

which is the exponential rate of growth of these configurations. If we now take the supremum over all such sequences, we arrive at the definition of complexity function $S(\rho)\colon [0,\infty) \to [0,\infty)$

$$S(\rho) = \sup_{(N_i,L_i)} \limsup_{i\to\infty} \frac{\ln J_{N_i,L_i}}{L_i} \tag{3}$$

where the supremum runs over all the sequences such that $N_i/L_i \to \rho$.

Remark 3. Whenever we encounter $J_{N,L} = 0$ for some (N, L), we will redefine it as $J_{N,L} = 1$ so that $\ln J_{N,L} = 0$ can be computed. Consequently, if there are no configurations with densities approaching a certain ρ, we get $S(\rho) = 0$. Also note that the lim sup can be replaced with lim since we can, if needed, pass to a subsequence.

Remark 4. This definition implies that the number of configurations with density $N/L \approx \rho$ grows as $e^{LS(\rho)}$ for large L. The density $\rho_\star$ at which the complexity function $S(\rho)$ attains its maximum, i.e. the density corresponding to the largest rate of growth, is called the *equilibrium density* and is a static setting analogue of jamming limit as it can be shown that

$$\rho_\star = \lim_{L\to\infty} \frac{\mathbb{E}[X_s(L)]}{L}.$$

where $X_s(L)$ is the number of occupied sites in the static (equilibrium) model in which each jammed configuration of length L is equally likely. Furthermore, the random variables $X_s(L)/L$ can be shown to converge in distribution to delta distribution concentrated at $\rho_\star$.

Remark 5. In most commonly encountered models the sup in the definition is superfluous, as any choice of the sequence (say $((N_L, L))_L$ where $N_L = \lfloor \rho L \rfloor$) will produce the same limit.

Remark 6. Lastly, note that the complexity function is bounded

$$S(\rho) \leq 2 \ln 2.$$

This follows from the trivial bound $J_{N,L} \leq 2^{2L}$. A more precise bound

$$S(\rho) \leq 2H(\rho/2)$$

follows from the inequality $J_{N,L} \leq \binom{2L}{N} \sim \exp\left(2L \cdot H(N/(2L))\right)$, where $H(p)$ is the Shannon's entropy function $H(p) = -p \ln p - (1-p) \ln(1-p)$, see [5].

Figure 3: Vertices in the graph encoding the Flory model on a ladder.

To be able to apply the method from [17], we first need to determine the bivariate generating function for this model. To this end, we apply the so-called transfer matrix method (see [29, §4.7], [11, §V], and [21, §2–4]). This is a well known method for counting words of a regular language. Notice that every jammed configuration on a ladder is composed of the nodes shown in Figure 3. The transfer matrix is then of the shape

$$A(x) = \begin{bmatrix} 0 & 0 & 0 & x & 0 & 0 \\ 1 & 0 & 0 & 0 & x & 0 \\ 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 & x \\ 1 & 0 & 0 & 0 & x & 0 \end{bmatrix},$$

where x is a formal variable associated with the number of atoms. Taking into consideration what the possible starting and ending nodes of a jammed configuration in the Flory model on a two-row ladder graph are, we define vectors

$$a(x) = (x, 0, x, 0, x^2, x^2)^T \qquad \text{and} \qquad b = (0, 1, 0, 1, 1, 1)^T.$$

Now the bivariate generating function enumerating all the jammed configurations in the Flory model on a two-row ladder of length L with precisely N atoms is given by

$$\begin{aligned} F_{\text{Flory}}(x, y) &= 1 + 2xy + \sum_{n=2}^{\infty} a(x)^T \cdot [A(x)]^{n-2} \cdot b \cdot y^n \\ &= 1 + 2xy + a(x)^T \sum_{n=0}^{\infty} [A(x) \cdot y]^n \cdot b \cdot y^2 \\ &= \frac{1 + xy - xy^2}{1 - xy - xy^2}, \end{aligned} \tag{4}$$

where y is a formal variable associated with the length of the configuration. Hence, we have just proved the following lemma.

Lemma 1. *The bivariate generating function enumerating the total number of jammed configurations in the Flory model on a two-row ladder of length L, where precisely N sites are occupied with atoms, is*

$$F_{Flory}(x, y) = \frac{1 + xy - xy^2}{1 - xy - xy^2}.$$

We will now briefly describe the method from [17], based on the classical Legendre transform, which enables one to derive the complexity function directly from the bivariate (rational) generating function for the sequence $J_{N,L}$.

Remark 7. In order to follow the procedure described here, the generating function one starts with does not actually need to be rational. It suffices for the function to be analytic in y on some disk around the origin for each $x > 0$. But in order to actually calculate the complexity, one has to be able to express the radius of convergence (or the modulus of the non-removable singularity closest to the origin) of that function in terms of x. This is why rational generating functions are particularly amenable to this approach.

Let

$$F(x,y) = \sum_L \left(\sum_N J_{N,L}\, x^N \right) y^L = \frac{p(x,y)}{q(x,y)}$$

be the bivariate generating function of the sequence $(J_{N,L})$, where p and q are coprime polynomials. For each fixed $x > 0$, let $y_0(x)$ be the root of the polynomial $q(x,y)$ with the smallest modulus. Then, by the Cauchy–Hadamard theorem from complex analysis (see e.g. [32, Theorem 2.4.3]), it follows

$$\limsup_{L\to\infty} \sqrt[L]{\sum_N J_{N,L}\, x^N} = |y_0(x)|^{-1}.$$

As $x > 0$ and the sum above runs over $0 \leq N \leq 2L$ we obtain the bounds

$$\sqrt[L]{\max_N \left(J_{N,L}\, x^N\right)} \leq \sqrt[L]{\sum_N J_{N,L}\, x^N} \leq \sqrt[L]{(2L+1) \max_N \left(J_{N,L}\, x^N\right)}$$

and consequently

$$\limsup_{L\to\infty} \sqrt[L]{\max_N \left(J_{N,L}\, x^N\right)} = \limsup_{L\to\infty} \sqrt[L]{\sum_N J_{N,L}\, x^N} = |y_0(x)|^{-1}.$$

After taking logarithms

$$\limsup_{L\to\infty} \frac{\ln\left(\max_N \left(J_{N,L}\, x^N\right)\right)}{L} = -\ln|y_0(x)|. \tag{5}$$

Next, note

$$\ln\left(\max_N \left(J_{N,L}\, x^N\right)\right) = \max_N \left(\ln J_{N,L} + N \ln x\right) = L \max_N \left(\frac{\ln J_{N,L}}{L} + \frac{N}{L}\ln x\right)$$

and so (5) reads

$$\limsup_{L\to\infty} \max_N \left(\frac{\ln J_{N,L}}{L} + \frac{N}{L}\ln x\right) = -\ln|y_0(x)|. \tag{6}$$

We next wish to show the following claim

Claim 1. *The complexity function $S(\rho)$ associated to the sequence $J_{N,L}$ satisfies the equation*

$$\sup_\rho \{S(\rho) + \rho \ln x\} = -\ln|y_0(x)|. \tag{7}$$

Proof. Fix $\rho \geq 0$ and let (N_i, L_i) be an arbitrary sequence such that $N_i/L_i \to \rho$. Clearly

$$\frac{\ln J_{N_i,L_i}}{L_i} + \frac{N_i}{L_i}\ln x \leq \max_N \left(\frac{\ln J_{N,L_i}}{L_i} + \frac{N}{L_i}\ln x\right)$$

and thus

$$\limsup_{i\to\infty}\left(\frac{\ln J_{N_i,L_i}}{L_i}\right) + \rho\ln x = \limsup_{i\to\infty}\left(\frac{\ln J_{N_i,L_i}}{L_i} + \frac{N_i}{L_i}\ln x\right) \leq$$
$$\leq \limsup_{i\to\infty}\max_N\left(\frac{J_{N,L_i}}{L_i} + \frac{N}{L_i}\ln x\right) \leq \limsup_{L\to\infty}\max_N\left(\frac{\ln J_{N,L}}{L} + \frac{N}{L}\ln x\right).$$

Since this holds for any sequence (N_i, L_i) with $N_i/L_i \to \rho$ we get

$$S(\rho) + \rho\ln x = \sup_{(N_i,L_i)}\limsup_{i\to\infty}\left(\frac{\ln J_{N_i,L_i}}{L_i}\right) + \rho\ln x \leq \limsup_{L\to\infty}\max_N\left(\frac{\ln J_{N,L}}{L} + \frac{N}{L}\ln x\right).$$

Since the latter inequality holds for any ρ we conclude

$$\sup_\rho\{S(\rho) + \rho\ln x\} \leq \limsup_{L\to\infty}\max_N\left(\frac{\ln J_{N,L}}{L} + \frac{N}{L}\ln x\right). \tag{8}$$

For the opposite inequality, let us take a sequence $((N_L, L))_L$ where for each L, $0 \leq N_L \leq 2L$ is the number at which the maximum in the right hand side of (8) is attained. Passing to a subsequence $((N_{L_i}, L_i))_i$, if needed, we may assume that the $\limsup$ on the right hand of (8) is actually $\lim$. After passing to a subsequence once more, if needed, we may assume that $N_{L_i}/L_i \to \tilde\rho$ for some $\tilde\rho \in [0, 2]$. Therefore

$$\limsup_{L\to\infty}\max_N\left(\frac{\ln J_{N,L}}{L} + \frac{N}{L}\ln x\right) = \lim_{i\to\infty}\left(\frac{\ln J_{N_{L_i},L_i}}{L_i} + \frac{N_i}{L_i}\ln x\right) =$$
$$= \lim_{i\to\infty}\left(\frac{\ln J_{N_{L_i},L_i}}{L_i}\right) + \tilde\rho\ln x \leq S(\tilde\rho) + \tilde\rho\ln x$$

where in the last step we used the fact that $((N_{L_i}, L_i))_i$ is just one of the sequences over which the supremum is taken in the definition of $S(\tilde\rho)$. Finally

$$\limsup_{L\to\infty}\max_N\left(\frac{\ln J_{N,L}}{L} + \frac{N}{L}\ln x\right) \leq S(\tilde\rho) + \tilde\rho\ln x \leq \sup_\rho\{S(\rho) + \rho\ln x\} \tag{9}$$

and putting the statements (6), (8) and (9) together gives the claim (7). □

If we set $x = e^t > 0$ then the relation (7) becomes

$$\sup_\rho\{t\rho - (-S(\rho))\} = -\ln|y_0(e^t)|$$

and it expresses the fact that the function $-\ln|y_0(e^t)|$ is the convex conjugate (or *Legendre-Fenchel transform*) of the function $-S(\rho)$, see [28, §12].

Since we know the function $-\ln|y_0(e^t)|$, and seek to find $-S(\rho)$, we actually need to invert the transformation. Luckily, the Legendre transformation is an involution and therefore

$$\sup_t \left\{t\rho - (-\ln|y_0(e^t)|)\right\} = -S(\rho),$$

which we can express in terms of x as

$$\sup_{x>0} \{\rho \ln x + \ln|y_0(x)|\} = -S(\rho). \tag{10}$$

Remark 8. The Legendre transformation is actually an involution only on the set of closed convex functions, and if applied twice on any function it returns the closed convex envelope of that function [28, Corollary 12.1.1 and Theorem 12.2]. As the complexity functions for similar models in the literature are known to be closed concave ($-S$ is closed convex), it is reasonable to think that the complexity function in our model must be as well. With this in mind, a more precise restatement of (10) would be

$$\sup_{x>0} \{\rho \ln x + \ln|y_0(x)|\} = \mathrm{cl}\left(\mathrm{conv}\left(-S(\rho)\right)\right).$$

Using the first derivative test, we see that x_0 at which maximum on the left hand side of (10) is attained must satisfy

$$\rho = -x_0 \frac{y_0'(x_0)}{y_0(x_0)} = \left[\frac{x}{y}\frac{\partial_x q}{\partial_y q}\right]_{x=x_0, y=y_0(x_0)} \tag{11}$$

where we used the implicit function theorem and the fact that $y_0(x)$ is defined by the relation $q(x, y_0(x)) = 0$. If we, now, for each $0 \le \rho \le 2$ solve (11) and $q(x_0, y_0) = 0$ for $x_0 = x_0(\rho)$ and $y_0 = y_0(x_0(\rho))$, then we can write the complexity function as

$$S(\rho) = -\rho \ln x_0 - \ln|y_0|. \tag{12}$$

In order to find the equilibrium density $\rho_\star$ of the model we need to solve the optimization problem

$$\rho_\star = \arg\max_\rho \{-\rho \ln x_0 - \ln|y_0|\}.$$

Note that the complexity function $S(\rho)$ can be parameterized using x as

$$S(\rho(x)) = x\frac{y_0'(x)}{y_0(x)} \ln x - \ln|y_0(x)|,$$

and it is easier to solve the optimization problem in this form. The solution $x = x_\star$ at which the maximum is attained must satisfy the equation

$$\frac{y_0'(x)}{y_0(x)} \ln x + x\frac{y_0''(x)y_0(x) - (y_0'(x))^2}{(y_0(x))^2} \ln x + x\frac{y_0'(x)}{y_0(x)}\frac{1}{x} - \frac{y_0'(x)}{y_0(x)} = 0.$$

or equivalently

$$\left(\frac{y_0'(x)}{y_0(x)} + x\frac{y_0''(x)y_0(x) - (y_0'(x))^2}{(y_0(x))^2}\right) \ln x = 0.$$

(a) $\frac{N}{L} = \frac{1}{2} + \frac{1}{18}$ (b) $\frac{N}{L} = 1$

Figure 4: Examples of two jammed configurations with extreme densities for the Flory model with $L = 9$.

From here, it is easy to see that $x_\star = 1$, and the corresponding $y_\star = y_0(1)$ is the solution of the equation $q(1, y_\star) = 0$. The equilibrium density $\rho_\star$ and the maximum value of the complexity function can then be calculated using (11) and (12) as

$$\begin{aligned} \rho_\star &= \frac{1}{y_\star} \frac{\partial_x q(1, y_\star)}{\partial_y q(1, y_\star)}, \\ S(\rho_\star) &= -\ln |y_\star|. \end{aligned} \tag{13}$$

Remark 9. Before we proceed to apply this method in order to compute the complexity of the Flory model on a two-row ladder, let us briefly discuss the support of S. As each occupied site in a maximal configuration must have all of its three neighbors unoccupied, and as each unoccupied site has at least one occupied neighbor, if follows $E \leq 3N$, where N and E stand for the number of occupied and unoccupied sites in a configuration, respectively. From here, $2L = E + N \leq 4N$ and therefore $\frac{N}{L} \geq \frac{1}{2}$. Also, to each occupied site corresponds the unique unoccupied site (above or below) which is on the same rung of the ladder. Hence, $N \leq E$ or, equivalently, $2N \leq N + E = 2L$ and therefore $\frac{N}{L} \leq 1$. One can actually show, more precisely, that $\left\lfloor \frac{L}{2} \right\rfloor + 1 \leq N \leq L$. Examples of the most and the least saturated maximal configurations for $L = 9$ are depicted in Figure 4.
The previous discussion implies that the complexity $S(\rho)$ can be non-zero only on the interval $\frac{1}{2} \leq \rho \leq 1$.

It is clear from the method described above that complexity is determined by the denominator of the bivariate generating function. In our case, the denominator of the expression (4) is

$$q(x, y) = 1 - xy - xy^2,$$

which coincides with the one that appears in [17, §4.2 *Isolated empty sites* (4.21)]. The name *Isolated empty sites* of the model discussed in [17, §4.2] already suggests that this is equivalent to the Flory model on a two-row ladder, because in the Flory model on the two-row ladder we are not allowed to have two empty columns in a row, and in non-empty columns we always have exactly one atom. Following the method described above, Krapivsky and Luck in [17, equation (4.28)] obtain the complexity function

$$S_{\text{Flory}}(\rho) = \rho \ln(\rho) - (1 - \rho) \ln(1 - \rho) - (2\rho - 1) \ln(2\rho - 1), \tag{14}$$

for $1/2 < \rho < 1$ (see Figure 5). The density for which the complexity function is maximized is

$$\rho_\star^{\text{Flory}} = \frac{5 + \sqrt{5}}{10}, \tag{15}$$

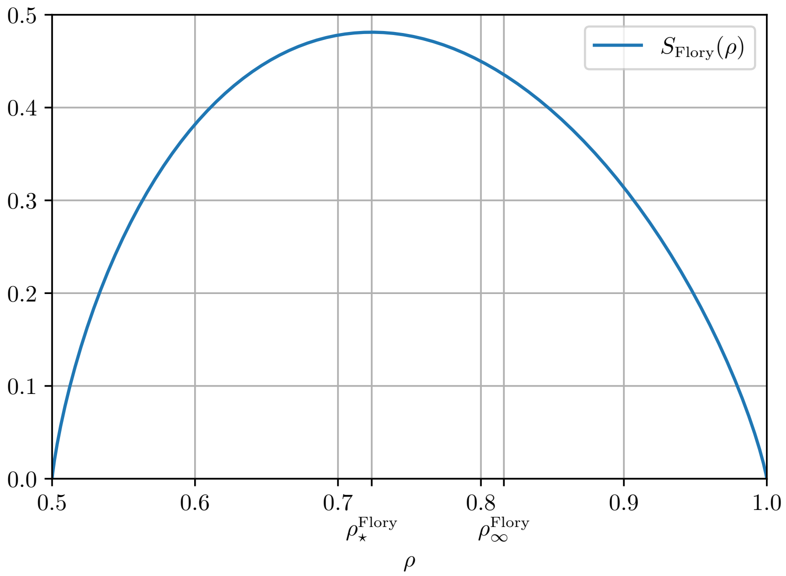


Figure 5: The complexity of the Flory model on the ladder. Also depicted on the plot are the jamming limit $\rho_\infty^{\text{Flory}} = 1 - (2e)^{-1}$, and the equilibrium density $\rho_\star^{\text{Flory}} = (5 + \sqrt{5})/10$.

and the value of the maximum is

$$S(\rho_\star^{\text{Flory}}) = \ln \left(\frac{1 + \sqrt{5}}{2} \right) . \tag{16}$$

This can be obtained from (13) by setting $x_\star = 1$ and $y_\star = \frac{\sqrt{5}-1}{2}$. The interpretation of the value calculated in (16) is that on the two-row ladder of length L, there are roughly

$$\left(\frac{1 + \sqrt{5}}{2} \right)^L$$

jammed configurations. Comparing the jamming limit $\rho_\infty^{\text{Flory}}$ from equation (2), which summarizes the dynamic model, and the value of $\rho_\star^{\text{Flory}}$ (i.e. the argument of the maximum of the complexity function) given in (15), which summarizes the equilibrium model, we see that Edwards hypothesis does not hold (see Figure 5).

Remark 10. Notice that the constant appearing in relation (16) is the famous golden ratio. It is in fact very easy to see that the sequence J_L counting the number of all jammed configurations of length L satisfies the well-known Fibonacci recursion

$$J_L^{\text{Flory}} = J_{L-1}^{\text{Flory}} + J_{L-2}^{\text{Flory}} .$$

3 Settlement model on a ladder

Recently, a combinatorial settlement planning model was introduced in [25] and the same model has been further studied in [8,16,26]. As in the case of Flory model, most of the results have been developed for the one-dimensional version of this model, the

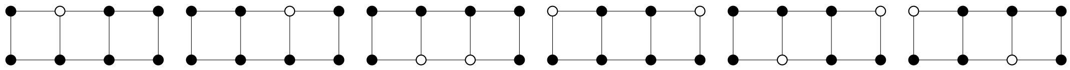


Figure 6: All the jammed configurations of the settlement model from [25] on a two-row ladder graph of length 4 ($J_4 = 6$, $J_{6,4} = 4$, $J_{7,4} = 2$).

so-called Riviera model. The complexity function for the Riviera model has been derived in [16], and in the same paper the authors discuss the dynamic model, and they obtain the jamming limit using simulations. In this paper, we are interested in the model from [25] on a ladder. As in the previous section, we first show some concrete examples of jammed configurations of this model. In Figure 6 are shown all the jammed configurations of combinatorial settlement model, introduced in [25], on a two-row ladder graph of length 4. Recall that there are no restrictions on the houses placed in the bottom row, while each house in the top row (which is not on the boundary) must have at least one of its immediate neighbors unoccupied. It is clear from Figure 6 that

$$X_s(4) \sim \begin{pmatrix} 6 & 7 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix},$$

where $X_s(4)$ is a random variable that counts the number of occupied sites in the static combinatorial settlement model on a two-row ladder graph of length 4. Denoting by $X_d(4)$ the corresponding random variable in the dynamic version of the model, and carefully analyzing in which order the houses had to be built to reach a particular jammed configurations, we have

$$X_d(4) \sim \begin{pmatrix} 6 & 7 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

This particular model has been considered in the static case in [8], but only bivariate generating function was developed. However, in light of the new method (introduced in [17]) for computing complexity function, we know that this result about the bivariate generating function can be used to obtain the desired complexity function. From [8, §4.1 and relation (A.4)], we have

$$F(x, y) = \frac{1 - xy + x^2y - 2x^3y^2 + x^4y^2 + x^4y^3 + x^5y^3 - x^8y^5}{1 - xy - x^3y^2 + x^4y^3 - x^5y^3 - x^6y^4 + x^9y^6}. \tag{17}$$

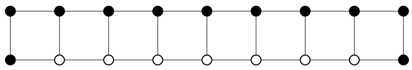
It is not hard to see what the extreme configurations look like for this model (see also [25, Theorem 4.5]) and to conclude

$$L + 2 \le N \le 2L - \lfloor L/3 \rfloor = \frac{5}{3}L + (L/3 - \lfloor L/3 \rfloor) \le \frac{5}{3}L + \frac{2}{3},$$

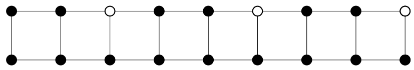
and from there

$$1 + \frac{2}{L} \le \frac{N}{L} \le \frac{5}{3} + \frac{2}{3L}.$$

In Figure 7 are given examples of the most and the least saturated maximal configurations for $L = 9$. The inequality above immediately implies that the support of the complexity function of this model will be contained in the interval $1 \le \rho \le 5/3$.



(a) $\frac{N}{L} = 1 + \frac{2}{9}$



(b) $\frac{N}{L} = \frac{5}{3}$

Figure 7: Examples of two jammed configurations with extreme densities for the settlement model with $L = 9$.

Using the same notation as in the previous section, we have

$$q(x, y) = 1 - xy - x^3y^2 + x^4y^3 - x^5y^3 - x^6y^4 + x^9y^6. \tag{18}$$

As it is not possible to explicitly solve for $y_0(x)$ in the equation $q(x, y) = 0$, the first step is to find a parametrization of this equation which will enable us to express its roots in terms of the new variables. To this end, we employ the parametrization

$$\begin{cases} t = y^{\frac{1}{3}}, \\ u = xy^{\frac{2}{3}}, \end{cases} \quad \text{i.e.} \quad \begin{cases} x = \dfrac{u}{t^2}, \\ y = t^3. \end{cases}$$

Plugging this into the equation $q(x, y) = 0$ (where $q(x, y)$ is given in (18)) gives us

$$(u^4 - u)t + (u^9 - u^6 - u^3 + 1) - u^5t^{-1} = 0.$$

Multiplying by t, we get

$$(u^4 - u)t^2 + (u^9 - u^6 - u^3 + 1)t - u^5 = 0.$$

From here it follows that

$$t_0^{\pm} = \frac{-(u^9 - u^6 - u^3 + 1) \pm \sqrt{u^{18} - 2u^{15} - u^{12} + 8u^9 - 5u^6 - 2u^3 + 1}}{2(u^4 - u)}. \tag{19}$$

These two expression for t_0 in terms of u enable us to express the roots $y_0(x)$ of the polynomial $q(x, y_0(x))$ for each $x > 0$ in terms of the parameter u. In fact, all the quantities of interest can now be expressed in terms of u in the following way

$$y_0 = [t_0(u)]^3, x_0 = \frac{u}{[t_0(u)]^2}, \quad \rho(u) = \left[\frac{x}{y}\frac{\partial_x q}{\partial_y q}\right]_{x=x_0, y=y_0}, \quad S(\rho) = -\rho \ln x_0 - \ln |y_0|.$$

We have, thus, proved the following theorem which provides a parameterized expression for the complexity function of the combinatorial settlement planning model on a two-row ladder.

Theorem 1. *The complexity function of the combinatorial settlement planning model on a two-row ladder is given as*

$$S(\rho) = -\rho \ln x_0 - \ln |y_0|,$$

where

$$\begin{cases} \rho(u) = \left.\frac{-9x^8y^5+6x^5y^3+5x^4y^2-4x^3y^2+3x^2y+1}{-6x^8y^5+4x^5y^3+3x^4y^2-3x^3y^2+2x^2y+1}\right|_{x=x_0, y=y_0}, \\ x_0 = u\,[t_0(u)]^{-2}, \\ y_0 = [t_0(u)]^3, \end{cases}$$

and $t_0(u)$ is given by (19).

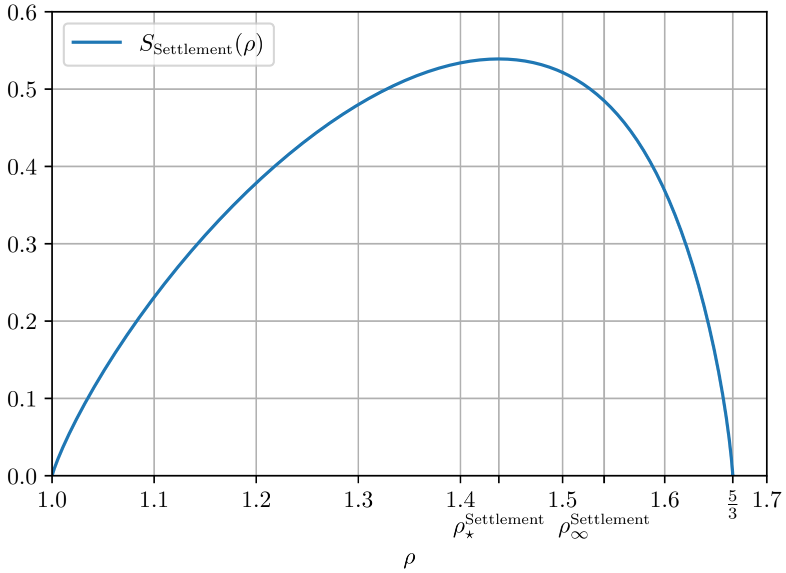


Figure 8: The complexity of the combinatorial settlement model on the ladder. Also depicted on the plot are (a Monte Carlo estimation of) the jamming limit $\rho_\infty^{\mathrm{Settlement}} \approx 1.54064$, and the equilibrium density $\rho_\star^{\mathrm{Settlement}} \approx 1.4374959$.

Remark 11. The equilibrium density $\rho_\star^{\mathrm{Settlement}}$ for this model can be calculated using (13). More precisely,

$$\rho_\star^{\mathrm{Settlement}} = \frac{1 + y_\star + y_\star^2 + 6y_\star^3 - 9y_\star^5}{1 + 2y_\star + 4y_\star^3 - 6y_\star^5} \approx 1.4374959$$

where $y_\star$ is the root of the polynomial $1 - y - y^2 - y^4 + y^6$ with the smallest modulus.

Remark 12. A closer inspection of the parametrization given in the theorem above and the equation (19) reveals that, in order to obtain the whole range of values $x > 0$, it suffices to take $0 < u \le u_0$, where $u_0 \approx 0.70633685$ is the positive root of the polynomial appearing under the square root in (19). For each such a value u, both t_0^+ and t_0^-, have to be used in order to produce the whole range $x > 0$.

Theorem 1 enables us to produce the graph in Figure 8 showing the complexity function of the settlement model. Note that, as previously discussed, the complexity is supported on $[1, \frac{5}{3}]$.
When it comes to the dynamic version of the combinatorial settlement planning model, there are no analytical results for the jamming limit. Even in the simplest case of Riviera model (which lives on a one-dimensional lattice), the exact solution is not known, and it is not clear whether it is even possible to derive one. One-dimensional RSA, and similar models, usually enjoy a peculiar property that is commonly referred to as the shielding property. The deposition of any elementary object splits the line into two half-lines which evolve independently from each other. This ensures the exact solvability of this class of models, by means of a common analytical approach based on tracking empty intervals (see [18, §7]). Clearly, the original Flory model has the shielding property. The Riviera model, on the other hand, does not enjoy the shielding property. On the contrary, the rule that at least

one of the two neighboring lots of each house should remain forever unbuilt couples both neighboring sites of any occupied one. The ensuing lack of exact solvability has observable consequences (see [16, §3]). The authors of [16] even write that this model (in all likelihood) cannot be solved by analytical means. In the case of the two-row ladder, the situation is even more involved, so we only provide Monte Carlo estimation of the jamming limit. The value is approximately $\rho_\infty^{\text{Settlement}} \approx 1.54064$, and comparing this value with $\rho_\star^{\text{Settlement}} \approx 1.4374959$, which is the argument of the maximum of the complexity function from Theorem 1 (see Figure 8) again shows the violation of Edwards hypothesis.

4 Conclusions

In this paper we use the novel approach developed in [17] to compute the complexity function (configurational entropy) of two similar, but essentially different, models on a two-row ladder graph. One of them is the famous Flory model, and the other one is the combinatorial settlement model introduced in [25]. Using the obtained configurational entropies, we compare the dynamic and static versions of these models and conclude weak violation of Edwards hypothesis.

Acknowledgments

We wish to thank Professors Tomislav Došlić, Pavel Krapivsky, and Jean-Marc Luck for fruitful and stimulating exchanges. We also thank the anonymous referee for careful reading of our manuscript and helpful comments.

References

[1] A. Baram and D. Kutasov: *Random sequential adsorption on a quasi-one-dimensional lattice: an exact solution*, Journal of Physics A: Mathematical and General **25**, no. 8, L493, 1992.

[2] A. Baule, F. Morone, H. J. Herrmann, and H. A. Makse: *Edwards statistical mechanics for jammed granular matter*, Reviews of modern physics **90**, no. 1, 015006, 2018.

[3] H. H. Chern, H. K. Hwang, and T. H. Tsai: *Random unfriendly seating arrangement in a dining table*, Advances in Applied Mathematicss **65**, 38–64, 2015.

[4] A. Crisanti, F. Ritort, A. Rocco, and M. Sellitto: *Inherent structures and nonequilibrium dynamics of one-dimensional constrained kinetic models: a comparison study*, The Journal of Chemical Physics **113**, no. 23, 10615–10634, 2000.

[5] I. Csiszár, P. C. Shields, *et al.*: *Information theory and statistics: A tutorial*, Foundations and Trends in Communications and Information Theory **1**, no. 4, 417–528, 2004.

[6] G. De Smedt, C. Godreche, and J. M. Luck: *Jamming, freezing and metastability in one-dimensional spin systems*, The European Physical Journal B-Condensed Matter and Complex Systems **27**, 363–380, 2002.

[7] D. Dean: *Metastable states of spin glasses on random thin graphs*, The European Physical Journal B-Condensed Matter and Complex Systems **15**, no. 3, 493–498, 2000.

[8] T. Došlić, M. Puljiz, S. Šebek, J. Žubrinić: *On a variant of Flory model*, https://arxiv.org/abs/2210.12411, 2022.

[9] T. Došlić, M. Puljiz, S. Šebek, and J. Žubrinić: *Complexity function of jammed configurations of Rydberg atoms*, https://arxiv.org/abs/2302.08791, 2023.

[10] Y. Fan and J. Percus: *Random sequential adsorption on a ladder*, Journal of statistical physics **66**, 263–271, 1992.

[11] P. Flajolet and R. Sedgewick: *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.

[12] P. J. Flory: *Intramolecular reaction between neighboring substituents of vinyl polymers*, Journal of the American Chemical Society **61**, no. 6, 1518–1521, 1939.

[13] K. Georgiou, E. Kranakis, and D. Krizanc: *Random maximal independent sets and the unfriendly theater seating arrangement problem*, Discrete Mathematics **309**, no. 16, 5120–5129, 2009.

[14] L. Gerin: *The Page-Rényi parking proces*, Electronic Journal of Combinatorics **22**, no. 4, Paper 4.4, 13, 2015.

[15] J. L. Jackson and E. W. Montroll: *Free Radical Statistics*, The Journal of Chemical Physics **28**, no. 6, 1101–1109, 2004.

[16] P. Krapivsky and J. Luck: *Jamming and metastability in one dimension: from the kinetically constrained Ising chain to the Riviera model*, arXiv preprint arXiv:2211.12815, 2022.

[17] P. Krapivsky and J. Luck: *A renewal approach to configurational entropy in one dimension*, https://arxiv.org/abs/2302.08852, 2023.

[18] P. Krapivsky, S. Redner, and E. Ben-Naim: *A kinetic view of statistical physics*, Cambridge University Press, Cambridge, 2010.

[19] A. Lefevre and D. S. Dean: *Metastable states of a ferromagnet on random thin graphs*, The European Physical Journal B-Condensed Matter and Complex Systems **21**, 121–128, 2001.

[20] A. Lefevre and D. S. Dean: *Tapping thermodynamics of the one-dimensional Ising model*, Journal of Physics A: Mathematical and General **34**, no. 6, L213, 2001.

[21] D. Lind and B. Marcus: *An introduction to symbolic dynamics and coding*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2021.

[22] C. Marvel and C. L. Levesque: *The structure of vinyl polymers: the polymer from methyl vinyl ketone*, Journal of the American Chemical Society **60**, no. 2, 280–284, 2001.

[23] R. Nord: *Irreversible random sequential filling of lattices by Monte Carlo simulation*, Journal of Statistical Computation and Simulation **39**, no. 4, 231–240, 1991.

[24] E. S. Page: *The distribution of vacancies on a line*, Journal of the Royal Statistical Society. Series B. Methodological **21**, 364–374, 1959.

[25] M. Puljiz, S. Šebek, and J. Žubrinić: *Combinatorial settlement planing*, To appear in Contributions to Discrete Mathematics, 2021.

[26] M. Puljiz, S. Šebek, and J. Žubrinić: *Packing density of combinatorial settlement planning models*, To appear in The American Mathematical Monthly, 2021.

[27] A. Rényi: *On a one-dimensional problem concerning space-filling*, Publ. Math. Inst. Hungar. Acad. Sci. **3**, 109–127, 1958.

[28] R. T. Rockafellar: *Convex analysis*, Princeton Landmarks in Mathematics, Reprint of the 1970 original, Princeton Paperbacks, 1997.

[29] R. P. Stanley: *Enumerative combinatorics. Volume 1*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2012.

[30] E. M. Tory, W. Jodrey, and D. Pickard: *Simulation of random sequential adsorption: Efficient methods and resolution of conflicting results*, Journal of Theoretical Biology **102**, no. 3, 439–445, 1983.

[31] J.S. Wang: *Series expansion and computer simulation studies of random sequential adsorption*, Colloids and Surfaces A: Physicochemical and Engineering Aspects **165**, no. 1–3, 325–343, 2000.

[32] H. S. Wilf: *generatingfunctionology*, Academic Press, 1990.

Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.08

Extended abstract on Local Irregularity Conjecture and cactus graphs

Jelena Sedlar and Riste Škrekovski

Abstract

A graph is locally irregular if the degrees of the end-vertices of every edge are distinct. An edge coloring of a graph G is locally irregular if every color induces a locally irregular subgraph of G. A colorable graph G is any graph which admits a locally irregular edge coloring. The locally irregular chromatic index $\chi'_{\text{irr}}(G)$ of a colorable graph G is the smallest number of colors required by a locally irregular edge coloring of G. The Local Irregularity Conjecture claims that all colorable graphs require at most 3 colors for a locally irregular edge coloring.

In this extended abstract, we summarize our results on the mentioned conjecture for the class of cactus graphs. First, we established that the conjecture holds for unicyclic graphs and cacti with vertex disjoint cycles. Then we observed that there exists a cactus graph, the so called bow-tie graph B, which is colorable and requires at least 4 colors for a locally irregular edge coloring [8]. As B is a cactus graph and all non-colorable graphs are also cacti, the class of cactus graphs seems to be a relevant for the Local Irregularity Conjecture. By further researching this class, we established that 4 colors are the maximum number of colors required by a locally irregular edge coloring of any colorable cactus graph [9]. Using the same approach, but with the more elaborated argument, we established that the bow-tie graph B is the only connected colorable cactus graph which requires 4 colors for a locally irregular edge coloring [10]. Our last result indicates that B is possibly the only connected graph with $\chi'_{\text{irr}}(B) = 4$, and consequently it could be the only graph which contradicts the present form of the Local Irregularity Conjecture.

Keywords: locally irregular edge coloring; Local Irregularity Conjecture; cactus graphs.

AMS subject classification (2020): 05C15.

(Jelena Sedlar) University of Split, Faculty of civil engineering, architecture and geodesy, Croatia, jelena.sedlar@gradst.hr

(Riste Škrekovski) University of Ljubljana, FMF, Ljubljana, Slovenia and Faculty of Information Studies, Novo Mesto, Slovenia, riste.skrekovski@fmf.uni-lj.si

1 Introduction

For all graphs mentioned in this paper it is tacitly assumed they are simple, finite and connected. A *locally irregular* graph is any graph in which the two end-vertices of every edge have distinct degrees. A *locally irregular k-edge coloring* (also called *k-liec* for short) is any k-edge coloring of G such that every color induces locally irregular subgraph of G. We say that a graph G is *colorable* if it admits a locally irregular edge coloring. For a colorable graph G, the smallest number k such that G admits a k-liec is called the *locally irregular chromatic index* of G and denoted by $\chi'_{\mathrm{irr}}(G)$. Since isolated vertices of G do not influence the local irregularity of an edge coloring, if such vertices arise in a graph by edge deletion we will ignore them. We first need to establish which graphs are colorable. For that purpose, any graph with edge disjoint cycles is called a *cactus* graph. We define a class $\mathfrak{T}$ of cacti as follows:

- the triangle K_3 belongs to $\mathfrak{T}$,
- for every graph G in $\mathfrak{T}$, a graph H which is also from $\mathfrak{T}$ is obtained as: let v be a vertex of G of degree 2 from a triangle in G, and let w be a vertex of degree one in a path of even length or in a graph consisting of a triangle and an odd-length path pending on one vertex of the triangle, then identifying v and w yields H.

Obviously, every graph G in $\mathfrak{T}$ is a special kind of cactus graph in which every cycle is a triangle, all triangles are vertex disjoint, and all vertices in G are of degree ≤ 3. Also, every vertex of degree 3 belongs to a triangle, a pair of triangles is connected by a path of odd length and there may be paths of even length pending at a vertex of a triangle. It is known [1] that paths of odd length, cycles of odd length and cactus graphs from $\mathfrak{T}$ are the only non-colorable connected graphs. As for colorable graphs, the following conjecture was proposed [1].

Conjecture 1 (Local Irregularity Conjecture)**.** *For every colorable connected graph G, $\chi'_{\mathrm{irr}}(G) \leq 3$.*

Many partial results support this conjecture, the conjecture is true for trees [2], graphs with vertex disjoint cycles [8], graphs in which the minimum degree is at least 10^{10} [7], k-regular graphs for $k \geq 10^7$ [1]. For general graphs, the first constant upper bound on $\chi'_{\mathrm{irr}}(G)$ that was found is 328 [4], and then it was decreased to 220 [6].

2 Preliminaries

Let us introduce some basic notions and notation related to colorings, which are already introduced in literature, mainly in [2].
The number of a-colored edges incident to a vertex u is called the *a-degree* of u and it is denoted by $d_G^a(u)$. Similarly we have b-degree, c-degree, etc. The *a-sequence* of a vertex $u \in V(G)$ is defined as $d_G^a(v_1), \ldots, d_G^a(v_k)$ where $v_1, \ldots, v_k$ are all neighbors

of u. We usually assume that neighbors v_i of u are denoted so that the a-sequence is non-increasing.
For a graph G and a subgraph G_0 of G, a k-liec ϕ of G is said to be an *extension* of k-liec ϕ^0 of G_0 if $\phi(e) = \phi^0(e)$ for every $e \in E(G_0)$.
Let G be a colorable graph which admits a k-liec for $k \geq 3$. If G contains a leaf u, then a graph G' obtained from G by appending an even length path P at u also admits a k-liec. An *ear* of a graph G is any subgraph P of G such that P is a path and for any internal vertex $u \in V(P)$ it holds that $d_G(u) = 2$. If G contains an ear $P_q = u_0u_1 \dots u_q$ of length $q \geq 3$, then at least one internal vertex u_i of P_q is bichromatic by a k-liec of G, so if G' is obtained from G by replacing an ear P_q by an ear P_{q+2r} of the length $q + 2r$, then G' also admits a k-liec. In both situations we say G is obtained by *trimming* G'.

Remark 1. Let G' be a colorable graph which admits a k-liec for $k \geq 3$. If G is obtained by trimming G', then G also admits a k-liec.

A graph G is *totally trimmed* if it does not contain a pending path of length ≥ 3 and an ear of length ≥ 5. Because of Observation 1, in the rest of the paper we will tacitly assume every graph G is totally trimmed.
A *shrub* is any tree rooted at its leaf. An *almost locally irregular k-edge coloring* of a shrub G, or *k-aliec* for short, is any k-liec of G or any edge coloring of G such that only the edge incident to the root is locally regular. The following are useful results for trees from [2].

Theorem 1. *Every shrub admits a* 2*-aliec.*

Theorem 2. *Every colorable tree T satisfies $\chi'_{\mathrm{irr}}(T) \leq 3$. Moreover, if $\Delta(T) \geq 5$, then $\chi'_{\mathrm{irr}}(T) \leq 2$.*

Let T be a tree with $\chi'_{\mathrm{irr}}(T) = 3$ and ϕ a 3-liec of T. If there is precisely one vertex u in T which is incident to edges of all three colors, and every shrub rooted at u is colored by at most two colors, than ϕ is called a *special* coloring and u is called the *rainbow root* of ϕ. The proof of Theorems 1 and 2 from [2] implies the following observation.

Remark 2. Let T be a colorable tree with $\chi'_{\mathrm{irr}}(T) = 3$. Every vertex of maximum degree in T is a neighbor of another vertex of maximum degree in T. Let u be a vertex of maximum degree in T, then there exists a special 3-liec of T for which u is the rainbow root. Also, for any shrub T_i of T rooted at u, there exists a 3-liec of T which uses color c only in T_i.

3 Our results

In this section we will present our results on unicyclic graphs and cacti from [8], [9] and [10]. Since the Local Irregularity Conjecture holds for trees, we first investigated unicyclic graphs given that they are obtained from trees by introducing a single edge. In [8] the following result is established.

Theorem 3. *Every colorable unicyclic graph G sattisfies $\chi'_{\mathrm{irr}}(G) \leq 3$.*

A natural question that arises from this result is whether the bound $\chi'_{\mathrm{irr}}(G) \leq 3$ is tight, i.e. are there colorable unicyclic graphs which are not 2-colorable. The family of cycles of length $4k+2$ are such graphs, but this family is not an isolated case, there exist other unicyclic graphs which require three colors, for example the graph from Figure 1. One can assure infinitely many such graphs for example by taking longer threads of suitable parity in the given graph.

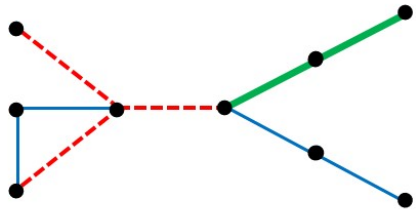


Figure 1: A colorable unicyclic graph distinct from cycle which requires 3 colors for locally irregular edge coloring.

The approach for unicyclic graphs usually extends to cactus graphs, so we further investigated that class of graphs. Let G be a cactus graph with at least two cycles, let C be a cycle in G and let u be a vertex from C. We say that u is a *root vertex* of C if the connected component of $G - E(C)$ which contains u is a cyclic graph. A cycle C of G is a *proper end-cycle* if $G - V(C)$ contains at most one cyclic connected component. Every cactus graph with vertex disjoint cycles contains at least two proper end-cycles, given it is not a unicyclic graph. Focusing the attention to a proper end-cycle of a cactus graph with vertex disjoint cycles, we obtained the following result [8].

Theorem 4. *Every colorable cactus graph G with vertex disjoint cycles satisfies $\chi'_{\mathrm{irr}}(G) \leq 3$.*

The attempt to extend the approach to cacti in which cycles may share a vertex yielded the so called bow-tie graph B, illustrated by Figure 2, and the following result.

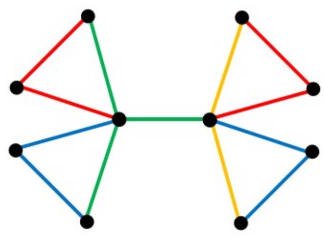


Figure 2: The bow-tie graph B and a locally irregular 4-edge coloring of it.

Proposition 1. *For the bow-tie graph B, it holds that $\chi'_{\mathrm{irr}}(B) = 4$.*

Proof. The edge coloring of the graph B shown in Figure 2 is locally irregular and it uses 4 colors, thus $\chi'_{\text{irr}}(G) \leq 4$. It remains to establish that B does not admit a locally irregular coloring with less than 4 colors. Let u and v denote two vertices of degree 5 in B. Let $u_1, \ldots, u_4$ (resp. $v_1, \ldots, v_4$) be the four vertices neighboring u (resp. v) distinct from v (resp. u), so that u_1 and u_2 (resp. v_1 and v_2) belong to a same triangle of B. Let ϕ be a locally irregular edge coloring of B which uses less than 4 colors. Notice that $\phi(u_1u_2)$ must be equal to precisely one of $\phi(u_1u)$ and $\phi(u_2u)$, we may assume $\phi(u_1u_2) = \phi(u_1u) \neq \phi(u_2u)$. Similarly, we may assume $\phi(u_3u_4) = \phi(u_3u) \neq \phi(u_4u)$. Let us denote $a = \phi(uv)$.
If $\phi(u_1u_2) = \phi(u_3u_4)$, then it must hold $a = \phi(u_1u_2) = \phi(u_3u_4)$, otherwise ϕ would not be locally irregular. Thus, the a-degree of u by ϕ equals 3.
If $\phi(u_1u_2) \neq \phi(u_3u_4)$, then it must hold $\phi(u_2u) = \phi(u_4u)$, otherwise ϕ would use at least 4 colors. Notice that $\phi(u_1u_2) = a$ or $\phi(u_3u_4) = a$ would imply ϕ is not a liec, thus it must be $a = \phi(u_2u) = \phi(u_4u)$, so again we obtain the a-degree of u by ϕ equals 3.
The analogous analisys of the edges in the two triangles of B containing v yields that a-degree of v also equals 3. Since uv is an edge in B colored by a and a-degree of both u and v equals 3, this is a contradiction with ϕ being locally irregular. $\square$

The above result implies that B contradicts the Local Irregularity Conjecture. This gives rise to the following questions: are 4 colors now enough for a liec of all colorable graphs, and are there other graphs beside B which require 4 colors. Since the class of cactus graphs yielded the first known counterexample, we attempted to answer these two questions for that class.
To establish that all colorable cacti admit a 4-liec, we need the following definitions. A *grape* G is any cactus graph with at least one cycle in which all cycles share a vertex u, and the vertex u is called the *root* of G. A *berry* G_i of a grape G is any subgraph of G induced by $V(G'_i) \cup \{u\}$, where u is the root of G and G'_i a connected component of $G - u$. Notice that a berry G_i can be either a unicyclic graph in which u is of degree 2 or a tree in which u is a leaf, so such berries will be called *unicyclic berries* and *acyclic berries*, respectively. A unicyclic berry G_i is said to be *triangular* if its cycle is the triangle.
An *end-grape* G_u of a cactus graph G is any subgraph of G such that:

- G_u is a grape rooted at u where u is the only vertex of G_u incident to edges from $G - E(G_u)$, and
- u is incident to either one edge from $G - E(G_u)$ or two such edges which then must belong to a same cycle of $G - E(G_u)$, and such edges are called the *exit edges* of G_u.

This notion is illustrated by Figure 3. Also, for an end-grape G_u rooted at u, the graph $G_0 = G - E(G_u)$ will be called the *root component* of G_u. In [9], we established the following theorem.

Theorem 5. *Every colorable cactus graph G satisfies $\chi'_{\text{irr}}(G) \leq 4$.*

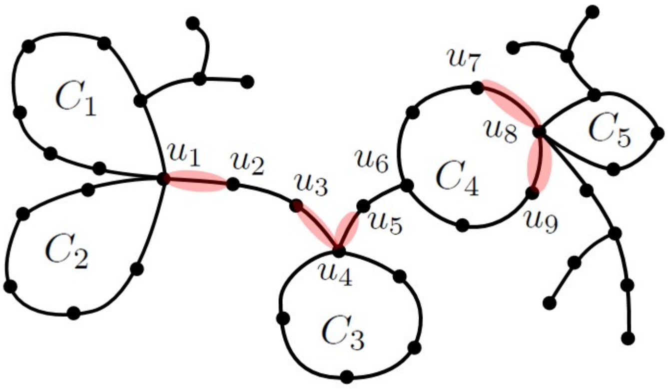


Figure 3: A cactus graph G with five cycles which contains two end-grapes, G_{u_1} and G_{u_8}. The end-grape G_{u_1} has two unicyclic berries and one exit edge u_1u_2. The end-grape G_{u_8} consists of one unicyclic and one acyclic berry and it has two exit edges u_8u_7 and u_8u_9. Notice that the cycle C_3 is not an end-grape of G since it has two exit edges u_4u_3 and u_4u_5 and they do not belong to the same cycle.

The result of Theorem 5 is obtained by induction on the number of cycles in G, using Theorem 3 as the basis of the induction. This approach enables one to assume that the root component G_0 of an end-grape G_u is already colored by a locally irregular edge coloring and then one has to extend such a coloring to the end-grape G_u by using at most 4 colors in all. This is done by removing precisely one edge incident to the root vertex u from every unicyclic berry of G_u, denote such set by E_u. Notice that $G_u - E_u$ becomes a tree and thus admits a 3-liec in colors a, b and c, the edges of E_u can then be colored by the fourth color d. The problem with local irregularity can thus arise only on edges incident to u in G. This is avoided by carefuly chosing the coloring of the tree $T = G_u - E_u$ using Observation 2.
This result can be further extended to a claim that every colorable cactus graph distinct from B requires at most three colors for the locally irregular edge coloring. Our argument of this claim is lengthy but uses the same approach as Theorem 5. The main difference is that in proving Theorem 5, no special attention needs to be given to the a- and b-degrees of the neighbors of u in T since there is fourth color d to use it for at least one of the two edges incident to u in G_0. When the fourth color must not be used, then a great care has to be taken of these a- and b-degrees in T because the same colors must be used for both edges incident to u in G_0. So, one has to avoid colors a and b for edges incident to u in G_u to spare them for G_0. That is not always possible, so special berries and alternative colorings for them need to be introduced, which is done in [10].

Primary coloring. A berry coloring by which all edges are incident to u have a same color, say c, is a desirable coloring since it is very convenient when gathering berries in a grape. For that purpose, it is also desirable that the c-degree of the neighbors of u is ≤ 2.

Definition 1. *Let G be a graph rooted at a vertex u with $d_G(u) \in \{1, 2\}$. Let v be* *a neighbor of u and w the other neighbor of u if $d_G(u) = 2$. A* standard primary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G with the following properties:*

- *every edge non-incident to u is locally irregular by $\phi_{a,b,c}$;*
- *u is monochromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(u) = \{c\}$;*
- *$d^c(v) \leq 2$ and $d^c(w) \leq 2$.*

It would be convenient to use a standard primary coloring for all berries, but berries $B_1, \ldots, B_7$ from Figure 4 (dashed edges are not included in a berry) do not admit such a coloring, so these berries are called *alternative berries*. For such berries, we introduce an alternative primary coloring as follows.

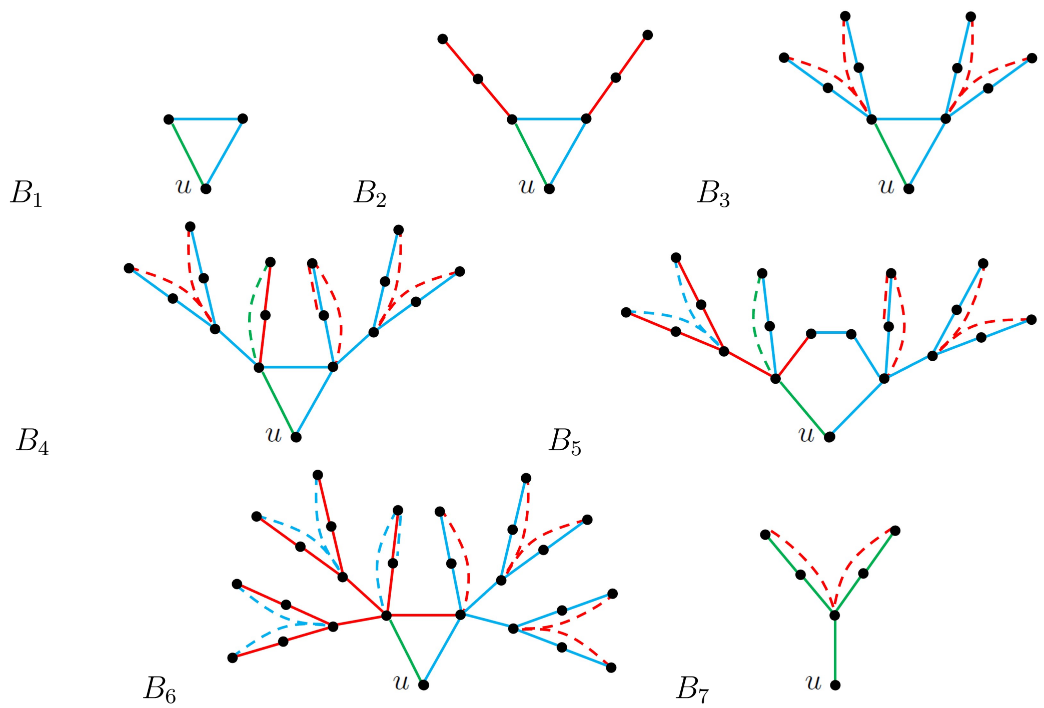


Figure 4: Alternative berries $B_1, \ldots, B_7$ rooted at u. Dashed edges are not included in the berry.

Definition 2. *Let G be a graph rooted at a vertex u with $d_G(u) \in \{1, 2\}$. Let v be a neighbor of u and w the other neighbor of u if $d_G(u) = 2$.*

- *If $d_G(u) = 1$, an* alternative primary coloring *of G is any 3-liec $\phi_{a,b,c}$ of G such that v is monochromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(v) = \{c\}$.*
- *If $d_G(u) = 2$, an* alternative primary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G with the following properties:*
  - *every edge $e \neq uv$ is locally irregular by $\phi_{a,b,c}$;*

- *u is bichromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(uv) = c$ and $\phi_{a,b,c}(uw) = a$;*
- *$d^c(v) \leq 2$ and $d^a(w) \in \{2, 4\}$;*
- *if $d^c(v) = 2$ then $d^a(w) = 4$.*

Alternative primary coloring of alternative berries is illustrated by Figure 4, all other berries admit a standard primary coloring [10]. In the sequel we will say shortly 'primary coloring', assuming standard primary coloring for standard berries and alternative primary coloring for alternative berries.

Secondary and tertiary coloring. So far we have one kind of coloring for every berry - a primary coloring. When we gather berries in an end-grape, if the problem with local irregularity arises for edges incident to the root u, it is convenient to have another kind of coloring for every berry, so that the change of coloring for some berries changes the color degrees of u and its neighbors. For standard berries, another kind of coloring is a so called secondary coloring, and for alternative berries it is tertiary coloring.

Definition 3. *Let $G \neq B_7$ be a graph rooted at a vertex u with $d_G(u) \in \{1, 2\}$. Let v be a neighbor of u and w the other neighbor of u if $d_G(u) = 2$.*

- *If $d_G(u) = 1$, a* secondary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G such that all edges not incident to u are locally irregular and $d^c(v) \neq 2$ where c is the color of uv by $\phi_{a,b,c}$.*
- *If $d_G(u) = 2$, a* secondary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G with the following properties:*
  - *every edge $e \neq uv$ is locally irregular by $\phi_{a,b,c}$;*
  - *u is bichromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(uv) = c$ and $\phi_{a,b,c}(uw) = a$;*
  - *$d^c(v) \leq 2$;*
  - *if $d^c(v) = 2$ then $d^a(w) \geq 3$.*

We exclude B_7 from the above definition, since we always want an alternative primary coloring of it.

Definition 4. *Let G be a graph rooted at a vertex u with $d_G(u) = 2$. Let v and w be the neighbors of u. A* tertiary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G with the following properties:*

- *every edge $e \neq uv$ is locally irregular by $\phi_{a,b,c}$;*
- *u is monochromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(u) = \{c\}$;*
- *$d^c(v) \leq 2$ and $d^c(w) \in \{3, 4\}$.*

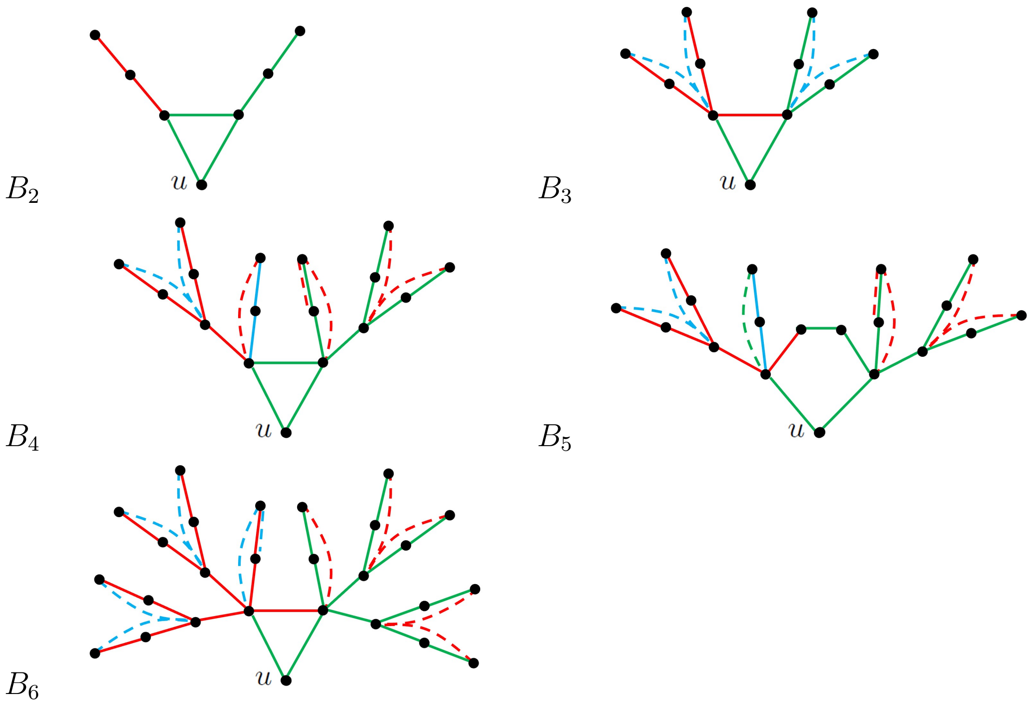


Figure 5: A tertiary coloring of berries $B_2, \ldots, B_6$ and its closures.

A tertiary coloring of $B_2, \ldots, B_6$ is shown in Figure 5 (without dashed lines). Notice that B_1 is the only berry which admits only one coloring.

Even with all these kinds of colorings, it turns out that the coloring of the root component G_0 cannot be extended to all possible end-grapes. To be more precise, there are six end-grapes to which it is not always possible to extend a coloring of G_0, we denote them by $A_1, \ldots, A_6$ and they are shown in Figure 6. For all other end-grapes, the following proposition [10] holds (here G_0' denotes a graph obtained from G_0 by adding a leaf neighbor to u).

Proposition 2. *Let G be a cactus graph with $\mathfrak{c} \geq 2$ cycles which is not a grape and which does not contain end-grapes $A_1, \ldots, A_6$. For an end-grape G_u of G with a root component G_0, there exists a colorable graph $\tilde{G}_0 \in \{G_0, G_0'\}$ such that every k-liec of $\tilde{G}_0$, for $k \geq 3$, can be extended to k-liec of G in which every berry of G_u is colored by a primary, secondary or tertiary coloring.*

If a cactus graph G contains only end-grapes $A_1, \ldots, A_6$, then by deleting an edge in every triangle of such end-grapes one can obtain a tree, a unicyclic graph or a cactus graph G^{op} which contains an end-grape distinct from $A_1, \ldots, A_6$. A tree and a unicyclic graph admit a 3-liec, and for G^{op} the induction hypothesis and Proposition 2 assure the existance of a 3-liec in which every berry is colored by a primary, secondary od tertiary coloring. Thus, it only remains to establish that a 3-liec of G^{op} can be extended to a 3-liec of G, i.e. to the removed edges of triangles in end-grapes of G. Since every berry of G^{op} is colored by a primary, secondary

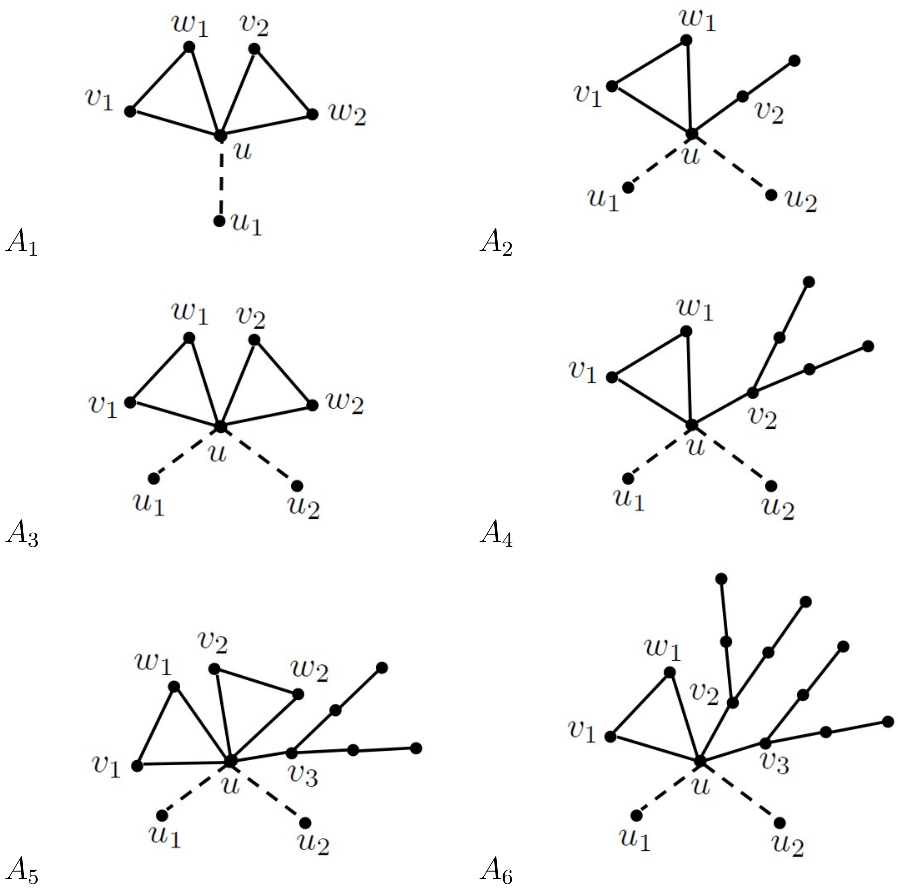


Figure 6: The singular end-grapes $A_1, \dots, A_6$.

or tertiary coloring, the question reduces to whether such berry colorings can be extended to the removed edges of G. The extension of alternative primary and tertiary coloring is shown in Figures 4 and 5, and the existence of the extension for a standard primary and a secondary coloring of standard berries is established in [10]. This finally yields the following result [10].

Theorem 6. *Let $G \neq B$ be a colorable cactus graph. Then $\chi'_{\mathrm{irr}}(G) \leq 3$.*

Our investigation of the Local Irregularity Conjecture makes us believe that B is the only graph with $\chi'_{\mathrm{irr}}(G) \geq 4$, so the following conjecture is proposed in [10].

Conjecture 2. *The bow-tie graph B is the only colorable connected graph with $\chi'_{\mathrm{irr}}(B) > 3$.*

Acknowledgments

Both authors acknowledge partial support of the Slovenian research agency ARRS program P1-0383 and ARRS project J1-1692. The first author also the support of Project KK.01.1.1.02.0027, a project co-financed by the Croatian Government and the European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme.

References

[1] O. Baudon, J. Bensmail, J. Przybyło, M. Woźniak, On decomposing regular graphs into locally irregular subgraphs, *Eur. J. Combin.* **49** (2015) 90–104.

[2] O. Baudon, J. Bensmail, É. Sopena, On the complexity of determining the irregular chromatic index of a graph, *J. Discret. Algorithms* **30** (2015) 113–127.

[3] J. Bensmail, F. Dross, N. Nisse, Decomposing degenerate graphs into locally irregular subgraphs, *Graphs Combin.* **36** (2020) 1869–1889.

[4] J. Bensmail, M. Merker, C. Thomassen, Decomposing graphs into a constant number of locally irregular subgraphs, *Eur. J. Combin.* **60** (2017) 124–134.

[5] H. Lei, X. Lian, Y. Shi, R. Zhao, Graph classes with locally irregular chromatic index at most 4, *J. Optim. Theory Appl.* **195** (2022) 903–918.

[6] B. Lužar, J. Przybyło, R. Soták, New bounds for locally irregular chromatic index of bipartite and subcubic graphs, *J. Comb. Optim.* **36** (2018) 1425–1438.

[7] J. Przybyło, On decomposing graphs of large minimum degree into locally irregular subgraphs, *Electron. J. Combin.* **23** (2016) 2–31.

[8] J. Sedlar, R. Škrekovski, Remarks on the Local Irregularity Conjecture, *Mathematics* **9(24)** (2021) 3209.

[9] J. Sedlar, R. Škrekovski, A note on the locally irregular edge colorings of cacti, *Discrete Math. Lett.* **11** (2023) 1–6.

[10] J. Sedlar, R. Škrekovski, Local Irregularity Conjecture vs. cacti, arXiv:2207.03941 [math.CO].

Proceedings of the 4^{th} Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.09

On circumradius equations of cyclic polygons

Dragutin Svrtan

Abstract

Finding formulas for the area or circumradius of polygons inscribed in a circle in terms of side lengths is a classical subject. For the area of a triangle we have the famous Heron's formula and for cyclic quadrilaterals we have the Brahmagupta's formula. A three decades ago D. P. Robbins found the minimal equations of degree 7 satisfied by the squared area of cyclic pentagons and hexagons by a method of undetermined coefficients and he wrote the result in a nice compact form. For the circumradius of cyclic pentagons and hexagons he did not publish the formulas because he was not able to put them into a compact form (in this paper we describe our compact form also for a heptagon and octagon). The Robbins approach could hardly be used for heptagons due to computational complexity of the approach (leading to a system with 143307 equations). In another approach with two collaborators a concise heptagon/octagon area formula was obtained in 2004. (not long after D. P. Robbins premature death) in the form of a quotient of two resultants (the quotient still hard to be written explicitly because it would have about one million terms–this approach uses covariants of binary quintics). It is not clear if this approach could be effectively used for cyclic polygons with nine or more sides. A nice survey on this and other Robbins conjectures is written by I. Pak. In this paper we shall explain a simple quadratic system, which seems to be new, for the circumradius and area of arbitrary cyclic polygons based on a Wiener-Hopf factorization of our new Laurent polynomial invariant of cyclic polygons. Explicit formulas, of degree 38, for the squared circumradius (and less explicit for the squared area) of cyclic heptagons /octagons are obtained. By solving our system in certain algebraic extensions we found a compact form of our circumradius heptagon/octagon formulas with remarkably small coefficients. In 2005. we have presented an intrinsic proof of the Robbins formulas for the area (and also for the circumradius and area times circumradius) of cyclic hexagons based on an intricate direct elimination of diagonals (the case of pentagon was treated in Ref. [7]) and using a new algorithm from Ref. [11]. In the early stage we used computations with MAPLE (which sometimes lasted several days!).

(Dragutin Svrtan) Faculty of Science, Department of Mathematics, Zagreb, Croatia, dragutin.svrtan@gmail.com

1 Introduction

Cyclic polygons are the polygons inscribed in a circle. In terms of their side lengths $a_1, a_2, \ldots, a_n$, their area S and circumradius r are given in case of triangles and quadrilaterals explicitly by the following well known formulas: the Heron's formula (60 B.C.) for the area and the circumradius r of triangles (by letting $A = (4S)^2, \rho = 1/r^2$) :

$$\begin{aligned} &A - (a+b+c)(a+b-c)(a-b+c)(-a+b+c) = 0, \\ &a^2b^2c^2\rho - (a+b+c)(a+b-c)(a-b+c)(-a+b+c) = 0 \end{aligned} \tag{1}$$

and the Brahmagupta's formula, (7 th c. A.D.) for the area and the circumradius of convex ($\varepsilon = 1$) and nonconvex ($\varepsilon = -1$) quadrilaterals:

$$A_\varepsilon - (a+b+c-\varepsilon d)(a+b-c+\varepsilon d)(a-b+c+\varepsilon d)(-a+b+c+\varepsilon d) = 0, \tag{2}$$

$$\begin{aligned} &(ab+\varepsilon cd)(ac+\varepsilon bd)(bc+\varepsilon ad)\rho_\varepsilon - \\ &(a+b+c-\varepsilon d)(a+b-c+\varepsilon d)(a-b+c+\varepsilon d)(-a+b+c+\varepsilon d) = 0. \end{aligned} \tag{3}$$

(Note that for $\varepsilon = 0$ (or $d = 0$) Brahmagupta's formula transforms into Heron's formula.) In a masterfully written (in german language) thirty pages long paper (and published in 1828 in Crelle's Journal) A. F. Möbius studied some properties of the polynomial equations for the circumradius of arbitrary cyclic polygons (convex and nonconvex) and produced a polynomial of degree $\delta_n = \frac{n}{2}\binom{n-1}{\lfloor (n-1)/2 \rfloor} - 2^{n-2}$ that relates the square of a circumradius (r^2) of a cyclic polygon to the squared side lengths. He also showed that the squared area rationally depends on $r^2, a_1, a_2, \ldots, a_n$. His approach is based, by a clever use of trigonometry, on the rationalization (in terms of the squared sines) of the sine of a sum of n angles (peripheral angles of a cyclic polygon). In this way one obtains a polynomial relating the circumradius to the side lengths squared. These polynomials, known also as generalized Heron r-polynomials, are a kind of generalized (symmetric) multivariable Chebyshev polynomials and are quite difficult to be computed explicitly. Möbius obtained nice form for the leading and constant terms for pentagons and hexagons, but no complete answer even for pentagons. By an argument involving series expansions (cf. [8]) he proved that the r^2-degree for cyclic n-polygon is equal to δ_n. In the final part of the paper he obtained for the squared area a rational function in $r^2, a_1, a_2, \ldots, a_n$ involving partial derivatives, with respect to side length variables, of all the coefficients of the Heron r-polynomial. So, in principle, one could get from this formula the area polynomial by using Viete formulas together with a heavy use of symmetric functions.
About thirty years ago David Robbins ([3, 4]) obtained, for the first time, concise explicit formulas for the areas of cyclic pentagons and hexagons (he mentioned that he computed also the circumradius polynomials for cyclic pentagons and hexagons but was not able to put either formula into a sensible compact form). In [3] two general conjectures (Conjecture1 and Conjecture2), naturally extending nice Möbius product formulas for the leading and constant terms for pentagons and hexagons are given. We shall give a verification of these conjectures up to $n = 8$.
One of the Additional Conjectures of Robbins, stating that the degree of the minimal A-polynomial equation for cyclic n-polygons $\alpha_n\left(16S^2, a_1^2, \ldots, a_n^2\right) = 0$, (i.e. of the

generalized Heron $A-$ polynomial), is equal to δ_n was established in [FP] first (by relating it to the Sabitov theory of volume polynomials of polyhedra, see nice survey article by Pak) and later in [8] (obtained by reviving the argument of Möbius and reproving the Robbins lower bound on the degrees of minimal polynomials, c.f. [17]). In Robbins work a method of undetermined coefficients is used for pentagons (70 unknowns) and hexagons (134 unknowns). This method seems to be inadequate for heptagons because one would need to handle a linear system with 143307 undetermined coefficients. By using a clever substitution (Robbins t_i 's) he was able to write the pentagon and hexagon area equations in a compact form. He wrote his formulas also as a discriminant of some (still mysterious) qubic. Along these lines in [8] it is found that for $(2m+1)$-gon or $(2m+2)-$ gon, the generalized Heron A-polynomial is the defining polynomial of a certain variety of binary $(2m-1)$-forms with $m-1$ double roots (in some sense it demystify Robbins cubic but its role is still mysterious). In [8] a formula for the area polynomial for heptagons and octagons is found in the form of a quotient of two resultants, one of which could be expanded explicitly so far. This exiting result was finished by two of the Robbins collaborators just few months later after Robbins passed away.
Another approach, which uses elimination of diagonals in cyclic polygons, is treated at length in [5] where among numerous results one also finds an explicit derivation of the Robbins area polynomial for pentagons by using some general properties, developed in that paper, together with a little use of one undetermined coefficient. Independently in [7], where an almost forgotten elegant Gauss quadratic pentagon area equation is revived, the Robbins pentagon area formula was obtained with a simpler system of equations by a direct elimination (and MAPLE of course) with no assistance of undetermined coefficient method. In [7] also the circumradius and the area times circumradius formulas for pentagons, in terms of symmetric functions of the side lengths squared, are explicitly computed. The diagonal elimination approach seems to be better suited for circumradius computations than for the area computations. By introducing diagonals into play the original side length variables are separated into groups (symmetry breaking) and, after eliminating diagonals, one needs to use immense computations with symmetric functions to regain the symmetry. In [11,18] we have designed an algorithm, which generalizes the basic algorithm for writing symmetric functions in terms of the elementary symmetric functions, which does not expresses everything in terms of the original variables. Instead it goes only down to the level of symmetric functions of the partial alphabets and leads to global symmetric function expansion. This enabled us to get r-polynomials for hexagons (and hopefully more in the future).
In this paper we illustrate yet another approach to the Robbins problem, especially well suited for obtaining Heron r-polynomials. We have discovered that Robbins problem is somehow related to a Wiener-Hopf factorization. We first associate a Laurent polynomial L_P to a cyclic polygon P, which is invariant under similarity of cyclic polygons (it is a kind of "conformal invariant"). Then there exists a (Wiener-Hopf) factorization of L_P into a product of two polynomials, $\gamma_+(1/z)$ and $\gamma_-(z)$, (in our case it will be $\gamma_- = \gamma_+ =: \gamma$) providing a complex realization of $\mathbf{P}$ is given. The factorization (i.e. $\gamma(z)$) is then given in terms of the elementary symmetric functions e_k of the vertex quotients, if we regard vertices of (a realization of) P as complex

numbers of equal moduli ($= r$). For (e_k)'s, viewed as the unknowns, we then obtain a system of n quadratic equations, arising from our Wiener-Hopf factorization, with $n-1$ unknowns (note that e_n is necessarily equal to 1 as a product of all the vertex quotients (we call this a "cocycle property" or simply "cocyclicity")). The consistency condition (obtained by eliminating all $e_k, k = 1..n-1$) for our "overdetermined" system will then give a relation between the coefficients of our conformal invariant L_P, which in turn will be nothing but the equation relating the inverse square radius of P with the elementary symmetric polynomials in the squares of the sides.
During of these investigations we found another type of substitutions by expressing the coefficients of L_P in terms of the inverse radius squared (ρ) and the elementary symmetric functions of side lengths squared. By using this substitutions, our Heron ρ-polynomials get remarkably small coefficients. Further simplifications we have obtained by doing computations in some quadratic algebraic extensions. In such quadratic extensions we can simplify our original system (having all but one equations quadratic) by replacing two quadratic equations by two linear ones). Also the final result can be written in a more compact form $\rho_n = A_n^2 - \Delta_n B_n^2$ (a Pell equation). Thus the number of terms is the final formula is roughly a square root of the number of terms in the fully expanded formula. With such tricks we have obtained so far, down to earth, explicit formulas for Heron ρ-polynomials, up to $n = 8$.

2 Equations for cyclic polygons via Wiener-Hopf factorization

Assume that a cyclic polygon P has its vertices on a circle centered at the origin in the complex plane. Suppose that these vertices are in order $v_1, \ldots, v_n$ and that the radius of the circle is r. Also let $v_{n+1} = v_1$ and define the vertex quotients by

$$q_j = \frac{v_{j+1}}{v_j}. \tag{4}$$

The geometric meaning of these vertex quotients are $q_j = \cos\varphi_j + I\sin\varphi_j = e^{I\varphi_j}$, where φ_j denotes the central angle $\sphericalangle(v_j O v_{j+1})$ of P. Then we have the following Cocycle identity:

$$\prod_{j=1}^{n} q_j = 1. \tag{5}$$

The side lengths a_j ($=$ the distance from v_j to v_{j+1}) of P are given by

$$\begin{aligned} a_j^2 &= |v_j - v_{j+1}|^2 = (v_j - v_{j+1})\overline{(v_j - v_{j+1})} = r^2\left(2 - \frac{v_{j+1}}{v_j} - \frac{v_j}{v_{j+1}}\right) \\ &= r^2\left(2 - \left(q_j + q_j^{-1}\right)\right). \end{aligned} \tag{6}$$

Now we associate to a cyclic polygon P, with side lengths $a_1, \ldots, a_n$, a Laurent polynomial $L_P(z)$ defined by the following formula:

$$L_P(z) := \prod_{j=1}^{n} \left(z + z^{-1} + 2 - a_j^2 \rho \right) \in \mathbb{C}\left[z, z^{-1} \right], \tag{7}$$

where $\rho = 1/r^2$ denotes the squared curvature of the circle circumscribed to P.
Note that this polynomial is a conformal invariant in the sense that if cyclic polygons P_1 and P_2 are similar, then $L_{P_1}(z) = L_{P_2}(z)$.
Basic notations:
Denote by e_k the elementary symmetric functions of $q_1, \ldots, q_n$ (vertex variables):

$$1 + e_1 t + e_2 t^2 + \cdots + e_n t^n = \prod_{j=1}^{n} (1 + q_j t) \tag{8}$$

and by ε_k the elementary symmetric functions of $a_1^2, \ldots, a_n^2$ (side lengths squared):

$$1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots + \varepsilon_n t^n = \prod_{j=1}^{n} \left(1 + a_j^2 t \right). \tag{9}$$

Lemma 1. *(Additive form of L_P). We have*

$$L_P(z) = \sum_{-n \le k \le n} \lambda_k z^k = \lambda_0 + \sum_{k=0}^{n} \lambda_k \left(z^k + z^{-k} \right), \tag{10}$$

where

$$\lambda_{-k} = \lambda_k = \sum_{i=k}^{n} \binom{2i}{i-k} (-1)^{n-i} \varepsilon_{n-i} \rho^{n-i} \quad (0 \le k \le n). \tag{11}$$

(Note that $\lambda_N = \lambda_{-n} = 1$.)

Proof. We compute

$$\begin{aligned}
L_P(z) &= \prod_{j=1}^{n} \left(z + z^{-1} + 2 - a_j^2 \rho \right) = \prod_{j=1}^{n} \left((1+z)^2 z^{-1} - a_j^2 \rho \right) \\
&= \sum_{0 \le i \le n} (1+z)^{2i} z^{-i} e_{n-i} \left(a_1^2, \ldots, a_n^2 \right) (-\rho)^{n-i} \\
&= \sum_{0 \le i \le n} \left(\sum_{0 \le j \le 2i} \binom{2i}{j} z^{i-j} \varepsilon_{n-i} (-\rho)^{n-i} \right) \\
&= \sum_{0 \le i \le n} \binom{2i}{i} \varepsilon_{n-i} (-\rho)^{n-i} + \sum_{1 \le k \le n} \left(\textstyle\sum_{k \le i \le n} \binom{2i}{i-k} \varepsilon_{n-i} (-\rho)^{n-i} \right) \left(z^k + z^{-k} \right).
\end{aligned}$$

By equating the coefficients the result follows. ☐

If we know the vertex coordinates $v_1, \ldots, v_n$ of P then in terms of the vertex quotients $q_j = v_{j+1}/v_j$ we can factor its Laurent polynomial L_P into a product of two polynomials, one in z and the other in z^{-1}.

Lemma 2. *(Multiplicative form of L_P) We have*

$$L_P(z) = \gamma\left(z^{-1}\right)\gamma(z), \tag{12}$$

where $\gamma(z)$ is the following polynomial

$$\gamma(z) = 1 + e_1 z + e_2 z^2 + \cdots + e_n z^n \tag{13}$$

with $e_1, \ldots, e_n$ denoting the elementary symmetric functions of vertex quotients $q_1, \ldots, q_n$ of the cyclic polygon P (note that $e_n = q_1 \cdots q_n = 1$).

Proof. We apply the identity

$$z + z^{-1} + q + q^{-1} = q^{-1}\left(1 + qz^{-1}\right)(1 + qz) \tag{14}$$

to each factor of the defining formula (7) of $L_P(z)$ and then use the cocycle identity (7). □

By combining both Lemma 1 and Lemma 2 we obtain the following

Theorem 1. *The quantities $e_0 = 1, e_1, e_2, \ldots, e_{n-1}, e_n = 1$, associated to a cyclic polygon P, defined by (8) satisfy the following quadratic system of equations:*

$$\sum_{j=0}^{k} e_{k-j} e_{n-j} = c_k, \quad k = 1..n, \tag{15}$$

or more explicitly:

$$\begin{gathered}
e_1 + e_{n-1} = c_1, \\
e_2 + e_1 e_{n-1} + e_{n-2} = c_2, \\
\vdots \\
e_{n-1} + e_{n-2} e_{n-1} + \cdots + e_1 e_2 + e_1 = c_{n-1}, \\
1 + e_1^2 + e_2^2 + \cdots + e_{n-1}^2 + 1 = c_n
\end{gathered} \tag{15'}$$

with $c_k = \lambda_{n-k}$, where the lambda's are defined by (11).

Proof. By comparing the coefficients of $z^{n-1}, z^{n-2}, \ldots, z, 1$ in the factorization resulting Lemma 1 and Lemma 2 which explicitly looks as:

$$\begin{gathered}
\left(1 + \frac{e_1}{z} + \frac{e_2}{z^2} + \cdots + \frac{e_n}{z^n}\right)\left(1 + e_1 z + e_2 z^2 + \cdots + e_n z^n\right) = \\
c_n + c_{n-1}\left(z + z^{-1}\right) + c_{n-2}\left(z^2 + z^{-2}\right) + \cdots + c_0\left(z^n + z^{-n}\right)
\end{gathered}$$

and using that $e_0 = e_n = 1$. □

Example 1. *For $n = 3$ we get the following system:*

$$\begin{aligned}
&e_1 + e_2 = c_1 \\
&e_2 + e_1 e_2 + e_1 = c_2 \\
&e_1^2 + e_2^2 + 2 = c_3
\end{aligned} \tag{Eq3}$$

with

$$c_1 = \sum_{i=2}^{3} \binom{2i}{i-2} (-1)^{3-i} \varepsilon_{3-i} \rho^{3-i} = -\varepsilon_1 \rho + 6,$$
$$c_2 = \sum_{i=1}^{3} \binom{2i}{i-1} (-1)^{3-i} \varepsilon_{3-i} \rho^{3-i} = \varepsilon_2 \rho^2 - 4\varepsilon_1 \rho + 15, \qquad (C3)$$
$$c_3 = \sum_{i=0}^{3} \binom{2i}{i} (-1)^{3-i} \varepsilon_{3-i} \rho^{3-i} = -\varepsilon_3 \rho^3 + 2\varepsilon_2 \rho^2 - 6\varepsilon_1 \rho + 20.$$

By eliminating e_1, e_2 from the (dependent!) system (Eq3) above we obtain

$$c_1^2 + 2c_1 - 2c_2 + 2 - c_3 = 0. \qquad (16)$$

By substituting for c_1, c_2, c_3 from (C3) into (16) we obtain

$$\rho^2 \left(\varepsilon_3 \rho + \varepsilon_1^2 - 4\varepsilon_2 \right) = 0.$$

Since $\rho\,(= 1/r^2)$ is nonzero we end up with the Heron formula (1) for inverse radius squared:

$$\varepsilon_3 \rho + \varepsilon_1^2 - 4\varepsilon_2 = 0$$

written in terms of elementary symmetric functions $\varepsilon_1 = a_1^2 + a_2^2 + a_3^2$, $\varepsilon_2 = a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2$, $\varepsilon_2 = a_1^2 a_2^2 a_3^2$.

This example shows the main feature of our Wiener-Hopf type approach to Robbins circumradius of cyclic polygons problem. We may hope that simply by eliminating $e_1, \ldots, e_{n-1}$ from the system (15) of Theorem 1 we would get an equation for the circumradius of general cyclic polygons. But elimination from such a "simple" quadratic system may be computationally very demanding even for a very powerful computers today. Further notation: The special values for $z = \pm 1$ of the polynomial $\gamma_P(z)$ we denote by

$$Y_n := \gamma_P(1) = 2 + e_1 + e_2 + \cdots + e_{n-1}, \qquad (17)$$
$$\Theta_n := \gamma_P(-1) = 1 + (-1)^n - e_1 + e_2 + \cdots + (-1)^{n-1} e_{n-1}, \qquad (18)$$
$$\Delta_n = \sum_{j=0}^{n} 4^{n-j} (-1)^j \varepsilon_j \rho^j.$$

Then, from the factorization $L_P(\pm 1) = \gamma_P(\pm 1)^2$ we immediately get

$$Y_n^2 = 2\left(c_1 + c_2 + \cdots + c_{n-2} + c_{n-1} + 1\right) + c_n = \Delta_n, \qquad (19)$$
$$\Theta_n^2 = (-1)^n \varepsilon_n \rho^n. \qquad (20)$$

If we adjoin to our quadratic system, from Theorem 1, two linear equations, resulting from (17) and (18):
Auxiliary equations:

$$\begin{aligned} e_1 + e_2 + \cdots + e_{n-1} &= Y_n - 2, \\ -e_1 + e_2 + \cdots + (-1)^{n-1} e_{n-1} &= \Theta_n - 1 - (-1)^n. \end{aligned} \qquad (21)$$

For example for $n = 3$ the two auxiliary equations are:

$$\begin{aligned} e_1 + e_2 &= Y_3 - 2 \quad && \text{with } Y_3^2 = 2\left(c_1 + c_2 + 1\right) + c_3, \\ -e_1 + e_2 &= \Theta_3 && \text{with } \Theta_3^2 = -\varepsilon_3 \rho^3 \end{aligned} \tag{22}$$

and we obtain immediately

$$c_1 + 2 - Y_3 = 0. \tag{23}$$

This gives us a new form of the classical Heron formula for the circumradius:

$$\rho_3 = \boxed{\rho^{-2}\left(A_3^2 - \Delta_3 B_3^2\right) = 0} \tag{24}$$

where

$$A_3 := c_1 + 2, \quad B_3 = 1, \text{ and } \Delta_3 = Y_3^2 = 2\left(c_1 + c_2 + 1\right) + c_3. \tag{25}$$

This new derivation of the classical Heron formula explains some features of our approach to Robbins problem. We are intending to write a final result in the form

$$\rho_n = \boxed{\rho^{-2^{n-2}}\left(A_n^2 - \Delta_n B_n^2\right) = 0}, \tag{26}$$

which is much shorter than if we would expand A_n^2 and B_n^2. Without auxiliary equations we would get the formula in the expanded form which may not be explicitly computable on a computer at our disposal.

Cyclic quadrilaterals $(n = 4)$

Now by eliminating e_1, e_2, e_3 from the basic system

$$Eq4 = \left\{e_1 + e_3 - c_1, e_2 + e_1 e_3 + e_2 - c_2, e_3 + e_2 e_3 + e_1 e_2 + e_1 - c_3, e_1^2 + e_2^2 + e_3^2 - c_4\right\}$$

we obtain

$$\boxed{c_1^4 - 2c_1^2 c_2 - c_1^2 c_4 - c_1^2 + 2c_1 c_3 + c_3^2 = 0}.$$

With only first auxiliary equations

$$e_1 + e_2 + e_3 = Y_4 - 2, \quad Y_4^2 = 2\left(c_1 + c_2 + c_3 + 1\right) + c_4$$

we get

$$\rho_4 = \rho^{-4}\left(A_4^2 - \Delta_4 B_4^2\right) = 0$$

where

$$A_4 := c_1^2 + c_1 + c_3, \quad B_4 = c_1.$$

Remark 1. If we substitute $c_1 = 8 - \varepsilon_1\rho,\ c_2 = \varepsilon_2\rho^2 - 6\varepsilon_1\rho + 28,\ c_3 = -\varepsilon_3\rho^3 + 4\varepsilon_2\rho^2 - 15\varepsilon_1\rho + 56,\ c_4 = \varepsilon_4\rho^4 - 2\varepsilon_3\rho^3 + 6\varepsilon_2\rho^2 - 20\varepsilon_1\rho + 70$, and $\varepsilon_4 = \eta_4^2$ we obtain

$$\begin{aligned}\rho_4 &= \left(\varepsilon_3\rho + \varepsilon_1^2 - 4\varepsilon_2 + \eta_4\left(8 - \varepsilon_1\rho\right)\right)\left(\varepsilon_3\rho + \varepsilon_1^2 - 4\varepsilon_2 - \eta_4\left(8 - \varepsilon_1\rho\right)\right)\\ &= \rho_4^+\rho_4^-\end{aligned}$$

where ρ_4^+corresponds to convex quadrilaterals and $\rho_4^-$to nonconvex quadrilaterals. Note also the following property:

$$\rho_4 = \rho_3^2 - \varepsilon_4\left(8 - \varepsilon_1\rho\right)^2.$$

Note that $8 - \varepsilon_1\rho$ can be interpreted as $-\rho_2$ (for a digon).

Cyclic pentagons $(n = 5)$

By eliminating $e_1, \ldots, e_4$ from the basic system for cyclic pentagon we obtain a polynomial in $c_1, \ldots, c_5$ having 119 terms and coefficients between -20 and 32. By substituting $c_{5-k} = \sum_{i=k}^{5}\binom{2i}{i-k}(-1)^{5-i}\varepsilon_{5-i}\rho^{5-i}(0 \le k \le 4)$ we obtain a ρ^8 times a polynomial of degree 7 in ρ having 81 terms and coefficients between -16384 and 8192.
By using auxiliary equations we obtain a much shorter expression (with coefficients $\pm1, \pm2, \pm3, \pm4$)

$$\rho_5 = \rho^{-8}\left(A_5^2 - B_5^2\Delta_5\right)$$

where

$$\begin{aligned}A_5 =& c_1^4 + \left(-3c_2 + 2c_3 + c_4 - 3\right)c_1^2 + \left(-2c_2 - 4c_4 + 2\right)c_1 +\\ &+ 2c_2^2 + \left(-2c_3 - 2c_4 + 4\right)c_2 + c_3^2 + 2c_3 - 2c_4 + \left(c_2 + 3\right)c_5 + 2,\\ B_5 =& -c_1^3 + 2c_1^2 + \left(2c_2 - c_3\right)c_1 - 2c_2 + 2c_4 - c_5 - 2,\\ \Delta_5 =& Y_5^2 = 2\left(c_1 + c_2 + c_3 + c_4 + 1\right) + c_5.\end{aligned}$$

$$\begin{aligned}&\rho_5^{elem} =\\ &\rho^{14}\epsilon_5{}^3 + \left(-2\,\epsilon_1\epsilon_3\epsilon_5{}^2 + \epsilon_2{}^2\epsilon_5{}^2 - 4\,\epsilon_4\epsilon_5{}^2\right)\rho^{12} + \left(2\,\epsilon_1{}^3\epsilon_5{}^2 - 2\,\epsilon_1{}^2\epsilon_2\epsilon_4\epsilon_5 + \epsilon_1{}^2\epsilon_3{}^2\epsilon_5 - 8\,\epsilon_1\epsilon_2\epsilon_5{}^2 + 8\,\epsilon_1\epsilon_3\epsilon_4\epsilon_5 - 2\,\epsilon_2\epsilon_3{}^2\epsilon_5 + 32\,\epsilon_3\epsilon_5{}^2\right)\rho^{10} +\\ &+ \left(-2\,\epsilon_1{}^4\epsilon_3\epsilon_5 + \epsilon_1{}^4\epsilon_4{}^2 + 8\,\epsilon_1{}^3\epsilon_4\epsilon_5 + 4\,\epsilon_1{}^2\epsilon_2\epsilon_3\epsilon_5 - 2\,\epsilon_1{}^2\epsilon_3{}^2\epsilon_4 - 16\,\epsilon_1{}^2\epsilon_5{}^2 - 32\,\epsilon_1\epsilon_3{}^2\epsilon_5 + 16\,\epsilon_2{}^2\epsilon_3\epsilon_5 + \epsilon_3{}^4 - 32\,\epsilon_2\epsilon_5{}^2 - 64\,\epsilon_3\epsilon_4\epsilon_5\right)\rho^8 +\\ &+ \left(\epsilon_1{}^6\epsilon_5 + 6\,\epsilon_1{}^4\epsilon_2\epsilon_5 - 4\,\epsilon_1{}^4\epsilon_3\epsilon_4 + 32\,\epsilon_1{}^3\epsilon_3\epsilon_5 - 32\,\epsilon_1{}^3\epsilon_4{}^2 - 32\,\epsilon_1{}^2\epsilon_2{}^2\epsilon_5 + 16\,\epsilon_1{}^2\epsilon_2\epsilon_3\epsilon_4 + 4\,\epsilon_1{}^2\epsilon_3{}^3 - 32\,\epsilon_1{}^2\epsilon_4\epsilon_5 + 32\,\epsilon_1\epsilon_3{}^2\epsilon_4 - 32\,\epsilon_2{}^3\epsilon_5 -\right.\\ &\left.- 16\,\epsilon_2\epsilon_3{}^3 + 256\,\epsilon_1\epsilon_5{}^2 + 128\,\epsilon_2\epsilon_4\epsilon_5 + 224\,\epsilon_3{}^2\epsilon_5\right)\rho^6 + \left(-2\,\epsilon_1{}^6\epsilon_4 - 64\,\epsilon_1{}^5\epsilon_5 + 16\,\epsilon_1{}^4\epsilon_2\epsilon_4 + 6\,\epsilon_1{}^4\epsilon_3{}^2 + 128\,\epsilon_1{}^3\epsilon_2\epsilon_5 + 64\,\epsilon_1{}^3\epsilon_3\epsilon_4 - 32\,\epsilon_1{}^2\epsilon_2{}^2\epsilon_4 -\right.\\ &\left.- 48\,\epsilon_1{}^2\epsilon_2\epsilon_3{}^2 - 576\,\epsilon_1{}^2\epsilon_3\epsilon_5 + 384\,\epsilon_1{}^2\epsilon_4{}^2 + 512\,\epsilon_1\epsilon_2{}^2\epsilon_5 - 256\,\epsilon_1\epsilon_2\epsilon_3\epsilon_4 + 96\,\epsilon_2{}^2\epsilon_3{}^2 - 512\,\epsilon_1\epsilon_4\epsilon_5 - 768\,\epsilon_2\epsilon_3\epsilon_5 - 128\,\epsilon_3{}^2\epsilon_4 - 768\,\epsilon_5{}^2\right)\rho^4 +\\ &+ \left(4\,\epsilon_1{}^6\epsilon_3 + 32\,\epsilon_1{}^5\epsilon_4 - 48\,\epsilon_1{}^4\epsilon_2\epsilon_3 + 736\,\epsilon_1{}^4\epsilon_5 - 256\,\epsilon_1{}^3\epsilon_2\epsilon_4 + 192\,\epsilon_1{}^2\epsilon_2{}^2\epsilon_3 - 2816\,\epsilon_1{}^2\epsilon_2\epsilon_5 - 256\,\epsilon_1{}^2\epsilon_3\epsilon_4 + 512\,\epsilon_1\epsilon_2{}^2\epsilon_4 - 256\,\epsilon_2{}^3\epsilon_3 +\right.\\ &\left.+ 6144\,\epsilon_1\epsilon_3\epsilon_5 - 2048\,\epsilon_1\epsilon_4{}^2 - 512\,\epsilon_2{}^2\epsilon_5 + 1024\,\epsilon_2\epsilon_3\epsilon_4 + 2048\,\epsilon_4\epsilon_5\right)\rho^2 + \epsilon_1{}^8 - 16\,\epsilon_1{}^6\epsilon_2 + 96\,\epsilon_1{}^4\epsilon_2{}^2 - 128\,\epsilon_1{}^4\epsilon_4 - 256\,\epsilon_1{}^2\epsilon_2{}^3 - 2048\,\epsilon_1{}^3\epsilon_5 +\\ &+ 1024\,\epsilon_1{}^2\epsilon_2\epsilon_4 + 256\,\epsilon_2{}^4 + 8192\,\epsilon_1\epsilon_2\epsilon_5 - 2048\,\epsilon_2{}^2\epsilon_4 - 16384\,\epsilon_3\epsilon_5 + 4096\,\epsilon_4{}^2\end{aligned}$$

Cyclic heptagons $(n = 7)$

In this case we have $\rho_7 = \rho^{-64}\left(A_7^2 - \Delta_7 B_7^2\right)$ (where here we have $\rho = r^{-1}$), $\Delta_7 = 2\left(c_1 + c_2 + \cdots + c_5 + c_6 + 1\right) + c_7$.

> ***CYCLIC HEPTAGON RADIUS EQUATION*** 20230519

> *The inverse circumradius equation for cyclic heptagon will be in the form*

> $\rho[7] = A[7]^2 - \Delta[7]\cdot B[7]^2$ *with notations*

> $\varepsilon[k] :=$ *the k_th elem. symm. poly. of side lengths* $a[1]^2, .., a[7]^2 : \varepsilon[0] := 1 :$

> $\rho := r^{-1} =$ *inverse circumradius* :

> **for** k *to* 7 **do** $c[k] = sum(\mathrm{binomial}(2\cdot 7 - 2\cdot j, k - j)\cdot(-1)^j\cdot \mathrm{epsilon}[j]\cdot \mathrm{rho}^{2\cdot j}, j = 0..k)$ **od**

$$-\rho^2\epsilon_1 + 14 = -\rho^2\epsilon_1 + 14$$

$$\rho^4\epsilon_2 - 12\rho^2\epsilon_1 + 91 = \rho^4\epsilon_2 - 12\rho^2\epsilon_1 + 91$$

$$-\rho^6\epsilon_3 + 10\rho^4\epsilon_2 - 66\rho^2\epsilon_1 + 364 = -\rho^6\epsilon_3 + 10\rho^4\epsilon_2 - 66\rho^2\epsilon_1 + 364$$

$$\rho^8\epsilon_4 - 8\rho^6\epsilon_3 + 45\rho^4\epsilon_2 - 220\rho^2\epsilon_1 + 1001 = \rho^8\epsilon_4 - 8\rho^6\epsilon_3 + 45\rho^4\epsilon_2 - 220\rho^2\epsilon_1 + 1001$$

$$-\rho^{10}\epsilon_5 + 6\rho^8\epsilon_4 - 28\rho^6\epsilon_3 + 120\rho^4\epsilon_2 - 495\rho^2\epsilon_1 + 2002 = -\rho^{10}\epsilon_5 + 6\rho^8\epsilon_4 - 28\rho^6\epsilon_3 + 120\rho^4\epsilon_2 - 495\rho^2\epsilon_1 + 2002$$

$$\rho^{12}\epsilon_6 - 4\rho^{10}\epsilon_5 + 15\rho^8\epsilon_4 - 56\rho^6\epsilon_3 + 210\rho^4\epsilon_2 - 792\rho^2\epsilon_1 + 3003 = \rho^{12}\epsilon_6 - 4\rho^{10}\epsilon_5 + 15\rho^8\epsilon_4 - 56\rho^6\epsilon_3 + 210\rho^4\epsilon_2 - 792\rho^2\epsilon_1 + 3003$$

$$-\rho^{14}\epsilon_7 + 2\rho^{12}\epsilon_6 - 6\rho^{10}\epsilon_5 + 20\rho^8\epsilon_4 - 70\rho^6\epsilon_3 + 252\rho^4\epsilon_2 - 924\rho^2\epsilon_1 + 3432 = -\rho^{14}\epsilon_7 + 2\rho^{12}\epsilon_6 - 6\rho^{10}\epsilon_5 + 20\rho^8\epsilon_4 - 70\rho^6\epsilon_3 + 252\rho^4\epsilon_2 - 924\rho^2\epsilon_1 + 3432 \quad (1)$$

>

> $\mathbf{\Delta[7]} := \boldsymbol{product}\left(4 - a[k]^2\cdot\rho^2, k = 1..7\right) = \boldsymbol{add}\left(4^{7-k}\cdot(-1)^k\cdot\rho^{2\cdot k}\epsilon_k, k = 0..7\right) = 2 + 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5 + 2c_6 + c_7$

> *Then in terms of elementary symmetric polynomials* $e[1], .., e[6]$ *of vertex quotients* $q[1], .., q[7]$ *our version of the*

> **New ρ − Robbins system for cyclic heptagons :**

> $e[0] := 1 : e[7] := 1 : \varepsilon[0] := 1 : c[0] := 1 :$

> *for* k *to* 7 *do* $eq[k] := sort(sum(e[j]\cdot e[7-k+j], j = 0..k) - c[k], [seq(c[k], k = 1..7)])$ *od*

$$eq_1 := -c_1 + e_6 + e_1$$

$$eq_2 := -c_2 + e_1e_6 + e_2 + e_5$$

$$eq_3 := -c_3 + e_1e_5 + e_2e_6 + e_3 + e_4$$

$$eq_4 := -c_4 + e_1e_4 + e_2e_5 + e_3e_6 + e_3 + e_4$$

$$eq_5 := -c_5 + e_1e_3 + e_2e_4 + e_3e_5 + e_4e_6 + e_2 + e_5$$

$$eq_6 := -c_6 + e_1e_2 + e_2e_3 + e_3e_4 + e_4e_5 + e_5e_6 + e_1 + e_6$$

$$eq_7 := -c_7 + e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + 2 \quad (2)$$

> $Eq[0] := add(e[j], j = 1..6) + 2 - Y[7]$

$$Eq_0 := e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + 2 - Y_7 \quad (3)$$

> $Fq[0] := add((-1)^{j-1}\cdot e[j], j = 1..6) - \Theta[7]$

$$Fq_0 := e_1 - e_2 + e_3 - e_4 + e_5 - e_6 - \Theta_7 \quad (4)$$

> *Remarks* : 1) $\boldsymbol{Y[7]^2} = \Delta[7] = 2\cdot(1 + add(c[k], k = 1..6)) + c[7]$:

> 2) $\boldsymbol{\Theta_7^2} = 2c_1 - 2c_2 + 2c_3 - 2c_4 + 2\cdot c_5 - 2\cdot c_6 + c_7 - 2 = -\varepsilon_7\cdot\rho^{14}$

>

> *So, we intend to work in a* ***double quadratic extension*** *defined by adjoining* $Y[7]$ *and* Θ_7 :

> *The elimination of all variables* $[e[6], e[5], e[4], e[3], e[2], e[1]]$ *from the* $\boldsymbol{\rho}$***–System7*** $:= \{\boldsymbol{seq}(\boldsymbol{eq}[\boldsymbol{k}], \boldsymbol{k} = \mathbf{1..7}), \boldsymbol{Eq}[\mathbf{0}], \boldsymbol{Fq}[\mathbf{0}]\}$ *lasted only* $t7 = 3035.781$, *on a Xeon WS (with 256 GB RAM)*.

> *The* ***implicit inverseradius equation*** *is obtained as* $op(2, op(K7[2]))$, *where*

>

> $st := time()$: $K7 := map(factor, eliminate(\{seq(eq[k], k = 1..7), Eq[0], Fq[0]\}, [e[6], e[5], e[4], e[3], e[2], e[1]])[2]); t7 = time() - st;$

$$t7 = 3035.781 \quad (5)$$

> $op(1, K7), op(1, op(2, K7))$

$$\left(\Theta_7^2 - Y_7^2 + 4c_2 + 4c_4 + 4c_6 + 4\right)\left(-c_1^3 + 2c_1^2 + 4c_1c_2 + 8Y_7 - 8c_1 - 8c_2 - 8c_3 - 16\right), -c_1^3 + 2c_1^2 \quad (6)$$

$$+ 4\, c_1\, c_2 + 8\, Y_7 - 8\, c_1 - 8\, c_2 - 8\, c_3 - 16$$

>

> *b1* := *sort*(*collect*(*op*(2, *K7*[2]), Theta[7]), Theta[7]) : *length*(*b1*)

363028 **(7)**

>

> *b2* := *simplify*(*b1*, { $Y_7^2 = \Theta_7^2 + 4\, c_2 + 4\, c_4 + 4\, c_6 + 4$ }) : *indets*(*b2*), *length*(*b2*), *degree*(*b2*, Theta[7])

$\{\Theta_7, Y_7, c_1, c_2, c_3, c_4, c_5, c_6\}$, 179163, 10 **(8)**

> *b3* := *map*(*factor*, *collect*(*b2*, Theta[7])) : *length*(*b3*)

316038 **(9)**

> *Note that b3 depends linearly on Y*[7]

> *b30* := *coeff*(*b3*, *Y*[7], 0) : *b31* := *coeff*(*b3*, *Y*[7]) : *length*(*b30*), *length*(*b31*)

173880, 107654 **(10)**

>

> *a7* := 2^{-8}·*simplify*(*b30*, { $\Theta_7^2 = 2\, c_1 - 2\, c_2 + 2\, c_3 - 2\, c_4 + 2\, c_5 - 2\, c_6 + c_7 - 2$ }) : *b7* := 2^{-8}·(*simplify*(*b31*, { $\Theta_7^2 = 2\, c_1 - 2\, c_2 + 2\, c_3 - 2\, c_4 + 2\, c_5 - 2\, c_6 + c_7 - 2$ })) :

> *indets*([*a7*, *b7*]), *length*(*a7*), *length*(*b7*)

$\{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$, 216981, 137841 **(11)**

> *Thus a7 and b7 do not depend on* Theta[7] *any more. By writing c*[*k*] *simply as ck*, *k* = 1 ..*k*

> *a7e* := *subs*(*seq*(*c*[*k*] = *c*||*k*, *k* = 1 ..7), *a7*) : *b7e* := *subs*(*seq*(*c*[*k*] = *c*||*k*, *k* = 1 ..7), *b7*) :

> *map*(*length*, [*a7*, *a7e*, *b7*, *b7e*, *A7h*, *B7h*])

[216981, 121216, 137841, 77021, **57588, 35958**] **(12)**

> *and converting to Horner form we obtain* "shorter polynomials" *A7h and B7h* :

> ***A7h* := *convert*(*a7e*, *horner*)**

(13)

$$A7h := 8 + (8 + (-10 + (-28 + (-15 - c7)\, c7)\, c7)\, c7)\, c7 + (12 + (28 + (11 + (-4 - c7)\, c7)\, c7)\, c7 + (-8 + (24 + (18 + 5\, c7)\, c7)\, c7 + (-16$$
$$+ (-4 + c7)\, c7 + (-8\, c7 + 4\, c6)\, c6)\, c6)\, c6)\, c6 + ((4 + (20 + (11 + c7)\, c7)\, c7)\, c7 + (-8 + (16 + (28 + 5\, c7)\, c7)\, c7 + ((-20$$
$$- 6\, c7)\, c7 + (-16\, c7 + 24 + 16\, c6)\, c6)\, c6)\, c6 + (-20 + (15 + 6\, c7)\, c7^2 + (-4 + (-50 - c7)\, c7 + (-6\, c7 + 4 + 4\, c6)\, c6)\, c6 + (-4 + ($$
$$-8 - 7\, c7)\, c7 + (8\, c7 - 4 - 8\, c6)\, c6 + (-c6 - 2\, c7 + 6 + (-2 - c5)\, c5)\, c5)\, c5)\, c5)\, c5 + (-12 + (-8 + (5 + 9\, c7)\, c7)\, c7 + (-16 + (-8$$
$$+ 20\, c7)\, c7 + ((8 + 5\, c7)\, c7 + (8\, c7 - 4\, c6)\, c6)\, c6)\, c6 + (8 + (-4 + (-28 + 4\, c7)\, c7)\, c7 + (8 + (-64 - 26\, c7)\, c7 + (-4\, c7 - 8$$
$$- 8\, c6)\, c6)\, c6 + (60 + (-2 - 25\, c7)\, c7 + (50\, c7 + 36 + 24\, c6)\, c6 + (12\, c6 + 4\, c7 + 36 + (6\, c6 + c7 - 7 - 2\, c5)\, c5)\, c5)\, c5)\, c5 + (-32$$
$$+ (8 + (36 + 3\, c7)\, c7)\, c7 + (-36 + (-16 - 22\, c7)\, c7 + (-12\, c7 + 4\, c6)\, c6)\, c6 + (-48 + (20 + 2\, c7)\, c7 + (68\, c7 - 16 + 8\, c6)\, c6 + ($$
$$-52 + (-4 + c7)\, c7 + (-4\, c7 - 48 - 12\, c6)\, c6 + (8\, c6 - 4\, c7 - 28 - 2\, c5)\, c5)\, c5)\, c5 + (36 + (-8 + (8 - c7)\, c7)\, c7 + (-24 + (16$$
$$- 2\, c7)\, c7 + (-28 + 4\, c7 + 8\, c6)\, c6)\, c6 + (40 + (-16 + 6\, c7)\, c7 + (8\, c7 - 24 - 8\, c6)\, c6 + (16\, c6 - 8\, c7 + 20)\, c5)\, c5 + (24 + (-4$$
$$+ 2\, c7)\, c7 + (-8\, c7 + 40 - 24\, c6)\, c6 + (16\, c6 - 8\, c7 + 8 - 4\, c5)\, c5 + (-8\, c5 + 24\, c6 + 4\, c7 - 16 - 8\, c4)\, c4)\, c4)\, c4)\, c4)\, c4 + (-8 + ($$
$$-20 + (-22 + (-13 - 5\, c7)\, c7)\, c7)\, c7 + (-8 + (-8 + (-8 - 11\, c7)\, c7)\, c7 + ((20 - 4\, c7)\, c7 + (24\, c7 - 8 - 8\, c6)\, c6)\, c6)\, c6 + ((15\, c7^2$$
$$- 4)\, c7 + (48 + (-32 + 2\, c7)\, c7 + (44\, c7 - 16 - 32\, c6)\, c6)\, c6 + (20 + (-12 - 21\, c7)\, c7 + (8\, c7 + 36 - 24\, c6)\, c6 + ((10 - c7)\, c7 + ($$
$$-4 - 4\, c6)\, c6 + (6\, c7 + 4)\, c5)\, c5)\, c5)\, c5 + (-8 + (60 + (108 + 42\, c7)\, c7)\, c7 + (-24 + (-24 - 2\, c7)\, c7 + (-12\, c7 - 24 - 8\, c6)\, c6)\, c6$$
$$+ (-16 + (-20 + (-50 + 3\, c7)\, c7)\, c7 + (-104 + (-24 + 2\, c7)\, c7 + (-4\, c7 + 8\, c6)\, c6)\, c6 + (-52 + (8 - 6\, c7)\, c7 + (-20 + 8\, c6)\, c6$$
$$+ (-16\, c6 + 4\, c7 - 12 - 6\, c5)\, c5)\, c5)\, c5 + (56 + (-12 + (2 + 2\, c7)\, c7)\, c7 + (16 + (-4 - 8\, c7)\, c7 + (-24\, c7 - 16)\, c6)\, c6 + (56 + (28$$
$$- 6\, c7)\, c7 + (24\, c7 + 72 + 24\, c6)\, c6 + (-24\, c6 - 4\, c7 + 28 + 12\, c5)\, c5)\, c5 + (-24 + (-104 + 2\, c7)\, c7 + (40\, c7 + 56 + 8\, c6)\, c6 + ($$
$$-24\, c6 - 12\, c7 - 8 + 24\, c5)\, c5 + (-8\, c5 - 16\, c6 - 16\, c7 - 24 + 8\, c4)\, c4)\, c4)\, c4)\, c4 + (-16 + (-8 + (-8 - 5\, c7)\, c7)\, c7 + (4 + (-30$$
$$+ (-11 - 3\, c7)\, c7)\, c7 + (16 + (12 - 3\, c7)\, c7 + (6\, c7 + 12)\, c6)\, c6)\, c6 + (-8 + (16 + (-12 + 3\, c7)\, c7)\, c7 + (12 + (-12 + 8\, c7)\, c7$$
$$+ (4\, c7 - 48 - 12\, c6)\, c6)\, c6 + (-12 + (40 - 11\, c7)\, c7 + (-11\, c7 + 20 + 8\, c6)\, c6 + (8\, c6 - c7 - 6 + 2\, c5)\, c5)\, c5)\, c5 + (44 + (30$$
$$+ (23 + 4\, c7)\, c7)\, c7 + (4 + (-6 + 14\, c7)\, c7 + (-4\, c7 + 4 - 12\, c6)\, c6)\, c6 + (-16 + (16 - 11\, c7)\, c7 + (16\, c7 + 92 + 12\, c6)\, c6 + ($$
$$-20\, c6 + 19\, c7 + 8 + 8\, c5)\, c5)\, c5 + (-12 + (-114 - 25\, c7)\, c7 + (2\, c7 + 32 - 12\, c6)\, c6 + (-36\, c6 + 8\, c7 + 44 + 8\, c5)\, c5 + (-20\, c5$$
$$+ 20\, c6 + 8\, c7 + 36\, c4)\, c4)\, c4)\, c4 + (12 + (16 + (11 + 2\, c7)\, c7)\, c7 + (16 + (-24 + c7)\, c7 + (-16\, c7 - 4 + 16\, c6)\, c6)\, c6 + (-16$$
$$+ (10 + 3\, c7)\, c7 + (-34\, c7 + 8 + 16\, c6)\, c6 + (18\, c6 - 4\, c7 - 28 - 2\, c5)\, c5)\, c5 + (15\, c7^2 + 8 + (12\, c7 + 36 - 20\, c6)\, c6 + (32\, c6$$
$$- 2\, c7 + 32 - 10\, c5)\, c5 + (-56\, c5 + 32\, c6 - 20\, c7 - 56 + 12\, c4)\, c4)\, c4 + (6 + (-9 + (-16 - c6)\, c6)\, c6 + (c7 + 4 + (3\, c7 + 20)\, c6$$
$$+ (-3\, c6 - 1 - 2\, c5)\, c5)\, c5 + (-31 + (-1 - 3\, c7)\, c7 + (c7 + 24 + 3\, c6)\, c6 + (2\, c6 + c7 - 26 + c5)\, c5 + (2\, c5 - 3\, c6 - 3\, c7 - 4$$
$$+ 9\, c4)\, c4)\, c4 + (-8 + (6 - c7)\, c7 + (-4 + 2\, c7 - 4\, c6)\, c6 + (-2\, c6 + 6\, c7 - 2)\, c5 + (-14\, c5 + 2\, c6 + 2\, c7 + 2 - 2\, c4)\, c4 + (-4\, c4$$
$$- 2\, c5 + 8\, c6 - c7 + (2 - c3)\, c3)\, c3)\, c3)\, c3)\, c3)\, c3)\, c3 + (-4 + (-24 + (-41 + (-25 - 2\, c7)\, c7)\, c7)\, c7 + (8 + (-28 + (-38$$
$$- 9\, c7)\, c7)\, c7 + (24 + (16 - 15\, c7)\, c7 + (20\, c7 + 8 - 4\, c6)\, c6)\, c6)\, c6 + (24 + (44 + (52 + 38\, c7)\, c7)\, c7 + ((40 + 40\, c7)\, c7 + (20\, c7$$
$$- 72 - 48\, c6)\, c6)\, c6 + (12 + (26 + (-11 - c7)\, c7)\, c7 + (-16 + (-20 - 4\, c7)\, c7 + (2\, c7 + 12\, c6)\, c6)\, c6 + (-16 + (-28 - 3\, c7)\, c7 + ($$
$$-8\, c7 + 16)\, c6 + (7\, c7 + 15 + 2\, c5)\, c5)\, c5)\, c5)\, c5 + (-24 + (12 + (38 + (13 - c7)\, c7)\, c7)\, c7 + (40 + (-36 + (-36 + c7)\, c7)\, c7 + (88$$
$$+ (-20 + 12\, c7)\, c7 + (4\, c7 + 8 - 16\, c6)\, c6)\, c6)\, c6 + (-80 + (-68 + (-48 + c7)\, c7)\, c7 + (-4\, c7^2 - 16 + (-20\, c7 - 16\, c6)\, c6)\, c6 + ($$
$$-84 + (-18 + 10\, c7)\, c7 + (8\, c7 - 32 + 4\, c6)\, c6 + (-8\, c6 + 24\, c7 + 36 - 10\, c5)\, c5)\, c5)\, c5 + (108 + (4 + (-10 - 2\, c7)\, c7)\, c7 + (-40$$
$$+ (36 - 16\, c7)\, c7 + (-28 + 48\, c6)\, c6)\, c6 + (104 + (-8 + 10\, c7)\, c7 + (8\, c7 + 16 + 24\, c6)\, c6 + (-36\, c6 - 2\, c7 + 60 - 24\, c5)\, c5)\, c5$$
$$+ (-16 + (-4 + 10\, c7)\, c7 + (-4\, c7 + 80 - 56\, c6)\, c6 + (-16\, c6 - 12\, c7 - 32 + 12\, c5)\, c5 + (8\, c5 + 32\, c6 - 56 - 8\, c4)\, c4)\, c4)\, c4)\, c4$$

$+ (-16 + (-14 + (-13 - 2\,c7)\,c7)\,c7^2 + ((-76 + (-40 - 9\,c7)\,c7)\,c7 + (40 + (-24 - 2\,c7)\,c7 + (12\,c7 + 16 - 8\,c6)\,c6)\,c6)\,c6 + (8$

$+ (64 + (8\,c7 + 16)\,c7)\,c7 + (16 + (32 + 40\,c7)\,c7 + (24\,c7 - 8 - 32\,c6)\,c6)\,c6 + (-4 + (28 - 16\,c7)\,c7 + (-28\,c7 + 80 + 4\,c6)\,c6$

$+ (-12 + 14\,c5)\,c5)\,c5)\,c5 + (88 + (44 + (82 - 3\,c7)\,c7)\,c7 + (8 + (128 + 6\,c7)\,c7 + (-20\,c7 - 8 + 24\,c6)\,c6)\,c6 + (-128 + (-144$

$- 36\,c7)\,c7 + (-48\,c7 - 16 + 48\,c6)\,c6 + (-80\,c6 + 28\,c7 - 92 + 56\,c5)\,c5)\,c5 + (-48 + (-76 - 18\,c7)\,c7 + (20\,c7 + 112)\,c6 + ($

$-96\,c6 + 64\,c7 + 160 + 92\,c5)\,c5 + (-24\,c6 + 76\,c7 - 56 + 8\,c4)\,c4)\,c4)\,c4 + (12 + (22 + (23 - 7\,c7)\,c7)\,c7 + (-44 + (-18$

$+ 2\,c7)\,c7 + (-52\,c7 - 4 + 28\,c6)\,c6)\,c6 + (29\,c7^2 - 72 + (-24\,c7 + 36 + 68\,c6)\,c6 + (-40 - 9\,c7 + (-c7 + 4)\,c6 + (4\,c6 - c7 - 18$

$+ 2\,c5)\,c5)\,c5)\,c5 + (-8 + (48 + 46\,c7)\,c7 + (50\,c7 + 20 + (2\,c7 - 40 + 4\,c6)\,c6)\,c6 + (84 + (-60 + c7)\,c7 + (-8\,c7 + 28 - 8\,c6)\,c6$

$+ (4\,c6 - 7\,c7 + 52 + 4\,c5)\,c5)\,c5 + (-100 + (-34 + 6\,c7)\,c7 + (-68 - 12\,c6)\,c6 + (16\,c6 + 4\,c7 - 56 + 4\,c5)\,c5 + (-24\,c5 + 12\,c6$

$+ 6\,c7 - 8 - 36\,c4)\,c4)\,c4)\,c4 + (2\,c7^2 + 20 + (-24 + (48 - 5\,c7)\,c7 + (-4\,c7 - 32 + 4\,c6)\,c6)\,c6 + (48 + (-48 + 5\,c7)\,c7 + (14\,c7$

$- 24 + 8\,c6)\,c6 + (-10\,c6 - 8\,c7 - 18 - 6\,c5)\,c5)\,c5 + (-84 + (-16 - c7)\,c7 + (4\,c7 + 52 - 8\,c6)\,c6 + (-8\,c6 - 14\,c7 - 24$

$+ 26\,c5)\,c5 + (40\,c5 - 4\,c6 - 16\,c7 - 44 + 24\,c4)\,c4)\,c4 + (-1 + (15 - 5\,c7)\,c7 + (13\,c7 - 4 - 7\,c6)\,c6 + (-26\,c6 + 21\,c7 + 26$

$+ 21\,c5)\,c5 + (-20\,c5 - 26\,c6 + 6\,c7 + 28 + 25\,c4)\,c4 + (-8\,c4 - 10\,c5 + 14\,c6 - 14\,c7 - 2 + (8\,c4 + 2\,c5 - c7 + 4 - 2\,c3)\,c3)\,c3)$

$c3)\,c3)\,c3)\,c3 + (-28 + (-16 + (21 + (20 + 2\,c7)\,c7)\,c7)\,c7 + (-20 + (-72 + (-12 + 5\,c7)\,c7)\,c7 + (16 + (-52 - c7)\,c7 + (-8\,c7$

$+ 4 - 4\,c6)\,c6)\,c6)\,c6 + (-8 + (4 + (30 + 3\,c7)\,c7)\,c7 + (56 + (36 + 16\,c7)\,c7 + (8\,c7 + 32 - 8\,c6)\,c6)\,c6 + (-16 + (-24 - 35\,c7)\,c7$

$+ (-38\,c7 - 16 - 16\,c6)\,c6 + (-12\,c6 - 4\,c7 - 44 + (-c6 + 9 - 2\,c5)\,c5)\,c5)\,c5)\,c5 + (108 + (32 - 22\,c7)\,c7 + (-56 + (24$

$+ 8\,c7)\,c7 + (-12\,c7 - 68 + 24\,c6)\,c6)\,c6 + (104 + (-40 - 40\,c7)\,c7 + (32\,c7 + 136 + 88\,c6)\,c6 + (64\,c7 + 56 + (2\,c7 + 8 + 4\,c6)\,c6$

$+ (4\,c6 + 4\,c7 + 48 - 5\,c5)\,c5)\,c5)\,c5 + (-116 + (-4 - 22\,c7)\,c7 + (100 + (8 - c7)\,c7 + (-4\,c7 - 4 - 4\,c6)\,c6)\,c6 + (-104 + (12$

$- 2\,c7)\,c7 + (-4\,c7 - 88)\,c6 + (-4\,c6 + 2\,c7 + 8 - 4\,c5)\,c5)\,c5 + (-60 + (8 + 3\,c7)\,c7 + (16\,c7 - 40 + 20\,c6)\,c6 + (-16\,c6 - 4\,c7$

$+ 16 + 8\,c5)\,c5 + (16\,c5 - 28\,c6 - 12\,c7 + 24 + 12\,c4)\,c4)\,c4)\,c4)\,c4 + (24 + (-12 + (-40 - 19\,c7)\,c7)\,c7 + (-16 + (12 - 14\,c7)\,c7$

$+ (-32\,c7 + 16 + 32\,c6)\,c6)\,c6 + (-24 + (-52 + 4\,c7)\,c7 + (128 + (64 + 2\,c7)\,c7 + (104 - 8\,c6)\,c6)\,c6 + (8 + (64 + c7)\,c7 + (-20$

$- 12\,c6)\,c6 + (12\,c6 + 10\,c7 - 40)\,c5)\,c5)\,c5 + (-48 + (12 + (-54 + c7)\,c7)\,c7 + (10\,c7^2 + 16 + (20\,c7 - 32 + 8\,c6)\,c6)\,c6 + (96$

$+ (-4 - 10\,c7)\,c7 + (-24\,c7 - 88 - 8\,c6)\,c6 + (12\,c6 - 48 - 20\,c5)\,c5)\,c5 + (-96 + (108 - 14\,c7)\,c7 + (-44\,c7 - 80 - 16\,c6)\,c6$

$+ (16\,c6 + 8\,c7 - 72 + 8\,c5)\,c5 + (48\,c5 + 24\,c6 + 32\,c7 + 72 - 16\,c4)\,c4)\,c4)\,c4 + (32 + (4 + (-28 - c7)\,c7)\,c7 + (-36 + (60$

$- 15\,c7)\,c7 + (-12\,c7 - 64 + 4\,c6)\,c6)\,c6 + (36 + (-36 + 9\,c7)\,c7 + (16\,c7 - 152 + 20\,c6)\,c6 + (4\,c6 - 18\,c7 - 36 - 42\,c5)\,c5)\,c5$

$+ (-44 + (134 + 19\,c7)\,c7 + (30\,c7 + 108 + 12\,c6)\,c6 + (20\,c6 - 16\,c7 - 72 - 48\,c5)\,c5 + (40\,c6 - 6\,c7 + 8 - 32\,c4)\,c4)\,c4 + (-8$

$+ (-8 - 13\,c7)\,c7 + (-4 - 4\,c6)\,c6 + (-48\,c6 + 34\,c7 + 24 + 56\,c5)\,c5 + (-40\,c5 - 52\,c6 + 24\,c7 + 64 - 16\,c4)\,c4 + (-11\,c7 - 5$

$+ (35 - c6)\,c6 + (2\,c7 + 10 - 2\,c5)\,c5 + (-12\,c5 + 2\,c6 + 6\,c7 - 3 - 21\,c4)\,c4 + (12\,c4 - 4\,c5 - 4\,c7 - 6)\,c3)\,c3)\,c3)\,c3)\,c3 + (28$

$+ (48 + (30 + 10\,c7)\,c7)\,c7 + (-56 + (64 + (38 - c7)\,c7)\,c7 + (-68 + (28 - 2\,c7)\,c7 + (4\,c7 + 8\,c6)\,c6)\,c6)\,c6 + ((-32 + (-30$

$- 2\,c7)\,c7)\,c7 + (24 + (-16 - 6\,c7)\,c7 + (32 + 8\,c6)\,c6)\,c6 + (36 + (-6\,c7 + 4)\,c7 + (-12\,c7 + 4 - 8\,c6)\,c6 + (16\,c6 + 12\,c7 + 4$

$+ 7\,c5)\,c5)\,c5)\,c5 + (-104 + (-20 + (-26 + 4\,c7)\,c7)\,c7 + (56 + (-20 + 10\,c7)\,c7 + (-16\,c7 - 16 - 40\,c6)\,c6)\,c6 + (28\,c7 - 80$

$+ (8\,c7 - 64)\,c6 + (32\,c6 + 22\,c7 + 4 - 12\,c5)\,c5)\,c5 + (12 + (12 - 5\,c7)\,c7 + (20\,c7 - 56 + 60\,c6)\,c6 + (-8\,c6 - 12\,c7 + 8$

$- 36\,c5)\,c5 + (16\,c5 - 48\,c6 - 16\,c7 + 64 + 20\,c4)\,c4)\,c4)\,c4 + ((24 + (-32 + 7\,c7)\,c7)\,c7 + (-16 + (-12 + 16\,c7)\,c7 + (12\,c7 - 48$

$- 16\,c6)\,c6)\,c6 + (24 + (16 + 10\,c7)\,c7 + (-8\,c7 - 48 - 8\,c6)\,c6 + (48\,c6 + 96 - 36\,c5)\,c5)\,c5 + (-104 + (-28 + 8\,c7)\,c7 + (-56\,c7$

$- 120 - 16\,c6)\,c6 + (32\,c6 - 24\,c7 - 144 - 76\,c5)\,c5 + (-16\,c5 + 48\,c6 - 40\,c7 + 176 - 32\,c4)\,c4)\,c4 + (-16 + (-36 + c7)\,c7 + ($

$-46\,c7 + 36)\,c6 + (-36\,c6 + 20\,c7 + 4 + (-c7 + 68 + 2\,c5)\,c5)\,c5 + (-12\,c7 + 92 + (2\,c7 + 36 + 4\,c6)\,c6 + (-4\,c6 - 8\,c7 + 68$

$+ 8\,c5)\,c5 + (20\,c5 - 8\,c6 - 10\,c7 + 8 + 20\,c4)\,c4)\,c4 + (16 + (-16 - 2\,c7)\,c7 + 24\,c6 + (32 + 6\,c5)\,c5 + (16\,c5 + 4\,c6 + 16\,c7 + 76$

$- 20\,c4)\,c4 + (-2\,c4 - 12\,c5 + 10\,c6 - 2\,c7 - 27 + 4\,c3)\,c3)\,c3)\,c3)\,c3 + (-4 + (4 + (2 + c7)\,c7)\,c7 + (68 + (16 + 5\,c7)\,c7 + (8\,c7$

$+ 28 - 12\,c6)\,c6)\,c6 + (-56 + (-28 - 12\,c7)\,c7 + (-44\,c7 - 88 - 40\,c6)\,c6 + (4\,c6 - 4 + (-16 - c5)\,c5)\,c5)\,c5 + (36 + (-20$

$+ 3\,c7)\,c7 + (-8\,c7 - 32 + 36\,c6)\,c6 + (32\,c6 + 44\,c7 + 40 + (4\,c6 + 2\,c7 - 12 - 4\,c5)\,c5)\,c5 + (-c7^2 + 60 + (-4\,c7 - 8 - 4\,c6)\,c6$

$+ (4\,c7 + 8\,c6 - 8\,c5)\,c5 + (-8\,c5 + 8\,c6 + 4\,c7 - 8 - 4\,c4)\,c4)\,c4)\,c4 + ((-4 + 36\,c7)\,c7 + (12\,c7 + 32 + 40\,c6)\,c6 + (-56 + (40$

$+ 2\,c7)\,c7 + (-16 - 8\,c6)\,c6 + (4\,c7 + 4)\,c5)\,c5 + (128 + (-32 + 4\,c7)\,c7 + (24 + 8\,c7)\,c6 + (32 - 12\,c5)\,c5 + (-16\,c5 - 8\,c6$

$- 12\,c7 - 48 + 8\,c4)\,c4)\,c4 + ((-4\,c7 - 32)\,c7 + (6\,c7 - 48 + 4\,c6)\,c6 + (-12\,c6 - 16\,c7 - 20)\,c5 + (32\,c5 - 32\,c6 + 8 + 8\,c4)\,c4$

$+ (-16\,c4 - 8\,c5 + 12\,c6 - 28 + 3\,c3)\,c3)\,c3)\,c3 + (12 + (-16 + (1 - c7)\,c7)\,c7 + ((-4 - 2\,c7)\,c7 + (4\,c7 + 20 + 8\,c6)\,c6)\,c6 + (24$

$+ (12 - 2\,c7)\,c7 + (-8\,c7 + 8 - 8\,c6)\,c6 + (4\,c6 + 6\,c7 + 4\,c5)\,c5)\,c5 + (8 + (-8\,c7 - 16\,c6)\,c6 + (16\,c6 + 16\,c7 + 16)\,c5 + (-16\,c5$

$+ 16\,c6 + 8\,c7 - 32 - 8\,c4)\,c4)\,c4 + (32 + (16 - 2\,c7)\,c7 + (16\,c7 + 16 + 8\,c6)\,c6 + (16\,c6 - 8\,c7 + 48 - 4\,c5)\,c5 + (16\,c5 - 16\,c6$

$- 64 + 16\,c4)\,c4 + (-4\,c4 + 4\,c5 - 2\,c7 - 4 - 4\,c3)\,c3)\,c3 + (-12 + (-3\,c7 - 4)\,c7 + (-4\,c7 - 12 - 12\,c6)\,c6 + (8\,c6 - 4\,c7 + 16$

$- 8\,c5)\,c5 + (-8\,c5 + 8\,c6 + 4\,c7 - 20)\,c4 + (8\,c4 - 8\,c6 - 4\,c7 - 32 - 8\,c3)\,c3 + (8\,c4 - 4 + 4\,c2)\,c2)\,c2)\,c2)\,c2)\,c2)\,c2)\,c2 + (8$

$+ (-8 + (-58 + (-50 - 12\,c7)\,c7)\,c7)\,c7 + (-8 + (60 + (96 + 40\,c7)\,c7)\,c7 + (-48 + (8 + 44\,c7)\,c7 + (-44\,c7 - 40$

$- 8\,c6)\,c6)\,c6)\,c6 + (-16 + (4 + (44 + (15 + c7)\,c7)\,c7)\,c7 + (8 + (-100 + (-44 - c7)\,c7)\,c7 + (-136\,c7 + 24 + (24 + 8\,c7$

$- 8\,c6)\,c6)\,c6)\,c6 + (16 + (-36 + (22 - 4\,c7)\,c7)\,c7 + (-4 + (-4 - 10\,c7)\,c7 + (24\,c7 + 68 - 16\,c6)\,c6)\,c6 + ((22 - 11\,c7)\,c7$

$+ (26\,c7 + 40 - 12\,c6)\,c6 + (-4\,c6 + 12\,c7 - 18 + 6\,c5)\,c5)\,c5)\,c5)\,c5 + (-32 + (4 + (58 + (2\,c7 + 8)\,c7)\,c7)\,c7 + (-80 + (48 + ($

$-36 - 4\,c7)\,c7)\,c7 + (-64 + (4 - 16\,c7)\,c7 - 16\,c6)\,c6)\,c6 + (64 + (-40 + (-42 + 4\,c7)\,c7)\,c7 + (-56 + (16 + 6\,c7)\,c7 + (64\,c7 + 56$

$+ 16\,c6)\,c6)\,c6 + (36 + (8 - 30\,c7)\,c7 + (132 - 56\,c6)\,c6 + (-64\,c6 + 14\,c7 - 56 + 42\,c5)\,c5)\,c5)\,c5 + (-40 + (112 + (42$

$+ 2\,c7)\,c7)\,c7 + (16 + (-104 + 4\,c7)\,c7 + (8\,c7 + 16 - 16\,c6)\,c6)\,c6 + (-120 + (56 + 12\,c7)\,c7 + (24\,c7 + 80 - 40\,c6)\,c6 + (-88\,c6$

$- 4\,c7 - 160 + 76\,c5)\,c5)\,c5 + (96 + (-16 - 6\,c7)\,c7 + (-8\,c7 + 16 + 8\,c6)\,c6 + (16\,c6 - 24\,c7 + 96 + 88\,c5)\,c5 + (-48\,c5 + 32\,c6$

$- 8 - 24\,c4)\,c4)\,c4)\,c4)\,c4 + (-8 + (-24 + (-14 - 20\,c7)\,c7)\,c7 + (16 + (104 + (-22 + 14\,c7)\,c7)\,c7 + ((72 + 6\,c7)\,c7$

$- 8\,c6^2)\,c6)\,c6 + (28 + (-36 + (25 + 3\,c7)\,c7)\,c7 + (-76 + (84 - 57\,c7)\,c7 + (-16\,c7 + 76 + 4\,c6)\,c6)\,c6 + (-56 + (-14 - 3\,c7)\,c7$

$+ (44\,c7 + 20 + 28\,c6)\,c6 + (-18\,c6 + 27\,c7 - 26 + (-c7 - 14 + 2\,c5)\,c5)\,c5)\,c5)\,c5 + (72 + (232 + (76 + 22\,c7)\,c7)\,c7 + (-40 + ($

$-240 - 54\,c7)\,c7 + (-8\,c7 + 56 + 24\,c6)\,c6)\,c6 + (8 + (-60 - 34\,c7)\,c7 + (32\,c7 + 144 + 24\,c6)\,c6 + (116 + (10 - c7)\,c7 + (4\,c7$

$+ 28 + 4\,c6)\,c6 + (-8\,c6 - 4\,c7 - 18 + 4\,c5)\,c5)\,c5)\,c5 + (8 + (-160 - 8\,c7)\,c7 + (48 + (2\,c7 + 8)\,c7 + (-24 - 8\,c6)\,c6)\,c6 + (60$

$+ (44 - 8\,c7)\,c7 + (8\,c7 + 156)\,c6 + (-16\,c6 + 12\,c7 - 16 + 8\,c5)\,c5)\,c5 + (-248 + (-96 - 10\,c7)\,c7 + (16\,c7 + 72 + 24\,c6)\,c6 + ($

$-32\, c5 + 24\, c7 - 120 - 4\, c5)\, c5 + (-56\, c6 + 32 + 40\, c4)\, c4)\, c4)\, c4)\, c4 + (12 + (-48 + (-83 - 22\, c7)\, c7)\, c7 + (24 + (16 - 9\, c7)\, c7$

$+ (32\, c7 + 76 - 8\, c6)\, c6)\, c6 + (32 + (6 + 41\, c7)\, c7 + (-40 + (108 + c7)\, c7 + (-4\, c7 - 24)\, c6)\, c6 + (38 + (-23 + 5\, c7)\, c7 + (-6\, c7$

$- 36 + 8\, c6)\, c6 + (8\, c6 - 14\, c7 - 42 + 6\, c5)\, c5)\, c5)\, c5 + (44 + (-40 + (36 - 5\, c7)\, c7)\, c7 + (12 + (4\, c7 - 12)\, c7 + (12\, c7$

$- 32)\, c6)\, c6 + (8 + (-60 + 17\, c7)\, c7 + (-4\, c7 - 184 - 24\, c6)\, c6 + (8\, c6 - 10\, c7 - 136 + 8\, c5)\, c5)\, c5 + (-96 + (104 - 12\, c7)\, c7 + ($

$-40\, c7 + 40 - 16\, c6)\, c6 + (48\, c6 + 24\, c7 - 32\, c5)\, c5 + (-56\, c5 + 32\, c6 + 44\, c7 + 120 + 16\, c4)\, c4)\, c4)\, c4 + (-16 + (-30 + (9$

$- 4\, c7)\, c7)\, c7 + (-32 + (-66 - 2\, c7)\, c7 + (-4\, c7 + 48 + 8\, c6)\, c6)\, c6 + (-10 + (-13 + 24\, c7)\, c7 + (-9\, c7 - 8 - 10\, c6)\, c6 + (-2\, c6$

$- 20\, c7 + 20 + 12\, c5)\, c5)\, c5 + (16 + (120 + 27\, c7)\, c7 + (-2\, c7 - 120)\, c6 + (48\, c6 - 45\, c7 - 48 + 22\, c5)\, c5 + (-22\, c5 + 16\, c6$

$- 30\, c7 - 16 - 56\, c4)\, c4)\, c4 + (-16 + (-5 - c7)\, c7 + (9\, c7 - 8)\, c6 + (-28\, c6 - 12\, c7 + 16 - 14\, c5)\, c5 + (60\, c5 + 6\, c6 + 11\, c7 + 14$

$+ 6\, c4)\, c4 + (17\, c7 + 26 + (-c7 - 38)\, c6 + (4\, c6 - c7 - 4 + 2\, c5)\, c5 + (4\, c6 - 3\, c7 + 30 - 12\, c4)\, c4 + (-2\, c4 + 10\, c5 + 2\, c6 - 6\, c7$

$- 4 + 4\, c3)\, c3)\, c3)\, c3)\, c3)\, c3)\, c3 + (16 + (-52 + (-54 + (-4 + 4\, c7)\, c7)\, c7)\, c7 + (40 + (96 + (-2 + 14\, c7)\, c7)\, c7 + (64 + (68$

$- 16\, c7)\, c7 + (-8\, c7 + 72 + 32\, c6)\, c6)\, c6)\, c6 + (16 + (176 + (154 + 12\, c7)\, c7)\, c7 + (-240 + (4 - 58\, c7)\, c7 + (-92\, c7 - 104$

$+ 56\, c6)\, c6)\, c6 + (-68 + (-60 - 22\, c7)\, c7 + (-28\, c7 - 8 + 124\, c6)\, c6 + (40 + (-40 + 3\, c7)\, c7 + (2\, c7 + 44 - 4\, c6)\, c6 + (-4\, c6 - 2$

$- 4\, c5)\, c5)\, c5)\, c5)\, c5 + (8 + (-8 + (-82 - 40\, c7)\, c7)\, c7 + (264 + (-240 - 46\, c7)\, c7 + (64\, c7 + 232 + 24\, c6)\, c6)\, c6 + (-144 + ($

$-144 + (22 - c7)\, c7)\, c7 + (144 + (156 - 6\, c7)\, c7 + (-8\, c7 - 32)\, c6)\, c6 + (76 + (76 + 8\, c7)\, c7 + (8\, c7 + 8 + 8\, c6)\, c6 + (24\, c6 + 2\, c7$

$- 20 - 12\, c5)\, c5)\, c5)\, c5 + (152 + (-16 + (-8 + 2\, c7)\, c7)\, c7 + (-88 + (72 + 12\, c7)\, c7 + (24\, c7 - 16 + 16\, c6)\, c6)\, c6 + (160 + (56$

$- 10\, c7)\, c7 + (-56\, c7 - 72 - 48\, c6)\, c6 + (32\, c6 - 24\, c7 - 132 - 4\, c5)\, c5)\, c5 + (-184 + (136 - 4\, c7)\, c7 + (-16\, c7 - 56 - 16\, c6)\, c6$

$+ (64\, c6 - 192 + 56\, c5)\, c5 + (48\, c5 - 16\, c6 - 8\, c7 + 16 + 16\, c4)\, c4)\, c4)\, c4)\, c4 + (48 + (-28 + (-46 - 15\, c7)\, c7)\, c7 + (8 + (-44$

$+ (54 + 3\, c7)\, c7)\, c7 + (-8 + (116 + 2\, c7)\, c7 + (-4\, c7 - 40 + 8\, c6)\, c6)\, c6)\, c6 + (100 + (124 + (99 - 2\, c7)\, c7)\, c7 + (-156 + (92$

$- 7\, c7)\, c7 + (-28\, c7 - 204 + 20\, c6)\, c6)\, c6 + (100 + (-170 + 21\, c7)\, c7 + (-4\, c7 - 116 + 20\, c6)\, c6 + (-12\, c6 - 22\, c7 + 42$

$- 24\, c5)\, c5)\, c5)\, c5 + (-96 + (-64 + (-10\, c7 + 8)\, c7)\, c7 + ((-8\, c7 + 16)\, c7 + (-40\, c7 - 176 - 16\, c6)\, c6)\, c6 + (-404 + (-264$

$+ 43\, c7)\, c7 + (-20\, c7 - 472 + 20\, c6)\, c6 + (80\, c6 - 16\, c7 - 64 - 60\, c5)\, c5)\, c5 + (-56 + (348 + 16\, c7)\, c7 + (4\, c7 - 24 + 120\, c6)\, c6$

$+ (188\, c6 + 40\, c7 + 284 - 124\, c5)\, c5 + (-164\, c5 - 128\, c6 + 40\, c7 + 176 - 80\, c4)\, c4)\, c4)\, c4 + (-140 + (-56 + (-22 - 20\, c7)\, c7)\, c7$

$+ (-68 + (-140 + 32\, c7)\, c7 + (52\, c7 - 24 - 24\, c6)\, c6)\, c6 + (120 + (104 + 61\, c7)\, c7 + (30\, c7 + 112 - 112\, c6)\, c6 + (-60\, c6 - 29\, c7$

$+ 182 + 40\, c5)\, c5)\, c5 + (20 + (248 + 3\, c7)\, c7 + (-120\, c7 - 452 + 8\, c6)\, c6 + (120\, c6 - 150\, c7 - 80 + 140\, c5)\, c5 + (72\, c5 - 136\, c6$

$- 56\, c7 + 164 - 120\, c4)\, c4)\, c4 + (-48 + (26 - 50\, c7)\, c7 + (-40\, c7 + 56 + 64\, c6)\, c6 + (-22\, c7 - 30 + (-c7 + 60 - 2\, c6)\, c6 + (2\, c6$

$+ 4\, c7 + 14)\, c5)\, c5 + (152 + (-4 + 3\, c7)\, c7 + (6\, c7 + 104)\, c6 + (-24\, c6 + 3\, c7 + 268 - 18\, c5)\, c5 + (10\, c5 - 16\, c6 + 6\, c7 + 56$

$+ 48\, c4)\, c4)\, c4 + (130 + (17 - 3\, c7)\, c7 + (-10\, c7 - 70 + 8\, c6)\, c6 + (14\, c6 - 30 - 20\, c5)\, c5 + (-18\, c5 + 4\, c6 + 22\, c7 + 64$

$- 4\, c4)\, c4 + (-16\, c4 + 32\, c5 - 8\, c6 - 7\, c7 - 34 + 4\, c3)\, c3)\, c3)\, c3)\, c3)\, c3 + ((-40 + (42 + (23 + c7)\, c7)\, c7)\, c7 + (80 + (-132 + (-2$

$+ 7\, c7)\, c7)\, c7 + (16 + (-16 + 6\, c7)\, c7 + (-8\, c6 - 12\, c7 - 48)\, c6)\, c6)\, c6 + (104 + (104 + (20 - 15\, c7)\, c7)\, c7 + (96 + (104$

$- 36\, c7)\, c7 + (-4\, c7 - 24 + 32\, c6)\, c6)\, c6 + (-116 + (-148 - c7)\, c7 + (52\, c7 - 140 + 44\, c6)\, c6 + (16\, c6 + 30\, c7 - 18\, c5)\, c5)\, c5)\, c5$

$+ (40 + (-136 + (-150 + 6\, c7)\, c7)\, c7 + (-232 + (16 + 26\, c7)\, c7 + (-144 - 24\, c6)\, c6)\, c6 + (208 + (4 + 26\, c7)\, c7 + (8\, c7 + 112$

$- 96\, c6)\, c6 + (-36\, c6 + 48\, c7 + 224 - 88\, c5)\, c5)\, c5 + (-328 + (92 + 14\, c7)\, c7 + (-16\, c7 - 96 + 24\, c6)\, c6 + (24\, c6 - 76\, c7 - 296$

$- 152\, c5)\, c5 + (56\, c5 - 24\, c6 - 76\, c7 + 200 + 32\, c4)\, c4)\, c4)\, c4 + (-120 + (-228 + (-126 - 13\, c7)\, c7)\, c7 + (152 + (56 + 24\, c7)\, c7$

$+ (60\, c7 + 40 - 64\, c6)\, c6)\, c6 + (160 + (104 + 37\, c7)\, c7 + (64\, c7 + 116 - 212\, c6)\, c6 + (-236\, c6 + 42\, c7 - 124 + (-2\, c6 - c7 + 18$

$+ 2\, c5)\, c5)\, c5)\, c5 + (72 + (128 - 70\, c7)\, c7 + (-80\, c7 - 152 + 168\, c6)\, c6 + (68 + (24 - c7)\, c7 + (4\, c7 + 32 + 12\, c6)\, c6 + (-4\, c6$

$- 6\, c7 + 68 + 6\, c5)\, c5)\, c5 + (464 + (124 - 6\, c7)\, c7 + (-16\, c7 + 16 - 8\, c6)\, c6 + (204 + 12\, c5)\, c5 + (4\, c5 + 32\, c6 + 24\, c7 + 40$

$- 24\, c4)\, c4)\, c4)\, c4 + (-100 + (72 - 56\, c7)\, c7 + (16 + (-60 + 3\, c7)\, c7 + (4\, c7 + 216 - 4\, c6)\, c6)\, c6 + (-280 + (98 - 8\, c7)\, c7 + ($

$-18\, c7 + 280 - 8\, c6)\, c6 + (28\, c6 - 2\, c7 - 14 + 34\, c5)\, c5)\, c5 + (17\, c7^2 + 568 + (36\, c7 + 132 + 20\, c6)\, c6 + (-16\, c6 - 14\, c7 + 8$

$+ 8\, c5)\, c5 + (-64\, c5 - 60\, c6 - 32\, c7 - 20 - 4\, c4)\, c4)\, c4 + (80 + (-130 - 3\, c7)\, c7 + (-34\, c7 - 80 + 16\, c6)\, c6 + (76\, c6 - 13\, c7$

$- 190 - 46\, c5)\, c5 + (-60\, c5 - 8\, c6 + 28\, c7 - 192 - 16\, c4)\, c4 + (-24\, c4 + 58\, c5 - 56\, c6 + 40\, c7 - 10 - 16\, c3)\, c3)\, c3)\, c3)\, c3 + ($

$-56 + (72 + (58 + 18\, c7)\, c7)\, c7 + (-200 + (80 + 8\, c7)\, c7 + (-24\, c7 - 144 - 32\, c6)\, c6)\, c6 + (88 + (-56 - 40\, c7)\, c7 + (-92\, c7 + 24$

$- 40\, c6)\, c6 + (-20 + (-60 + 3\, c7)\, c7 + (4\, c7 - 40 - 4\, c6)\, c6 + (-12\, c6 + 80 - 4\, c5)\, c5)\, c5)\, c5 + (-144 + (-12 + (70 - c7)\, c7)\, c7$

$+ (80 + (40 - 6\, c7)\, c7 + (-12\, c7 + 72 - 8\, c6)\, c6)\, c6 + (-104 + (48 + 12\, c7)\, c7 + (36\, c7 + 112 + 24\, c6)\, c6 + (-16\, c6 + 4\, c7 + 140$

$- 20\, c5)\, c5)\, c5 + (360 + (-140 + 2\, c7)\, c7 + (80 + 8\, c7 + 8\, c6)\, c6 + (-32\, c6 - 12\, c7 + 80 - 44\, c5)\, c5 + (-24\, c5 + 8\, c6 + 4\, c7 - 72$

$- 8\, c4)\, c4)\, c4)\, c4 + (-80 + (-8 + (30 + c7)\, c7)\, c7 + (-64 + (-76 + 10\, c7)\, c7 + (20\, c7 + 160 + 8\, c6)\, c6)\, c6 + (176 + (92$

$- 13\, c7)\, c7 + (-40\, c7 + 444 - 28\, c6)\, c6 + (-4\, c6 - 22\, c7 + 204 + 54\, c5)\, c5)\, c5 + (64 + (-376 + 4\, c7)\, c7 + (-48\, c7 - 224$

$- 64\, c6)\, c6 + (-8\, c6 - 28\, c7 - 188 + 136\, c5)\, c5 + (76\, c5 + 72\, c6 - 28\, c7 - 80 + 32\, c4)\, c4)\, c4 + (156 + (-84 + 39\, c7)\, c7 + (4\, c7$

$+ 84 - 4\, c6)\, c6 + (104\, c6 - 48\, c7 - 16 - 62\, c5)\, c5 + (56\, c5 + 124\, c6 - 36\, c7 - 396 + 120\, c4)\, c4 + (32\, c4 - 10\, c5 - 160\, c6 + 66\, c7$

$+ 24 - 12\, c3)\, c3)\, c3)\, c3 + ((40 + (54 - 9\, c7)\, c7)\, c7 + (104 + (48 - 22\, c7)\, c7 + (4\, c7 + 88 + 24\, c6)\, c6)\, c6 + (-160 + (-60$

$- 12\, c7)\, c7 + (20\, c7 - 72 + 40\, c6)\, c6 + (16\, c6 + 12\, c7 - 28 + 24\, c5)\, c5)\, c5 + (272 + (4\, c7 + 4)\, c7 + (16\, c7 + 104 - 16\, c6)\, c6 + ($

$-24\, c6 + 60\, c7 + 192 + 60\, c5)\, c5 + (-8\, c5 + 32\, c7 - 232 - 8\, c4)\, c4)\, c4 + (72 + (76 + 44\, c7)\, c7 + (48\, c7 - 24 - 64\, c6)\, c6 + (-72\, c6$

$- 52\, c7 - 188 - 116\, c5)\, c5 + (-140\, c5 - 16\, c6 - 48\, c7 - 336 - 32\, c4)\, c4 + (-108\, c4 - 120\, c5 - 112\, c6 + 84\, c7 - 224 + 72\, c3)\, c3)$

$c3 + (64 + (4 - 32\, c7)\, c7 + (-32\, c7 - 32\, c6)\, c6 + (-56\, c6 - 48\, c7 - 8)\, c5 + (24\, c5 - 24\, c6 + 28\, c7 - 200 + 32\, c4)\, c4 + (-16\, c4$

$+ 60\, c5 + 152\, c6 + 124\, c7 + 48 + 144\, c3)\, c3 + (40\, c3 + 56\, c4 - 24\, c5 - 16\, c6 - 16\, c7 - 56 + 24\, c2)\, c2)\, c2)\, c2)\, c2)\, c2)\, c2 + (-28$

$+ (-4 + (-3 + (-21 - 5\, c7)\, c7)\, c7)\, c7 + (-32 + (158 + (160 + 15\, c7)\, c7)\, c7 + (-64 + (-90 - 21\, c7)\, c7 + (-40\, c7 - 16$

$+ 28\, c6)\, c6)\, c6)\, c6 + (4 + (-40 + (-21 - 6\, c7)\, c7)\, c7 + (76 + (-268 - 38\, c7)\, c7 + (60\, c7 + 136 + 88\, c6)\, c6)\, c6 + (44 + (-2 + (c7$

$- 7)\, c7)\, c7 + (79\, c7 + 76 + (-5\, c7 - 72 - 2\, c6)\, c6)\, c6 + (8 + (14 + 2\, c7)\, c7 + (4\, c7 - 110 + 10\, c6)\, c6 + (-4\, c6 - 4\, c7 - 27$

$- 8\, c5)\, c5)\, c5)\, c5)\, c5 + (-8 + (138 + (8 + 9\, c7)\, c7)\, c7 + (-88 + (12 + (-22 - c7)\, c7)\, c7 + (-136 + (-4\, c7 + 80)\, c7 + (2\, c7 - 28$

$+ 12\, c6)\, c6)\, c6)\, c6 + (92 + (-52 + (-14 + 5\, c7)\, c7)\, c7 + (24 + (164 + 12\, c7)\, c7 + (-16\, c7 - 32 - 4\, c6)\, c6)\, c6 + (8 + (-79$

$+ 3\, c7)\, c7 + (-41\, c7 - 132 + 10\, c6)\, c6 + (36\, c6 - 17\, c7 - 24 + 2\, c5)\, c5)\, c5)\, c5 + (12 + (58 + (24 - 4\, c7)\, c7)\, c7 + (-58\, c7 - 16 + ($

$-16\, c7 + 24 - 28\, c6)\, c6)\, c6 + (-116 + (12 + 27\, c7)\, c7 + (-60 + 44\, c6)\, c6 + (86\, c6 - 38\, c7 - 24 - 30\, c5)\, c5)\, c5 + (104 + (-70$

$-2\,c7)\,c7+(-10\,c7-36+28\,c6)\,c6+(68\,c6-20\,c7+124-70\,c5)\,c5+(-92\,c5+12\,c6+16\,c7-36+8\,c4)\,c4)\,c4)\,c4)\,c4$
$+(24+(60+(-58+(8-2\,c7)\,c7)\,c7)\,c7+(112+(40+(29+5\,c7)\,c7)\,c7+(-44+(48+2\,c7)\,c7+(-8\,c7-92)\,c6)\,c6)\,c6+($
$-32+(-98+(-37+3\,c7)\,c7)\,c7+(16+(-40-19\,c7)\,c7+(-8\,c7-156+20\,c6)\,c6)\,c6+(-30+(43+15\,c7)\,c7+(4\,c7-168$
$-6\,c6)\,c6+(8\,c6-10\,c7+4-6\,c5)\,c5)\,c5)\,c5+(104+(-104+(-21+12\,c7)\,c7)\,c7+(-276+(-160-19\,c7)\,c7+(24\,c7$
$+100+24\,c6)\,c6)\,c6+(152+(90+12\,c7)\,c7+(10\,c7+208-20\,c6)\,c6+(86\,c6-76\,c7+220-38\,c5)\,c5)\,c5+(-272+(-116$
$+13\,c7)\,c7+(20\,c7+16-76\,c6)\,c6+(-28\,c6-26\,c7-84-88\,c5)\,c5+(36\,c5-8\,c6-52\,c7-104+140\,c4)\,c4)\,c4+(32$
$+(-168+(-92+c7)\,c7)\,c7+(92+(185+23\,c7)\,c7+(12\,c7-24-28\,c6)\,c6)\,c6+(-18+(111-20\,c7)\,c7+(-82\,c7-108$
$+4\,c6)\,c6+(-66\,c7-32+(117-c6)\,c6+(2\,c7+66-2\,c5)\,c5)\,c5)\,c5+(-88+(113+3\,c7)\,c7+(-55\,c7+180+(-c7$
$+60)\,c6)\,c6+(-32+(46+2\,c7)\,c7+(2\,c7-82+2\,c6)\,c6+(7-12\,c5)\,c5)\,c5+(139\,c7+332+(-8\,c7-80)\,c6+(24\,c6$
$-18\,c7+98+c5)\,c5+(22\,c5+40\,c6-15\,c7+72-24\,c4)\,c4)\,c4)\,c4+(-62+(-9+7\,c7)\,c7+(-22+(-81+c7)\,c7+90\,c6)\,c6$
$+(-22+(26-5\,c7)\,c7+(-2\,c7+108)\,c6+(-4\,c6+14\,c7+60-12\,c5)\,c5)\,c5+(150+(55-3\,c7)\,c7+(8\,c7-122+6\,c6)\,c6$
$+(-4\,c6+10\,c7-104-12\,c5)\,c5+(52\,c5-40\,c7-120-22\,c4)\,c4)\,c4+(-45+(-37-8\,c7)\,c7+(9\,c7+63-2\,c6)\,c6+($
$-16\,c6+31\,c7+6-34\,c5)\,c5+(26\,c5-16\,c6+8\,c7-9+10\,c4)\,c4+(-8\,c4-14\,c5-18\,c6+21\,c7+18+(-c4-2\,c5-c6$
$-11)\,c3)\,c3)\,c3)\,c3)\,c3)\,c3+(32+(42+(-56+(11-6\,c7)\,c7)\,c7)\,c7+(124+(38+(2-12\,c7)\,c7)\,c7+(24+(-74+6\,c7)\,c7$
$+(10\,c7-24-4\,c6)\,c6)\,c6)\,c6+(-64+(80+(61+4\,c7)\,c7)\,c7+(-348+(-308+42\,c7)\,c7+(12\,c7-192-36\,c6)\,c6)\,c6$
$+(80+(-39+40\,c7)\,c7+(19\,c7+204-38\,c6)\,c6+(-16\,c6-9\,c7+108+6\,c5)\,c5)\,c5)\,c5+(-12+(-402+(-66$
$-33\,c7)\,c7)\,c7+(216+(72+34\,c7)\,c7+(88\,c7+260-48\,c6)\,c6)\,c6+(-292+(184+86\,c7)\,c7+(-56\,c7-128-248\,c6)\,c6$
$+(16\,c7-120+(-c7-100)\,c6+(4\,c6-c7+18+2\,c5)\,c5)\,c5)\,c5+(104+(196+14\,c7)\,c7+(-60\,c7-100+(4\,c6+2\,c7$
$-12)\,c6)\,c6+(104+(-32+c7)\,c7+(-160-8\,c7-8\,c6)\,c6+(4\,c6-7\,c7+74+4\,c5)\,c5)\,c5+(32+(106+6\,c7)\,c7+(-104$
$-12\,c6)\,c6+(16\,c6+4\,c7+156+4\,c5)\,c5+(-24\,c5+12\,c6+6\,c7+88-36\,c4)\,c4)\,c4)\,c4)\,c4+(12+(-12+(144$
$+24\,c7)\,c7)\,c7+(-212+(280+12\,c7)\,c7+(-28\,c7-100-20\,c6)\,c6)\,c6+(-128+(286-34\,c7)\,c7+(-236+(-306-c7)\,c7$
$+(4\,c7+28+4\,c6)\,c6)\,c6+(92+(-116-7\,c7)\,c7+(378-4\,c6)\,c6+(-8\,c6+6\,c7+186+8\,c5)\,c5)\,c5)\,c5+(-460+(196+($
$-44+3\,c7)\,c7)\,c7+(316+(-52-5\,c7)\,c7+(-24\,c7+208-12\,c6)\,c6)\,c6+(-348+(114-2\,c7)\,c7+(34\,c7+536+36\,c6)\,c6$
$+(-18\,c6+16\,c7+356+18\,c5)\,c5)\,c5+(852+(-96+23\,c7)\,c7+(52\,c7-112+24\,c6)\,c6+(-124\,c6-38\,c7+276-26\,c5)\,c5$
$+(-4\,c5-52\,c6-28\,c7-236-72\,c4)\,c4)\,c4)\,c4+(-72+(87+(36+c7)\,c7)\,c7+(60+(26+15\,c7)\,c7+(23\,c7+100$
$-8\,c6)\,c6)\,c6+(418+(157-48\,c7)\,c7+(-8\,c7+182-10\,c6)\,c6+(2\,c6+60\,c7-71+36\,c5)\,c5)\,c5+(132+(-397$
$-45\,c7)\,c7+(-30\,c7-396-64\,c6)\,c6+(-152\,c6+82\,c7-332+36\,c5)\,c5+(250\,c5+32\,c6+37\,c7-444+152\,c4)\,c4)\,c4$
$+(52+(-110-9\,c7)\,c7+(51\,c7+152-4\,c6)\,c6+(-16\,c6+114\,c7-34-120\,c5)\,c5+(-40\,c5+16\,c6-37\,c7-38$
$+120\,c4)\,c4+(2\,c6-24\,c7-59+(-2\,c6-c7-64+2\,c5)\,c5+(16\,c5+4\,c6+3\,c7-86+12\,c4)\,c4+(2\,c4-6\,c5-4\,c6+6\,c7$
$-12-5\,c3)\,c3)\,c3)\,c3)\,c3)\,c3+(12+(-138+(31-9\,c7)\,c7)\,c7+(88+(-124-88\,c7)\,c7+(164+(-52+c7)\,c7+(-4\,c7+60$
$-12\,c6)\,c6)\,c6)\,c6+(-52+(164+(-57+c7)\,c7)\,c7+(136+(24+6\,c7)\,c7+(24\,c7+220)\,c6)\,c6+(-228+(13-6\,c7)\,c7$
$+(14\,c7+168-14\,c6)\,c6+(-38\,c6-3\,c7-6-2\,c5)\,c5)\,c5)\,c5+(-96+(-60+(-70+5\,c7)\,c7)\,c7+(152\,c7-32+(22\,c7$
$+108+44\,c6)\,c6)\,c6+(632+(188-26\,c7)\,c7+(-44\,c7+424-36\,c6)\,c6+(-44\,c6-10\,c7+112+68\,c5)\,c5)\,c5+(-108+($
$-138+5\,c7)\,c7+(-22\,c7-100-52\,c6)\,c6+(4\,c6-72\,c7-396+142\,c5)\,c5+(208\,c5+132\,c6-24\,c7+36)\,c4)\,c4)\,c4+(52$
$+(-28+(-14+9\,c7)\,c7)\,c7+(472+(284-47\,c7)\,c7+(-64\,c7+80+28\,c6)\,c6)\,c6+(-64+(18-69\,c7)\,c7+(26\,c7-432$
$+152\,c6)\,c6+(2\,c6+35\,c7-800+22\,c5)\,c5)\,c5+(688+(-252-35\,c7)\,c7+(140\,c7+348+88\,c6)\,c6+(-112\,c6+136\,c7$
$+152-110\,c5)\,c5+(-40\,c5+168\,c6+156\,c7-480+44\,c4)\,c4)\,c4+(108+(-119+61\,c7)\,c7+(134\,c7-124-132\,c6)\,c6+($
$-228\,c6+84\,c7-228+(-c6-308-2\,c5)\,c5)\,c5+(-54\,c7-288+(-c7-168)\,c6+(14\,c6+6\,c7-628-5\,c5)\,c5+(-42\,c5$
$-4\,c6-9\,c7-144-36\,c4)\,c4)\,c4+(-572+(-105+4\,c7)\,c7+(4\,c7+176-4\,c6)\,c6+(-16\,c6-8\,c7+152)\,c5+(12\,c5+6\,c6$
$-20\,c7-172-6\,c4)\,c4+(28\,c4-12\,c5-8\,c6+5\,c7+170-10\,c3)\,c3)\,c3)\,c3)\,c3+(-20+(216+(60+21\,c7)\,c7)\,c7+(-304$
$+(-60-8\,c7)\,c7+(-56\,c7-208+32\,c6)\,c6)\,c6+(224+(-136-17\,c7)\,c7+(52\,c7+176+196\,c6)\,c6+(124\,c6-27\,c7+20$
$+48\,c5)\,c5)\,c5+(-80+(-92+76\,c7)\,c7+(24\,c7+40-180\,c6)\,c6+(-72\,c6-84\,c7-200+(-c7-156+2\,c5)\,c5)\,c5+($
$-170\,c7-228+(116+2\,c7+4\,c6)\,c6+(-4\,c6-8\,c7-260+8\,c5)\,c5+(20\,c5-8\,c6-10\,c7-112+20\,c4)\,c4)\,c4)\,c4+(200$
$+(-236+34\,c7)\,c7+(-68\,c7-216-384\,c6)\,c6+(700+(-94-c7)\,c7+(4\,c7-300+4\,c6)\,c6+(-12\,c6+2\,c7-32$
$-8\,c5)\,c5)\,c5+(-1336+(40-12\,c7)\,c7+(-24\,c7-108-8\,c6)\,c6+(40\,c6+36\,c7-156+34\,c5)\,c5+(36\,c5+36\,c6+16\,c7$
$+176+20\,c4)\,c4)\,c4+(-172+(227+6\,c7)\,c7+(13\,c7+472)\,c6+(-30\,c6+14\,c7+762+15\,c5)\,c5+(16\,c5+56\,c6-12\,c7$
$+488-28\,c4)\,c4+(48\,c4-32\,c5+34\,c6-28\,c7+184+8\,c3)\,c3)\,c3)\,c3+(68+(80+22\,c7)\,c7+(20+(6+c7)\,c7+(-4\,c7$
$-112-12\,c6)\,c6)\,c6+(-308+(-92+4\,c7)\,c7+(28\,c7-268+8\,c6)\,c6+(-32\,c6+c7-92-26\,c5)\,c5)\,c5+(100+(258$
$-3\,c7)\,c7+(24\,c7+232+20\,c6)\,c6+(-64\,c6+20\,c7+196-48\,c5)\,c5+(-52\,c5-76\,c6+2\,c7-52+4\,c4)\,c4)\,c4+(-448$
$+(152-28\,c7)\,c7+(-20\,c7-120+16\,c6)\,c6+(-96\,c6+20\,c7-96+38\,c5)\,c5+(-48\,c5-80\,c6+20\,c7+568-100\,c4)\,c4$
$+(-24\,c4-22\,c5+128\,c6-51\,c7-124+22\,c3)\,c3)\,c3+(12+(2-35\,c7)\,c7+(6\,c7+76+96\,c6)\,c6+(36\,c6+28\,c7+40$
$+104\,c5)\,c5+(124\,c5-32\,c6+50\,c7+204+36\,c4)\,c4+(28\,c4+44\,c5+108\,c6-60\,c7+484-32\,c3)\,c3+(-108\,c3+12\,c4$
$-60\,c5-92\,c6-90\,c7-36-40\,c2)\,c2)\,c2)\,c2)\,c2)\,c2)\,c2+(-28+(64+(45-11\,c7)\,c7)\,c7+(-28+(12+(58+4\,c7)\,c7)\,c7$
$+(-116+(-152-c7)\,c7+(4\,c7+16-4\,c6)\,c6)\,c6)\,c6+(32+(-198+(-67-8\,c7)\,c7)\,c7+(108+(8-2\,c7)\,c7+(70\,c7$
$+260-52\,c6)\,c6)\,c6+(-18+(121-23\,c7)\,c7+(47\,c7+26-116\,c6)\,c6+(-46+(48-c7)\,c7+(16+2\,c6)\,c6+40\,c5)\,c5)\,c5)$
$c5+(44+(120+(12+22\,c7)\,c7)\,c7+(-232+(132-16\,c7)\,c7+(-16\,c7-104-44\,c6)\,c6)\,c6+(-20+(34-23\,c7)\,c7+(172$
$+(-2+3\,c7)\,c7+(-4+2\,c7-4\,c6)\,c6)\,c6+(-44+(-76-3\,c7)\,c7+(-6\,c7+36-4\,c6)\,c6+(4\,c6+2\,c7+124+8\,c5)\,c5)\,c5)$
$c5+(-16+(-28+(10-3\,c7)\,c7)\,c7+(28+(-4\,c7-44)\,c7+(4\,c7+4)\,c6)\,c6+(-40+(40+7\,c7)\,c7+(24\,c7+100+4\,c6)\,c6$
$+(-24\,c6+2\,c7+88-6\,c5)\,c5)\,c5+(68+(-60-4\,c7)\,c7+(76-16\,c6)\,c6+(-28\,c6+14\,c7+52-4\,c5)\,c5+(-4\,c5+12\,c7$
$-48+16\,c4)\,c4)\,c4)\,c4)\,c4+(40+(-46+(8\,c7-9)\,c7)\,c7+(-12+(84-16\,c7)\,c7+(40+(-28-c7)\,c7+(8-4\,c6)\,c6)\,c6)\,c6$
$+(-154+(-29-54\,c7)\,c7+(254+(2-2\,c7)\,c7+(20\,c7+136)\,c6)\,c6+(-2+(142-10\,c7)\,c7+(6\,c7+42-20\,c6)\,c6+($

$-4\,c6+12\,c7-32+16\,c5)\,c5)\,c5+(-44+(-160+(22+4\,c7)\,c7)\,c7+(36+(-14-10\,c7)\,c7+(16\,c7+84+8\,c6)\,c6)\,c6$
$+(488+(53-22\,c7)\,c7+(39\,c7+128-38\,c6)\,c6+(-82\,c6+30\,c7-142+32\,c5)\,c5)\,c5+(-292+(-20\,c7-56)\,c7+(38\,c7$
$+64-56\,c6)\,c6+(-136\,c6+63\,c7-284+94\,c5)\,c5+(142\,c5+8\,c6-26\,c7+44+44\,c4)\,c4)\,c4)\,c4+((-12+(-6$
$+6\,c7)\,c7)\,c7+(246+(182+c7)\,c7+(-114-25\,c7-6\,c6)\,c6)\,c6+(-84+(-94-11\,c7)\,c7+(-20\,c7-222+56\,c6)\,c6$
$+(20\,c6+4\,c7-166-20\,c5)\,c5)\,c5+(174+(-2+c7)\,c7+(24\,c7+74+40\,c6)\,c6+(-50\,c6+116\,c7-152-76\,c5)\,c5+($
$-66\,c5+18\,c6+21\,c7+200-124\,c4)\,c4)\,c4+(26+(-34+31\,c7)\,c7+(-14\,c7-74-22\,c6)\,c6+(-32\,c6-36\,c7-12+(-c7$
$-8+2\,c5)\,c5)\,c5+(-190+(-34-c7)\,c7+32\,c6+(4\,c6+2\,c7-60+8\,c5)\,c5+(-12\,c5-8\,c6-86-8\,c4)\,c4)\,c4+(-38\,c7$
$-84+(3\,c7+48-2\,c6)\,c6+(-4\,c6-2\,c7+36+14\,c5)\,c5+(-4\,c5-8\,c6-3\,c7+62-6\,c4)\,c4+(2\,c4-4\,c5+10\,c6+4\,c7$
$+10+6\,c3)\,c3)\,c3)\,c3)\,c3+(16+(16+(25-15\,c7)\,c7)\,c7+(-100+(40+(-3\,c7-4)\,c7)\,c7+(52+(12+c7)\,c7+(8\,c7$
$+140+4\,c6)\,c6)\,c6)\,c6+(-204+(-52+(44+6\,c7)\,c7)\,c7+(-160+(-136+17\,c7)\,c7+(-22\,c7+136-20\,c6)\,c6)\,c6+(328$
$+(54-c7)\,c7+(-32\,c7+240+20\,c6)\,c6+(-12\,c6-14\,c7-44-8\,c5)\,c5)\,c5)\,c5+(-52+(-16+(78-14\,c7)\,c7)\,c7+(544$
$+(144-10\,c7)\,c7+(40-12\,c7+16\,c6)\,c6)\,c6+(-260+(102-8\,c7)\,c7+(-70\,c7-704+120\,c6)\,c6+(60\,c6+15\,c7-446$
$+36\,c5)\,c5)\,c5+(388+(128-18\,c7)\,c7+(-16\,c7-64+96\,c6)\,c6+(68\,c6+32\,c7+160+120\,c5)\,c5+(-24\,c5+56\,c7-68$
$-100\,c4)\,c4)\,c4)\,c4+(-76+(528+(196+14\,c7)\,c7)\,c7+(-240+(-202-23\,c7)\,c7+(-48\,c7-308+68\,c6)\,c6)\,c6+(60+($
$-345+20\,c7)\,c7+(-13\,c7-428+154\,c6)\,c6+(42\,c6-16\,c7+294-192\,c5)\,c5)\,c5+(20+(-250+8\,c7)\,c7+(86\,c7-224$
$-92\,c6)\,c6+(9\,c7-278+(-c7+112-2\,c6)\,c6+(2\,c6+4\,c7-248)\,c5)\,c5+(-108+(-306+3\,c7)\,c7+(6\,c7+36)\,c6+($
$-24\,c6+3\,c7-386-18\,c5)\,c5+(10\,c5-16\,c6+6\,c7-436+48\,c4)\,c4)\,c4)\,c4+(366+(-10+42\,c7)\,c7+(-200+(96-c7)\,c7$
$+(-c7-282+2\,c6)\,c6)\,c6+(390+(-136+7\,c7)\,c7+(4\,c7-394+2\,c6)\,c6+(-8\,c6-14\,c7+20-8\,c5)\,c5)\,c5+(-1266+($
$-210-3\,c7)\,c7+(-26\,c7+240-18\,c6)\,c6+(26\,c6+4\,c7+522-8\,c5)\,c5+(-32\,c5+66\,c6+67\,c7+170+86\,c4)\,c4)\,c4+(88$
$+(134+11\,c7)\,c7+(12\,c7+158+6\,c6)\,c6+(-12\,c6-54\,c7+316+84\,c5)\,c5+(-56\,c5+12\,c6-52\,c7+506-22\,c4)\,c4+($
$-24\,c4-12\,c5+82\,c6-52\,c7-92+44\,c3)\,c3)\,c3)\,c3)\,c3+(52+(-112+(-103-21\,c7)\,c7)\,c7+(376+(-304+16\,c7)\,c7$
$+(48\,c7+184-8\,c6)\,c6)\,c6+(-32+(-64+41\,c7)\,c7+(168\,c7-76-72\,c6)\,c6+(-118+(114-c7)\,c7+(-c7-182$
$+2\,c6)\,c6+(4\,c6-210)\,c5)\,c5)\,c5+(248+(-128-50\,c7)\,c7+(-224+(20+2\,c7)\,c7+(8\,c7-228+8\,c6)\,c6)\,c6+(448+($
$-90-10\,c7)\,c7+(-32\,c7-436-20\,c6)\,c6+(32\,c6+5\,c7-256+24\,c5)\,c5)\,c5+(-20\,c6+68\,c7-736+(32\,c6-172$
$+22\,c5)\,c5+(4\,c5-8\,c6+208)\,c4)\,c4)\,c4+(324+(64+(-88+c7)\,c7)\,c7+(168+(136-11\,c7)\,c7+(-24\,c7-172$
$-4\,c6)\,c6)\,c6+(-1152+(-131+26\,c7)\,c7+(41\,c7-514+34\,c6)\,c6+(4\,c6-12\,c7-140-48\,c5)\,c5)\,c5+(192+(464$
$+7\,c7)\,c7+(40\,c7+1092+68\,c6)\,c6+(48\,c6-8\,c7+1276-166\,c5)\,c5+(-358\,c5-140\,c6+46\,c7+392-84\,c4)\,c4)\,c4+($
$-364+(251-17\,c7)\,c7+(-54\,c7-34+6\,c6)\,c6+(-4\,c6-100\,c7-116+262\,c5)\,c5+(-46\,c5-174\,c6+32\,c7+378$
$-168\,c4)\,c4+(-36\,c4+124\,c5+184\,c6-48\,c7-148+42\,c3)\,c3)\,c3)\,c3+(-16+(-8+(-33+9\,c7)\,c7)\,c7+(-440+(-152$
$+30\,c7)\,c7+(-104-32\,c6)\,c6)\,c6+(336+(-20+18\,c7)\,c7+(-24\,c7+648-68\,c6)\,c6+(-32\,c6-31\,c7+428-24\,c5)\,c5)\,c5$
$+(-688+(36+20\,c7)\,c7+(-48\,c7-136-40\,c6)\,c6+(64\,c6-90\,c7-188-46\,c5)\,c5+(12\,c5+12\,c6-80\,c7+312$
$+60\,c4)\,c4)\,c4+(-100+(-52-75\,c7)\,c7+(-122\,c7+144+152\,c6)\,c6+(320\,c6-7\,c7+756+444\,c5)\,c5+(618\,c5+32\,c6$
$+206\,c7-12+344\,c4)\,c4+(194\,c4-22\,c5-140\,c6+33\,c7+1278-330\,c3)\,c3)\,c3+(-164+(128+32\,c7)\,c7+(60\,c7+28$
$+160\,c6)\,c6+(192\,c6+74\,c7-312+8\,c5)\,c5+(-20\,c5-8\,c6+20\,c7+804-124\,c4)\,c4+(-236\,c4-862\,c5-844\,c6-266\,c7$
$-204-496\,c3)\,c3+(184\,c3-140\,c4+36\,c5+68\,c6-28\,c7+280-212\,c2)\,c2)\,c2)\,c2)\,c2)\,c2+(18+(30+(23+(-3$
$+2\,c7)\,c7)\,c7)\,c7+(-69+(-62+(21+2\,c7)\,c7)\,c7+(-38+(-17-7\,c7)\,c7+(-c7+28+2\,c6)\,c6)\,c6)\,c6+(18+(-88+(-56$
$-c7)\,c7)\,c7+(256+(160-11\,c7)\,c7+(14\,c7+2+2\,c6)\,c6)\,c6+(-55+(25-16\,c7)\,c7+(2\,c7-296+8\,c6)\,c6+(18\,c6-c7$
$-56-10\,c5)\,c5)\,c5)\,c5+(17+(143+(17+16\,c7)\,c7)\,c7+(-134+(1-15\,c7)\,c7+(-25\,c7-160+24\,c6)\,c6)\,c6+(108+($
$-129-5\,c7)\,c7+(10\,c7+56+98\,c6)\,c6+(20\,c6-82\,c7+4-26\,c5)\,c5)\,c5+(-30+(-104+2\,c7)\,c7+(-c6^{2}+5\,c7+3)\,c6$
$+(13\,c7-2+(3\,c7+70)\,c6+(-3\,c6-121-2\,c5)\,c5)\,c5+(-23+(-48-3\,c7)\,c7+(c7+48+3\,c6)\,c6+(2\,c6+c7-108$
$+c5)\,c5+(2\,c5-3\,c6-3\,c7-30+9\,c4)\,c4)\,c4)\,c4)\,c4+(-14+(-95+(-39-3\,c7)\,c7)\,c7+(114+(-86-8\,c7)\,c7+(-4\,c7$
$-18+14\,c6)\,c6)\,c6+(80+(-82-7\,c7)\,c7+(71\,c7+24+(-c7+28)\,c6)\,c6+(-116+(31+2\,c7)\,c7+(-132+2\,c6)\,c6+($
$-3\,c7-52-2\,c5)\,c5)\,c5)\,c5+(42+(-75+32\,c7)\,c7+(-10+(-11+c7)\,c7+(2\,c7-68)\,c6)\,c6+(-109\,c7+214+(-6\,c7$
$-164-4\,c6)\,c6+(4\,c6-6\,c7-120-4\,c5)\,c5)\,c5+(-364+(51-10\,c7)\,c7+(-6\,c7+18+12\,c6)\,c6+(46\,c6+5\,c7-6$
$+2\,c5)\,c5+(-10\,c5+18\,c6-16\,c7+194-2\,c4)\,c4)\,c4+(8+(40+(-17-c7)\,c7)\,c7+(19+(12-c7)\,c7+(-6\,c7-47$
$+2\,c6)\,c6)\,c6+(-322+(-82+16\,c7)\,c7+(-6\,c7+102+10\,c6)\,c6+(6\,c6-24\,c7+140-10\,c5)\,c5)\,c5+(231+(122$
$+12\,c7)\,c7+(-18\,c7+82+26\,c6)\,c6+(78\,c6-35\,c7+164-24\,c5)\,c5+(-86\,c5+18\,c6-25\,c7+149-66\,c4)\,c4)\,c4+(-54$
$+(37+11\,c7)\,c7+(-17\,c7-2\,c6)\,c6+(10\,c6-52\,c7+54+30\,c5)\,c5+(40\,c5+24\,c6+17\,c7-160+28\,c4)\,c4+(-26\,c6$
$+17\,c7+52+(-14-c5)\,c5+(-2\,c5-c7+50-2\,c4)\,c4+(2\,c5+2\,c6-c7-16+2\,c3)\,c3)\,c3)\,c3)\,c3)\,c3+(-45+(81+(24$
$-5\,c7)\,c7)\,c7+(-82+(-14+37\,c7)\,c7+(5\,c7-46+(c7-30+6\,c6)\,c6)\,c6)\,c6+(44+(-89+69\,c7)\,c7+(-198+(-68$
$-c7)\,c7+(-22\,c7-186-2\,c6)\,c6)\,c6+(222+(-21+9\,c7)\,c7+(-4\,c7-52+22\,c6)\,c6+(14\,c6-c7+22-10\,c5)\,c5)\,c5)\,c5$
$+(132+(239+(26-3\,c7)\,c7)\,c7+(94+(-69+10\,c7)\,c7+(-5\,c7-86-14\,c6)\,c6)\,c6+(-724+(-9+17\,c7)\,c7+(-20\,c7$
$-324+14\,c6)\,c6+(42\,c6-2\,c7+150-26\,c5)\,c5)\,c5+(275+(81+5\,c7)\,c7+(-2\,c7+78+15\,c6)\,c6+(7\,c7+488-71\,c5)\,c5$
$+(-142\,c5-76\,c6+21\,c7-66-27\,c4)\,c4)\,c4)\,c4+(-66+(245+(47-10\,c7)\,c7)\,c7+(-708+(-351+13\,c7)\,c7+(42\,c7$
$+124-6\,c6)\,c6)\,c6+(246+(-27+41\,c7)\,c7+(6\,c7+824-128\,c6)\,c6+(-24\,c6+862+2\,c5)\,c5)\,c5+(-456+(45$
$+16\,c7)\,c7+(-33\,c7-62-124\,c6)\,c6+(36\,c6-82\,c7+54+148\,c5)\,c5+(78\,c5-84\,c6-73\,c7-68+92\,c4)\,c4)\,c4+(21$
$+(122-49\,c7)\,c7+(-22\,c7+56+56\,c6)\,c6+(106\,c6-3\,c7+324+224\,c5)\,c5+(38\,c6+134\,c7-234+(-2\,c6-c7+438$
$+2\,c5)\,c5+(16\,c5+4\,c6+3\,c7+346+12\,c4)\,c4)\,c4+(678+(176-c7)\,c7+(-c7-244+2\,c6)\,c6+(6\,c6+6\,c7-326$
$-8\,c5)\,c5+(-4\,c5-4\,c6+3\,c7+20)\,c4+(-14\,c4+4\,c5-2\,c6-2\,c7-186+2\,c3)\,c3)\,c3)\,c3)\,c3+(59+(-357+(-113$
$-26\,c7)\,c7)\,c7+(384+(103+11\,c7)\,c7+(56\,c7+394-43\,c6)\,c6)\,c6+(-264+(264-2\,c7)\,c7+(-11\,c7+44-152\,c6)\,c6+($
$-67\,c6+63\,c7-121+18\,c5)\,c5)\,c5+(128+(134-62\,c7)\,c7+(c7+92+159\,c6)\,c6+(2\,c6+7\,c7+272+307\,c5)\,c5+(232\,c7$

$$- 43 + (-22 - c6)\, c6 + (2\, c7 + 368 - 2\, c5)\, c5 + (-12\, c5 + 2\, c6 + 6\, c7 + 290 - 21\, c4)\, c4)\, c4)\, c4 + (-286 + (213 - 82\, c7)\, c7$$
$$+ (75\, c7 + 324 + 430\, c6)\, c6 + (116\, c7 - 1320 + (-c7 + 326)\, c6 + (4\, c6 - c7 - 104 + 2\, c5)\, c5)\, c5 + (2180 + (102 + 3\, c7)\, c7$$
$$+ (10\, c7 - 78 + 8\, c6)\, c6 + (-20\, c6 - 25\, c7 - 362 - 8\, c5)\, c5 + (-40\, c6 - 22\, c7 - 294 - 36\, c4)\, c4)\, c4 + (-236 + (-181 - 5\, c7)\, c7$$
$$+ (-12\, c7 - 729 - 10\, c6)\, c6 + (24\, c6 + 21\, c7 - 1380 - 36\, c5)\, c5 + (42\, c5 - 32\, c6 + 22\, c7 - 947 + 22\, c4)\, c4 + (-12\, c4 + 24\, c5$$
$$- 60\, c6 + 39\, c7 + 8 - 24\, c3)\, c3)\, c3)\, c3 + (-180 + (-210 + 6\, c7)\, c7 + (-58\, c7 - 108 + (c7 + 69 + 6\, c6)\, c6)\, c6 + (826 + (39 - c7)\, c7$$
$$+ (-18\, c7 + 400 - 2\, c6)\, c6 + (20\, c6 - c7 + 63 + 14\, c5)\, c5)\, c5 + (-434 + (-324 + 4\, c7)\, c7 + (-24\, c7 - 704 - 8\, c6)\, c6 + (72\, c6$$
$$- 26\, c7 - 816 + 56\, c5)\, c5 + (126\, c5 + 96\, c6 - 19\, c7 + 4 + 18\, c4)\, c4)\, c4 + (696 + (-279 + 31\, c7)\, c7 + (18\, c7 + 64 - 30\, c6)\, c6$$
$$+ (72\, c6 + 27\, c7 + 338 - 138\, c5)\, c5 + (68\, c5 + 112\, c6 - 20\, c7 - 550 + 98\, c4)\, c4 + (40\, c4 - 18\, c5 - 180\, c6 + 63\, c7 + 837$$
$$- 36\, c3)\, c3)\, c3 + (-64 + (42 + 60\, c7)\, c7 + (4\, c7 - 155 - 139\, c6)\, c6 + (-96\, c6 - 12\, c7 - 236 - 238\, c5)\, c5 + (-280\, c5 + 2\, c6$$
$$- 116\, c7 + 17 - 160\, c4)\, c4 + (-72\, c4 + 4\, c5 - 48\, c6 + 50\, c7 - 1446 + 158\, c3)\, c3 + (306\, c3 + 22\, c4 + 350\, c5 + 388\, c6 + 165\, c7$$
$$+ 211 - 35\, c2)\, c2)\, c2)\, c2)\, c2)\, c2 + (12 + (-17 + (4\, c7 - 12)\, c7)\, c7 + (-4 + (1 - c7)\, c7 + (16 + (-4 - c7)\, c7 - 30\, c6)\, c6)\, c6 + (48$$
$$+ (36 - 36\, c7)\, c7 + (136 + (108 - c7)\, c7 + (6\, c7 - 66)\, c6)\, c6 + (11\, c7 - 128 + (2\, c7 - 74 - 8\, c6)\, c6 + (8\, c6 + 72)\, c5)\, c5)\, c5 + (-14$$
$$+ (5 + (-2 + 3\, c7)\, c7)\, c7 + (-142 + (-38 - 4\, c7)\, c7 + (-c7 - 2\, c6)\, c6)\, c6 + (134 + (-56 + 9\, c7)\, c7 + (22\, c7 + 344 - 30\, c6)\, c6 + ($$
$$-8\, c6 - 12\, c7 + 138 + 4\, c5)\, c5)\, c5 + (-110 + (-43 + 2\, c7)\, c7 + (7\, c7 + 50 - 38\, c6)\, c6 + (-10\, c6 + 24\, c7 - 16 - 34\, c5)\, c5 + (20\, c5$$
$$+ 4\, c6 - 6\, c7 - 6 + 40\, c4)\, c4)\, c4)\, c4 + (-8 + (-165 + (-43 - 2\, c7)\, c7)\, c7 + (142 + (49 + 6\, c7)\, c7 + (11\, c7 + 100 - 16\, c6)\, c6)\, c6$$
$$+ (12 + (111 - 14\, c7)\, c7 + (9\, c7 + 112 - 38\, c6)\, c6 + (-14\, c6 + 15\, c7 - 130 + 72\, c5)\, c5)\, c5 + (-8 + (95 + 9\, c7)\, c7 + (-12\, c7 + 82$$
$$+ 2\, c6)\, c6 + (-52\, c6 + 9\, c7 + 36 + 94\, c5)\, c5 + (102\, c7 + (-50 - c7)\, c6 + (4\, c6 - c7 + 150 + 2\, c5)\, c5 + (4\, c6 - 3\, c7 + 140$$
$$- 12\, c4)\, c4)\, c4)\, c4 + (-64 + (10 - 10\, c7)\, c7 + (-20\, c7 + 2 + 58\, c6)\, c6 + (-202 + (24 - c7)\, c7 + 118\, c6 + (4\, c7 - 22)\, c5)\, c5 + (448$$
$$+ (65 + c7)\, c7 + (2\, c7 - 118)\, c6 + (-8\, c6 + 2\, c7 - 238)\, c5 + (18\, c5 - 14\, c6 - 12\, c7 - 132 - 18\, c4)\, c4)\, c4 + (-102 + (-35$$
$$- 2\, c7)\, c7 + (-34 - 4\, c6)\, c6 + (-4\, c6 + 18\, c7 - 86 - 20\, c5)\, c5 + (24\, c5 - 12\, c6 + 16\, c7 - 164 + 12\, c4)\, c4 + (2\, c4 - 2\, c5 - 24\, c6$$
$$+ 12\, c7 + 68 - 14\, c3)\, c3)\, c3)\, c3)\, c3 + (-46 + (111 + (6\, c7 + 56)\, c7)\, c7 + (-184 + (110 - 3\, c7)\, c7 + (-11\, c7 - 52 + 12\, c6)\, c6)\, c6$$
$$+ (46 + (54 + 3\, c7)\, c7 + (-78\, c7 - 40 + 22\, c6)\, c6 + (114\, c6 - 51\, c7 + 106 + 76\, c5)\, c5)\, c5 + (-22 + (146 - c7)\, c7 + (-22\, c7 - 58$$
$$+ (-c7 + 140 - 2\, c6)\, c6)\, c6 + (-438 + (70 + 2\, c7)\, c7 + (6\, c7 + 192 + 2\, c6)\, c6 + (-8\, c6 - 2\, c7 + 130 - 4\, c5)\, c5)\, c5 + (436 + (-46$$
$$+ 3\, c7)\, c7 + (4\, c7 - 22 - 6\, c6)\, c6 + (-20\, c6 + 2\, c7 + 28 - 4\, c5)\, c5 + (2\, c5 + 2\, c6 + 9\, c7 - 196 + 6\, c4)\, c4)\, c4)\, c4 + (-120 + (-131$$
$$+ 62\, c7)\, c7 + (-326 + (-128 + 2\, c7)\, c7 + (8\, c7 + 110)\, c6)\, c6 + (970 + (38 - 14\, c7)\, c7 + (-7\, c7 + 128 - 14\, c6)\, c6 + (-2\, c6 + 16\, c7$$
$$- 138 + 16\, c5)\, c5)\, c5 + (-514 + (-202 - 5\, c7)\, c7 + (8\, c7 - 610 - 26\, c6)\, c6 + (-52\, c6 + 11\, c7 - 936 + 66\, c5)\, c5 + (170\, c5 + 60\, c6$$
$$- 3\, c7 - 326 + 66\, c4)\, c4)\, c4 + (172 + (-131 - 9\, c7)\, c7 + (28\, c7 - 82 + 4\, c6)\, c6 + (-18\, c6 + 82\, c7 + 192 - 136\, c5)\, c5 + (-18\, c5$$
$$+ 46\, c6 - 19\, c7 + 166 + 30\, c4)\, c4 + (-28\, c4 - 32\, c5 - 42\, c6 - 2\, c7 + 214)\, c3)\, c3)\, c3 + (48 + (-126 + (-23 - 2\, c7)\, c7)\, c7 + (506$$
$$+ (188 - 10\, c7)\, c7 + (10 - 3\, c7 + 14\, c6)\, c6)\, c6 + (-398 + (88 - 16\, c7)\, c7 + (-2\, c7 - 810 + 54\, c6)\, c6 + (24\, c6 + 14\, c7 - 542$$
$$+ 4\, c5)\, c5)\, c5 + (510 + (-7 - 20\, c7)\, c7 + (20\, c7 + 80 + 68\, c6)\, c6 + (-48\, c6 + 12\, c7 + 100 + 24\, c5)\, c5 + (-42\, c5 - 10\, c6 + 51\, c7$$
$$- 12 - 68\, c4)\, c4)\, c4 + (-106 + (3 + 41\, c7)\, c7 + (64\, c7 + 38 - 86\, c6)\, c6 + (-200\, c6 + 22\, c7 - 794 - 378\, c5)\, c5 + (-580\, c5 - 16\, c6$$
$$- 230\, c7 + 934 - 462\, c4)\, c4 + (-108\, c4 + 320\, c5 + 318\, c6 - 169\, c7 - 1614 + 380\, c3)\, c3)\, c3 + (126 + (-197 + 8\, c7)\, c7 + (-50\, c7$$
$$+ 22 - 200\, c6)\, c6 + (-132\, c6 - 56\, c7 + 740 + 38\, c5)\, c5 + (110\, c5 + 46\, c6 - 51\, c7 - 1170 + 194\, c4)\, c4 + (530\, c4 + 1550\, c5$$
$$+ 1108\, c6 + 174\, c7 + 628 + 376\, c3)\, c3 + (-978\, c3 + 48\, c4 - 118\, c5 - 110\, c6 + 83\, c7 - 406 + 548\, c2)\, c2)\, c2)\, c2)\, c2 + (1 + (-27$$
$$+ (-16 + 3\, c7)\, c7)\, c7 + (37 + (-8\, c7 + 20)\, c7 + (2\, c7 - 9 + (3 - c6)\, c6)\, c6)\, c6 + (8 + (14 - 22\, c7)\, c7 + (33\, c7 + 2 + (4\, c7$$
$$+ 20)\, c6)\, c6 + (-79 + (-8 - 2\, c7)\, c7 + (9 - 4\, c6)\, c6 + (6 + 2\, c5)\, c5)\, c5)\, c5 + (-41 + (-87 + 4\, c7)\, c7 + (-32 + (-9 - 3\, c7)\, c7$$
$$+ (2\, c7 + 17 + 2\, c6)\, c6)\, c6 + (180 + (-39 - 2\, c7)\, c7 + (10\, c7 + 82 - 4\, c6)\, c6 + (-10\, c6 - 101 + 4\, c5)\, c5)\, c5 + (-90 + (-4\, c7$$
$$- 8)\, c7 + (c7 - 22 - 2\, c6)\, c6 + (-2\, c6 + 6\, c7 - 134 + 6\, c5)\, c5 + (28\, c5 + 10\, c6 - 6\, c7 + 16 + 7\, c4)\, c4)\, c4)\, c4 + (20 + (-68 + (-13$$
$$+ 2\, c7)\, c7)\, c7 + (180 + (75 - c7)\, c7 + (-6\, c7 - 32)\, c6)\, c6 + (-122 + (11 - 9\, c7)\, c7 + (-222 + 24\, c6)\, c6 + (-2\, c7 - 188)\, c5)\, c5$$
$$+ (100 + (16 - 3\, c7)\, c7 + (-c7 - 26 + 34\, c6)\, c6 + (-2\, c6 + 8\, c7 - 8 - 32\, c5)\, c5 + (-18\, c5 + 6\, c6 + 2\, c7 + 100 - 42\, c4)\, c4)\, c4 + ($$
$$-14 + (-28 + 8\, c7)\, c7 + (-4\, c7 + 8 - 2\, c6)\, c6 + (-10\, c7 - 62 - 52\, c5)\, c5 + (-84\, c5 + 18\, c6 - 40\, c7 + 116 + (-2\, c5 - c6 - 99$$
$$- c4)\, c4)\, c4 + (64\, c6 - 46\, c7 - 182 + (96 - c7 + 2\, c5)\, c5 + 24\, c4 + (c4 + c6 + 44)\, c3)\, c3)\, c3)\, c3 + (-9 + (184 + (53 + 9\, c7)\, c7)\, c7$$
$$+ (-210 + (-51 - 7\, c7)\, c7 + (-22\, c7 - 171 + 20\, c6)\, c6)\, c6 + (100 + (-98 + 10\, c7)\, c7 + (-10\, c7 - 48 + 48\, c6)\, c6 + (24\, c6 - 32\, c7$$
$$+ 121 - 32\, c5)\, c5)\, c5 + (-70 + (-48 + 14\, c7)\, c7 + (-10\, c7 - 76 - 36\, c6)\, c6 + (24\, c6 + 12\, c7 - 46 - 126\, c5)\, c5 + (-164\, c5$$
$$- 102\, c7 + 96 + (2\, c5 - c7 - 152 + 8\, c4)\, c4)\, c4)\, c4 + (88 + (-109 + 45\, c7)\, c7 + (-36\, c7 - 46 - 156\, c6)\, c6 + (-134\, c6 - 32\, c7$$
$$+ 846 + 64\, c5)\, c5 + (-44\, c7 - 1182 + (-c7 + 104 - 2\, c6)\, c6 + (2\, c6 + 4\, c7 + 330)\, c5 + (-10\, c5 + 16\, c6 + 9\, c7 + 208$$
$$+ 16\, c4)\, c4)\, c4 + (312 + (62 + c7)\, c7 + (2\, c7 + 292 + 6\, c6)\, c6 + (-4\, c6 - 14\, c7 + 696 + 16\, c5)\, c5 + (-28\, c5 + 12\, c6 - 12\, c7 + 512$$
$$- 7\, c4)\, c4 + (-4\, c5 + 28\, c6 - 16\, c7 - 132 + 13\, c3)\, c3)\, c3)\, c3 + (107 + (183 - 28\, c7)\, c7 + (83\, c7 + 199 + (-33 - c6)\, c6)\, c6 + (29\, c7$$
$$- 734 + (3\, c7 - 250)\, c6 + (-3\, c6 + 41 - 2\, c5)\, c5)\, c5 + (493 + (145 - c7)\, c7 + (9\, c7 + 540 + c6)\, c6 + (-30\, c6 + 9\, c7 + 714$$
$$- 19\, c5)\, c5 + (-84\, c5 - 51\, c6 + 12\, c7 + 93 - 21\, c4)\, c4)\, c4 + (-424 + (159 - 10\, c7)\, c7 + (-7\, c7 + 42 + 14\, c6)\, c6 + (-20\, c6 - 32\, c7$$
$$- 476 + 96\, c5)\, c5 + (-30\, c5 - 46\, c6 + 8\, c7 + 36 - 34\, c4)\, c4 + (-7\, c4 + 14\, c5 + 85\, c6 - 24\, c7 - 1035 + 16\, c3)\, c3)\, c3 + (129 + (-62$$
$$- 37\, c7)\, c7 + (-5\, c7 + 74 + 81\, c6)\, c6 + (70\, c6 - 11\, c7 + 280 + 197\, c5)\, c5 + (254\, c5 + 26\, c6 + 105\, c7 - 470 + 193\, c4)\, c4 + (56\, c4$$
$$- 108\, c5 - 74\, c6 + 23\, c7 + 1756 - 193\, c3)\, c3 + (-294\, c3 - 111\, c4 - 572\, c5 - 515\, c6 - 108\, c7 - 394 + 307\, c2)\, c2)\, c2)\, c2)\, c2 + (6$$
$$+ (-25 - 11\, c7)\, c7 + (40 + (1 - c7)\, c7 + (-2 + c7 - 2\, c6)\, c6)\, c6 + (-16 + (-10 - 3\, c7)\, c7 + (14\, c7 + 14 - 2\, c6)\, c6 + (-12\, c6$$
$$+ 2\, c7 - 10 - 8\, c5)\, c5)\, c5 + (-10 + (-35 + 5\, c7)\, c7 + (2\, c7 + 54 - 22\, c6)\, c6 + (-20\, c6 - 10\, c7 + 98 - 32\, c5)\, c5 + (-74 + (16$$
$$- c7)\, c7 + (-c7 + 4 + 2\, c6)\, c6 + (4\, c6 + 10)\, c5 + (-3\, c7 + 46 - 2\, c4)\, c4)\, c4)\, c4 + (22 + (40 - 13\, c7)\, c7 + (30\, c7 + 84 + (-22$$
$$- c7)\, c6)\, c6 + (-218 + (4 + 2\, c7)\, c7 + (-16 + 2\, c6)\, c6 + (-3\, c7 + 60 - 2\, c5)\, c5)\, c5 + (154 + (28 + c7)\, c7 + (-4\, c7 + 86 + 4\, c6)\, c6$$
$$+ (12\, c6 - 2\, c7 + 184 - 8\, c5)\, c5 + (-22\, c5 - 6\, c6 - 3\, c7 + 68 - 14\, c4)\, c4)\, c4 + (-32 + (14 + 3\, c7)\, c7 + (-4\, c7 + 32 - 2\, c6)\, c6$$
$$+ (4\, c6 - 14\, c7 - 56 + 18\, c5)\, c5 + (8\, c5 + 4\, c7 - 96 + 4\, c4)\, c4 + (12\, c4 + 3\, c7 - 60 - 4\, c3)\, c3)\, c3)\, c3 + (-12 + (73 + 18\, c7)\, c7 + ($$
$$-222 + (-80 + c7)\, c7 + (c7 + 8 - 2\, c6)\, c6)\, c6 + (188 + (-40 + 7\, c7)\, c7 + (2\, c7 + 316 - 18\, c6)\, c6 + (-4\, c6 - 2\, c7 + 208)\, c5)\, c5 + ($$
$$-180 + (-14 + 7\, c7)\, c7 + (-24 - 2\, c7 - 30\, c6)\, c6 + (8\, c6 + 10\, c7 - 36 - 4\, c5)\, c5 + (18\, c5 + 2\, c6 - 10\, c7 - 88 + 28\, c4)\, c4)\, c4 + (72$$

$+(-4-7\,c7)\,c7+(-12\,c7-56+14\,c6)\,c6+(32\,c6-2\,c7+284+124\,c5)\,c5+(200\,c5-10\,c6+94\,c7-584+194\,c4)\,c4$
$+(12\,c4-180\,c5-152\,c6+86\,c7+736-152\,c3)\,c3)\,c3+(-44+(129-20\,c7)\,c7+(26\,c7-64+104\,c6)\,c6+(38\,c6+18\,c7$
$-580-26\,c5)\,c5+(-106\,c5-26\,c6+14\,c7+780-130\,c4)\,c4+(-396\,c4-1010\,c5-570\,c6-51\,c7-552-64\,c3)\,c3$
$+(1228\,c3+76\,c4+180\,c5+72\,c6-63\,c7+292-676\,c2)\,c2)\,c2)\,c2+(1+(-29+(-c7-8)\,c7)\,c7+(34+(10+c7)\,c7+(3\,c7$
$+20-3\,c6)\,c6)\,c6+(-20+(7-2\,c7)\,c7+(3\,c7+12-6\,c6)\,c6+(-3\,c6+4\,c7-21+6\,c5)\,c5)\,c5+(14+(-c7+4)\,c7+(2\,c7$
$+18+c6)\,c6+(-6\,c6-14+15\,c5)\,c5+(26\,c5-c6+14\,c7-18+(23-c4)\,c4)\,c4)\,c4+(-8+(21-8\,c7)\,c7+(7\,c7-12$
$+18\,c6)\,c6+(16\,c6+2\,c7-152-6\,c5)\,c5+(-62\,c5-20\,c6+c7+200+(2\,c5-2\,c6-c7-42-2\,c4)\,c4)\,c4+(-9\,c7-71+($
$-32-c6)\,c6+(2\,c7-110-2\,c5)\,c5+(4\,c5-2\,c6+2\,c7-84+c4)\,c4+(-4\,c6+2\,c7+36-2\,c3)\,c3)\,c3)\,c3+(-30+(-62$
$+14\,c7)\,c7+(-36\,c7-92+13\,c6)\,c6+(74\,c6-20\,c7+270-37\,c5)\,c5+(-27\,c7-202+(-c7-158)\,c6+(4\,c6-c7-232$
$+2\,c5)\,c5+(22\,c5+12\,c6-2\,c7-55+8\,c4)\,c4)\,c4+(130+(c7-30)\,c7+(-36+c7-2\,c6)\,c6+(2\,c6+10\,c7+204$
$-24\,c5)\,c5+(4\,c5+6\,c6-c7+90+4\,c4)\,c4+(-4\,c4-2\,c5-16\,c6+3\,c7+434-2\,c3)\,c3)\,c3+(-80+(37+10\,c7)\,c7+(c7$
$-10-21\,c6)\,c6+(-20\,c6+7\,c7-136-76\,c5)\,c5+(-112\,c5-14\,c6-48\,c7+388-99\,c4)\,c4+(-18\,c4+82\,c5+68\,c6$
$-32\,c7-1000+99\,c3)\,c3+(128\,c3+118\,c4+418\,c5+318\,c6+30\,c7+322-430\,c2)\,c2)\,c2)\,c2+((-11-3\,c7)\,c7+(11\,c7$
$+32-2\,c6)\,c6+(-28+(-c7+4)\,c7+(-38+2\,c6)\,c6-26\,c5)\,c5+(26+(-c7+4)\,c7+(2+4\,c6)\,c6+(2-2\,c7)\,c5+(-2\,c5$
$+22-4\,c4)\,c4)\,c4+(2\,c7-12+(c7+8)\,c6+(-c7-30-14\,c5)\,c5+(-24\,c5+4\,c6-13\,c7+94-26\,c4)\,c4+(2\,c4+26\,c5$
$+22\,c6-12\,c7-110+20\,c3)\,c3)\,c3+(6+(-38+8\,c7)\,c7+(-8\,c7+30-24\,c6)\,c6+(-4\,c6-2\,c7+182+4\,c5)\,c5+(36\,c5$
$+4\,c6+3\,c7-236+38\,c4)\,c4+(124\,c4+280\,c5+126\,c6+9\,c7+188-16\,c3)\,c3+(-650\,c3-68\,c4-102\,c5-20\,c6+19\,c7$
$-116+442\,c2)\,c2)\,c2+(3+(7-2\,c7)\,c7+(5\,c7+13-2\,c6)\,c6+(-8\,c6+3\,c7-34+6\,c5)\,c5+(26\,c5+16\,c6+2\,c7+27+($
$-c6+9-2\,c5-c4)\,c4)\,c4+(6\,c6+c7-16+(-c7-26+2\,c5)\,c5-20\,c4+(c4+c6-59)\,c3)\,c3+(21+(-10-c7)\,c7+2\,c6^2$
$+(-c7+28+2\,c6+14\,c5)\,c5+(24\,c5+2\,c6+11\,c7-116+23\,c4)\,c4+(2\,c4-22\,c5-20\,c6+10\,c7+264-23\,c3)\,c3+($
$-26\,c3-55\,c4-154\,c5-101\,c6-3\,c7-131+273\,c2)\,c2)\,c2+((-c7+4)\,c7+(-4+c7+2\,c6)\,c6-20\,c5+(-4\,c5-c7+26$
$-4\,c4)\,c4+(-14\,c4-28\,c5-10\,c6-c7-22+4\,c3)\,c3+(156\,c3+20\,c4+24\,c5+2\,c6-2\,c7+24-156\,c2)\,c2+(-2+c7+($
$-2-c5)\,c5+(-2\,c5-c7+12-2\,c4)\,c4+(2\,c5+2\,c6-c7-26+2\,c3)\,c3+(2\,c3+12\,c4+28\,c5+16\,c6+26-90\,c2)\,c2$
$+(28\,c2-14\,c3-2\,c4-2\,c5-2+(15\,c2-c4-2\,c5-c6-2+(-2-c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1$

> ***B7h ≔ convert(b7e, horner)***

$B7h := -4+(-8+(11+(17+6\,c7)\,c7)\,c7)\,c7+(-8+(-12+(-12-2\,c7)\,c7)\,c7+(-11\,c7^2+(4\,c7+8+4\,c6)\,c6)\,c6)\,c6+(4+(-4$ **(14)**
$+(-17-11\,c7)\,c7)\,c7+(4+(-4-c7)\,c7+(16\,c7-4-4\,c6)\,c6)\,c6+(4+(10-5\,c7)\,c7+(16+10\,c7-4\,c6)\,c6+((4$
$+2\,c7)\,c7+(-10-c7-2\,c6)\,c6+(4\,c6-3\,c7+3-c5)\,c5)\,c5)\,c5)\,c5+(8+(12+(-16-8\,c7)\,c7)\,c7+(16+(-8+10\,c7)\,c7$
$+(-20\,c7+8)\,c6)\,c6+(-20+(16+(29-c7)\,c7)\,c7+(-4+(28-3\,c7)\,c7+(20+4\,c6)\,c6)\,c6+(-28+(6\,c7-12)\,c7+(2\,c7$
$-56-4\,c6)\,c6+(12\,c6-7\,c7+6-6\,c5)\,c5)\,c5)\,c5+(12+(-8+(-12+c7)\,c7)\,c7+(16+(4\,c7+16)\,c7+(4\,c7+4)\,c6)\,c6$
$+(36+(-4\,c7-28)\,c7+(-20\,c7-12-24\,c6)\,c6+(32\,c6+6\,c7+48-6\,c5)\,c5)\,c5+(-24+(8-2\,c7)\,c7+(-8+8\,c6)\,c6$
$+(20\,c6+12\,c7-20-20\,c5)\,c5+(-16\,c6-4\,c7+8\,c4)\,c4)\,c4)\,c4)\,c4+(8+(16+(4+(3+c7)\,c7)\,c7)\,c7+(4+(2\,c7$
$+17)\,c7^2+(-12+(-3\,c7-4)\,c7+(-4\,c7-4+4\,c6)\,c6)\,c6)\,c6+(-8+(4+(4-5\,c7)\,c7)\,c7+(4+(-22+2\,c7)\,c7+(2\,c7$
$-16+4\,c6)\,c6)\,c6+(-12+(6+9\,c7)\,c7+(-11\,c7+2-2\,c6)\,c6+(-2\,c6-4\,c7-8+4\,c5)\,c5)\,c5)\,c5+(4+(-60+(-55$
$-11\,c7)\,c7)\,c7+(20+(28+5\,c7)\,c7+(12\,c7-4-20\,c6)\,c6)\,c6+(44+(14+26\,c7)\,c7+(6\,c7+4+8\,c6)\,c6+(-4\,c6-15\,c7$
$+22+10\,c5)\,c5)\,c5+(-40+(4\,c7+52)\,c7+(-28\,c7-4+44\,c6)\,c6+(-28\,c6-24\,c7-60-18\,c5)\,c5+(32\,c5-28\,c6+28\,c7$
$+28)\,c4)\,c4)\,c4+((6+(c7+4)\,c7)\,c7+(12+(-4+6\,c7)\,c7+(-8+2\,c7-4\,c6)\,c6)\,c6+(-4+(2-2\,c7)\,c7+(-12\,c7+32$
$-4\,c6)\,c6+(10\,c6-5\,c7-4+(-c6+c7+8-c5)\,c5)\,c5)\,c5+(-32+(-6\,c7+18)\,c7+(-14\,c7-32+16\,c6)\,c6+(-4\,c7-28$
$+(-3\,c7+6+2\,c6)\,c6+(4\,c6+2\,c7+22-c5)\,c5)\,c5+(48+(36+3\,c7)\,c7+(-2\,c7-40)\,c6+(-4\,c6-7\,c7+6)\,c5+(10\,c5$
$+8\,c6-2\,c7-20-8\,c4)\,c4)\,c4)\,c4+(-8+(-10-8\,c7)\,c7+(9\,c7+10+(3\,c7-2\,c6)\,c6)\,c6+(4+(10-3\,c7)\,c7+(-4$
$-4\,c6)\,c6+(4\,c6+5\,c7-4-2\,c5)\,c5)\,c5+(-6+(-7-c7)\,c7+(16+6\,c6)\,c6+(-6\,c6+6\,c7-8-12\,c5)\,c5+(14\,c5-10\,c6$
$+c7-2\,c4)\,c4)\,c4+(5\,c7+3+(-8-5\,c7+5\,c6)\,c6+(3\,c6-c7-13-c5)\,c5+(5\,c5-10\,c6+6\,c7+24+c4)\,c4+(-8\,c4$
$+6\,c5+2\,c6-3\,c7-2+(c4-c5-c6+c7+1)\,c3)\,c3)\,c3)\,c3)\,c3)\,c3+(4+(20+(19+5\,c7)\,c7)\,c7+((8+(28+c7)\,c7)\,c7+($
$-16+(-12+3\,c7)\,c7+(-16-4\,c6)\,c6)\,c6)\,c6+(-16+(-36+(-32-10\,c7)\,c7)\,c7+(8+(-32-10\,c7)\,c7+(8\,c7+32$
$+8\,c6)\,c6)\,c6+(15\,c7^2-4+(16+10\,c7-4\,c6)\,c6+(-12\,c6+11\,c7-2+(-3+c5)\,c5)\,c5)\,c5)\,c5+(8+(-20-22\,c7)\,c7+($
$-16+(24+6\,c7)\,c7+(4\,c7-40-16\,c6)\,c6)\,c6+(68+(60+17\,c7)\,c7+(-16\,c7-20\,c6)\,c6+(-4\,c6-26\,c7+20+(-4\,c6$
$-2\,c7+4+2\,c5)\,c5)\,c5)\,c5+(-68+(20+8\,c7)\,c7+(4\,c7-16+28\,c6)\,c6+(-84+(-20+c7)\,c7+(4\,c7+8+4\,c6)\,c6$
$+4\,c5)\,c5+(48+(-24-2\,c7)\,c7+(-8\,c7-24-8\,c6)\,c6+(8\,c6+4\,c7+36-8\,c5)\,c5+(-12\,c5+16\,c6+8\,c7+16$
$-8\,c4)\,c4)\,c4)\,c4)\,c4+(-4+(-4+(1+8\,c7)\,c7)\,c7+(12+(28+13\,c7)\,c7+(4\,c7+4-12\,c6)\,c6)\,c6+(4+(-10-14\,c7)\,c7$
$+(-34\,c7-20-8\,c6)\,c6+(6+(-3-2\,c7)\,c7+(3\,c7+26+6\,c6)\,c6+(-8\,c6-2\,c7-14+2\,c5)\,c5)\,c5)\,c5+(-18\,c7^2-48+($
$-12+(-84-c7)\,c7+(-4\,c7+48-4\,c6)\,c6)\,c6+(24+(88-2\,c7)\,c7+(6\,c7+132+4\,c6)\,c6+(-16\,c6+3\,c7-2$
$+20\,c5)\,c5)\,c5+(72+(4+9\,c7)\,c7+(24\,c7-36+12\,c6)\,c6+(-40\,c6-14\,c7-52+18\,c5)\,c5+(20\,c5-28\,c6-28\,c7-32$
$+20\,c4)\,c4)\,c4)\,c4+(-4+(-8+(-19+4\,c7)\,c7)\,c7+(8+(10+c7)\,c7+(2\,c7+44)\,c6)\,c6+(12+(26-13\,c7)\,c7+(-3\,c7$
$-22-10\,c6)\,c6+(22\,c6+14\,c7+2-3\,c5)\,c5)\,c5+(8+(-17\,c7-24)\,c7+(-4\,c7-8\,c6)\,c6+(16\,c6+24\,c7-84-54\,c5)\,c5$
$+(6\,c5+32\,c6+4\,c7+44+24\,c4)\,c4)\,c4+(10+(-5+5\,c7)\,c7+(-6\,c7-32+6\,c6)\,c6+(2\,c6+6\,c7-4-24\,c5)\,c5+(-16\,c5$
$-8\,c6+24\,c7+32-26\,c4)\,c4+(-6\,c7-7+(-8+c6)\,c6+(-c6-2\,c7+16+3\,c5)\,c5+(5\,c5+2\,c6-3\,c7-4-3\,c4)\,c4+($
$-2\,c4-4\,c5+3\,c7+8-c3)\,c3)\,c3)\,c3)\,c3)\,c3+(12+(4+(-22-9\,c7)\,c7)\,c7+(16+(40-2\,c7)\,c7+(16\,c7+4)\,c6)\,c6+(8$
$+(16+(-3+c7)\,c7)\,c7+(-44+(2\,c7-40)\,c7+(-36-4\,c7-8\,c6)\,c6)\,c6+(16+(4\,c7+14)\,c7+(10\,c7+44+4\,c6)\,c6+($
$-6\,c6-9\,c7+8-c5)\,c5)\,c5)\,c5+(-64+(-12+(34-3\,c7)\,c7)\,c7+(8+(-16-8\,c7)\,c7+(4\,c7+64+16\,c6)\,c6)\,c6+(-72$
$+(3\,c7+12)\,c7+(12\,c7-4+12\,c6)\,c6+(-24\,c6-22\,c7-60+14\,c5)\,c5)\,c5+(116+(-20+3\,c7)\,c7+(-12\,c7-16$
$-36\,c6)\,c6+(8\,c7+68+32\,c5)\,c5+(-20\,c5+40\,c6+16\,c7-24-20\,c4)\,c4)\,c4)\,c4+(-20+(20+(33+4\,c7)\,c7)\,c7+(-12$
$+(-16-6\,c7)\,c7+(-4\,c7+32+16\,c6)\,c6)\,c6+(-24+(26-8\,c7)\,c7+(-24\,c7-32-16\,c6)\,c6+(34\,c6-6\,c7-6$
$+14\,c5)\,c5)\,c5+(60+(14\,c7-24)\,c7+(44\,c7-52-40\,c6)\,c6+(44\,c6-4\,c7-20-14\,c5)\,c5+(-36\,c5+24\,c6-44\,c7-20$
$+16\,c4)\,c4)\,c4+(16+(12\,c7-28)\,c7+(14\,c7-8-16\,c6)\,c6+(8\,c6-7\,c7+18+(-20+c7-3\,c5)\,c5)\,c5+(-40\,c7-24+($

-2 c7 - 40 - 4 c6) c6 + (10 c6 + 7 c7 + 36 - 8 c5) c5 + (-14 c5 + 4 c6 - 16) c4) c4 + (2 + (6 + 2 c7) c7 + (3 c7 - 14 - 2 c6) c6 + (
-12 c6 - 2 c7 + 34 + 8 c5) c5 + (14 c5 - 11 c7 + 34 + 14 c4) c4 + (4 c4 - 3 c5 + 8 c6 - 5 c7 - 7 + 2 c3) c3) c3) c3) c3 + (-20 + (-32
+ (-3 c7 - 4) c7) c7 + (16 + (-36 - 12 c7) c7 + (-4 c7 + 44 + 16 c6) c6) c6 + (-8 + (24 + 16 c7) c7 + (20 c7 + 8 + 8 c6) c6 + (6 c7
- 28 + (-2 + c5) c5) c5) c5 + (16 c7 + 88 + (-24 - 48 c6) c6 + (8 c6 - 24 c7 + 60 + (-4 c6 - 2 c7 - 4 + 4 c5) c5) c5 + (-84 + (12
+ c7) c7 + (4 c7 + 32 + 4 c6) c6 + (-8 c6 - 4 c7 - 28 + 8 c5) c5 + (8 c5 - 8 c6 - 4 c7 - 8 + 4 c4) c4) c4) c4 + (32 + (-16 - 3 c7) c7
+ (36 c7 - 12 - 36 c6) c6 + ((-38 - 2 c7) c7 + (-36 + 8 c6) c6 + (-2 c6 - c7 - 58 - 4 c5) c5) c5 + (-12 + (32 - 5 c7) c7 + (-12 c7
+ 60 - 4 c6) c6 + (12 c6 + 14 c7 + 48 - 2 c5) c5 + (-12 c5 + 16 c6 + 16 c7 - 8 - 12 c4) c4) c4 + (26 c7 + (4 c6 + 6 c7) c6 + (-22 c6
+ 13 c7 + 48 + 10 c5) c5 + (-24 c5 - 16 c6 + 8 c7 - 16 - 12 c4) c4 + (-6 c5 + 16 c6 - 15 c7 - 22 - c3) c3) c3) c3 + (12 + (12 + (c7
- 7) c7) c7 + (-40 + (-4 + 2 c7) c7 + (-4 c7 - 44 - 8 c6) c6) c6 + (36 + (-4 + 2 c7) c7 + (8 c7 + 32 + 8 c6) c6 + (-4 c6 - 6 c7 + 4
- 4 c5) c5) c5 + (8 c7 - 64 + (24 + 8 c7 + 16 c6) c6 + (-16 c6 - 16 c7 - 44) c5 + (16 c5 - 16 c6 - 8 c7 + 24 + 8 c4) c4) c4 + (-16
+ (-4 - 9 c7) c7 + (-16 c7 + 24 + 4 c6) c6 + (-12 c6 + 10 c7 + 48 + 22 c5) c5 + (20 c7 - 16 - 12 c4) c4 + (24 c4 + 6 c5 + 28 c6
- 4 c7 - 14 c3) c3) c3 + (-12 + (4 + 3 c7) c7 + (4 c7 + 16 + 12 c6) c6 + (-8 c6 + 4 c7 - 24 + 8 c5) c5 + (8 c5 - 8 c6 - 4 c7 + 32) c4
+ (12 c4 - 20 c5 - 16 c6 - 12 c7 - 12 - 12 c3) c3 + (12 c3 - 8 c4 + 12 - 4 c2) c2) c2) c2) c2) c2) c2 + (-4 + (12 + (29 + (23
+ 2 c7) c7) c7) c7 + (16 + (-48 + (-70 - 12 c7) c7) c7 + (32 + (4 + 3 c7) c7 + (16 c7 - 12 c6) c6) c6) c6 + (16 + (-20 + (-8 c7
- 24) c7) c7 + (-20 + (114 + 34 c7) c7 + (10 c7 - 32 - 28 c6) c6) c6 + (-8 + (10 + 7 c7) c7 + (-34 c7 - 28 + 4 c6) c6 + (20 c6
- 2 c7 - 12 + (-c7 + 5 + 2 c5) c5) c5) c5) c5 + (24 + (-32 + (-8 c7 - 24) c7) c7 + (8 + (36 + 38 c7) c7 + (-20 c7 + 8
+ 24 c6) c6) c6 + (-36 + (46 + 20 c7) c7 + (-114 c7 + 12 - 8 c6) c6 + (-20 + (12 - 2 c7) c7 + (6 c7 + 32 + 4 c6) c6 + (-16 c6 + 12
+ 4 c5) c5) c5) c5 + (4 + (-44 + (-12 + c7) c7) c7 + (24 + (-4 + 2 c7) c7 + (-4 c7 - 12 - 8 c6) c6) c6 + (28 + (-4 - 10 c7) c7
+ (4 c7 + 52 + 16 c6) c6 + (-16 c6 + 18 c7 + 24 + 8 c5) c5) c5 + (-48 + (-2 c7 + 32) c7 + (8 c7 - 16 + 24 c6) c6 + (-48 c6 + 4 c7
- 48 - 4 c5) c5 + (32 c5 - 24 c6 - 4 c7 + 16 + 8 c4) c4) c4) c4) c4 + (12 + (-16 + (13 + c7) c7) c7 + (-48 + (22 - 7 c7) c7 + (
-54 c7 + 4 + 32 c6) c6) c6 + (-8 + (10 + (3 - 2 c7) c7) c7 + (10 + (-29 + 4 c7) c7 + (c7 + 84 - 6 c6) c6) c6 + (28 + (-6 + c7) c7
+ (-5 c7 + 44 + 4 c6) c6 + (5 c6 + 3 c7 - c5) c5) c5) c5 + (-72 + (-70 + (-29 - 3 c7) c7) c7 + (76 + (116 + 5 c7) c7 + (14 c7
- 20) c6) c6 + (26 + (33 + c7) c7 + (-21 c7 - 94 - 4 c6) c6 + (8 c6 + c7 - 52 - 3 c5) c5) c5 + (52 + (70 + 10 c7) c7 + (-38 c7 - 60
- 16 c6) c6 + (18 c6 + 2 c7 - 66 - 8 c5) c5 + (32 c5 + 40 c6 + 20 c7 + 88 - 56 c4) c4) c4) c4 + (-8 + (36 + (4 c7 + 48) c7) c7 + (
-50 + (1 - 10 c7) c7 + (-9 c7 - 16 + 10 c6) c6) c6 + (-4 + (-12 c7 - 64) c7 + (24 c7 - 10 + 2 c6) c6 + (-25 c6 + 24 c7 - 10
- 22 c5) c5) c5 + (62 + (-39 - 7 c7) c7 + (29 c7 - 66 - 16 c6) c6 + (6 c6 + 14 c7 + 30 + 7 c5) c5 + (-4 c5 + 22 c6 - 22 c7 - 14
- 48 c4) c4) c4 + (8 + (-8 - 3 c7) c7 + (40 c7 + 42 - 26 c6) c6 + (14 c7 + 14 + (-45 + c6) c6 + (-c6 - 2 c7 - 2 + 3 c5) c5) c5 + (
-70 c7 - 38 + (-c7 + 18) c6 + (-2 c6 + 3 c7 + 45 + 3 c5) c5 + (-11 c5 - 8 c6 + 9 c7 + 52) c4) c4 + (19 + (-3 + 4 c7) c7 + (-3 c7
+ 7) c6 + (6 c6 - 12 c7 + 4 + 8 c5) c5 + (-4 c5 + 6 c6 - c7 - 7 + 2 c4) c4 + (7 c4 + c5 + c6 - 5 c7 - 15 + (-1 + c3) c3) c3) c3) c3) c3)
c3) c3 + (-4 + (4 + (31 - 13 c7) c7) c7 + (-56 + (-4 + (-2 - c7) c7) c7 + (-32 + (-16 - 3 c7) c7 + (24 + 4 c6) c6) c6) c6 + (4 + (
-110 + (-58 + 4 c7) c7) c7 + (136 + (4 c7 + 48) c7 + (2 c7 + 112 + 4 c6) c6) c6 + (44 + (72 + c7) c7 + (-12 c7 - 76 - 24 c6) c6
+ (12 c6 - 2 c7 - 28 - 3 c5) c5) c5) c5 + (-56 + (84 + (14 c7 + 48) c7) c7 + (-56 + (44 - 6 c7) c7 + (-16 c7 - 40 + 8 c6) c6) c6
+ (52 + (50 - 28 c7) c7 + (-8 c7 - 80 + 20 c6) c6 + (92 c6 - 36 c7 - 44 - 4 c5) c5) c5 + (-4 + (-56 - 8 c7) c7 + (20 c7 - 32
- 36 c6) c6 + (60 c6 + 18 c7 - 24 - 8 c5) c5 + (-20 c5 + 48 c6 - 28 c7 + 80 - 24 c4) c4) c4) c4 + (-20 + (26 + (18 + 6 c7) c7) c7
+ ((-12 - 28 c7) c7 + (-10 c7 - 40 + 12 c6) c6) c6 + (-6 + (-97 - 28 c7) c7 + (67 c7 - 86 + 44 c6) c6 + (-24 c6 + 37 c7 - 4 + (c6
- 51 - c5) c5) c5) c5 + (212 + (-136 + 31 c7) c7 + (38 c7 + 20 - 64 c6) c6 + (56 c7 + 252 + (c7 - 122 - 6 c6) c6 + (4 c7 - 116
- 3 c5) c5) c5 + (-228 + (-90 - 3 c7) c7 + (6 c7 + 144 + 8 c6) c6 + (4 c6 + c7 - 78 - 4 c5) c5 + (10 c5 - 24 c6 - 18 c7 - 44
+ 48 c4) c4) c4) c4 + (70 + (11 + 16 c7) c7 + (34 c7 + 56 + (-3 c7 - 60 + 2 c6) c6) c6 + (-94 + (-44 + 4 c7) c7 + (4 c7 - 142
+ 4 c6) c6 + (-12 c6 - 5 c7 + 75 - 4 c5) c5) c5 + (-82 + (-49 - 9 c7) c7 + (-12 c7 + 92 - 6 c6) c6 + (26 c6 + 10 c7 + 136) c5 + (
-50 c5 + 34 c6 + 35 c7 + 216 - 22 c4) c4) c4 + (-14 + (4 + 17 c7) c7 + (-c7 + 72 - 6 c6) c6 + (8 c6 - 41 c7 + 83 + 27 c5) c5 + (
-12 c5 + 24 c6 - 16 c7 - 166 - 70 c4) c4 + (53 c4 - 6 c5 + c6 + 3 c7 - 36 + (-9 c4 - c5 + c6 - 4 + 2 c3) c3) c3) c3) c3) c3 + (-8
+ (28 + (-25 - 6 c7) c7) c7 + (-4 + (36 + c7) c7 + (12 c7 - 32 - 12 c6) c6) c6 + (-60 + (-56 + 30 c7) c7 + (22 c7 - 108
- 16 c6) c6 + (24 c7 + 104 + (26 - 3 c7 + 2 c6) c6 + (8 c6 - 4 c7 - 24 + 2 c5) c5) c5) c5 + (84 + (16 + 41 c7) c7 + (44 + (-24
+ c7) c7 + (-28 + 4 c7 + 4 c6) c6) c6 + (-76 + (2 c7 - 68) c7 + (-10 c7 - 176 - 12 c6) c6 + (4 c6 - 5 c7 - 102) c5) c5 + (40 + (-8
- 3 c7) c7 + (84 - 16 c7 - 20 c6) c6 + (56 c6 + 34 c7 + 152 - 14 c5) c5 + (-76 c5 + 28 c6 + 12 c7 - 60 - 12 c4) c4) c4) c4 + (68
+ (104 + (69 - 4 c7) c7) c7 + (-90 c7 - 40 + (-4 c7 - 84 - 8 c6) c6) c6 + (-72 + (-121 + 16 c7) c7 + (12 c7 + 12 c6) c6 + (-26 c6
- c7 + 172 + 2 c5) c5) c5 + (-196 + (-6 + 7 c7) c7 + (30 c7 + 156 + 32 c6) c6 + (-50 c6 - 20 c7 + 242 + 42 c5) c5 + (-22 c5
- 124 c6 - 18 c7 - 72 + 28 c4) c4) c4 + (-62 + (-11 c7 + 20) c7 + (-15 c7 + 170) c6 + (16 c6 - 50 c7 + 86 + 75 c5) c5 + (148 c5
- 14 c6 - 16 c7 - 234 + 34 c4) c4 + (22 c6 + 63 c7 + 32 + (c6 - 104 - c5) c5 + (7 c5 - 4 c6 - 3 c7 + 24 + 24 c4) c4 + (-10 c4
+ 8 c5 + c7 - 41 + c3) c3) c3) c3) c3 + (60 + (-64 + (-24 - 3 c7) c7) c7 + (88 + (-40 + 8 c7) c7 + (8 c7 + 28 - 8 c6) c6) c6 + (-16
+ (40 + 14 c7) c7 + (-20 - 24 c6) c6 + (-48 c6 + 3 c7 - 6 - 12 c5) c5) c5 + (-44 + (36 - 27 c7) c7 + (-12 c7 + 64 + 44 c6) c6 + (
-14 c7 + 92 + 36 c5) c5 + (32 c5 - 56 c6 + 56 c7 - 96 + 20 c4) c4) c4 + (-80 + (96 - 52 c7) c7 + (16 c7 + 92 + 72 c6) c6 + (50 c6
- 11 c7 - 184 + (34 + c5) c5) c5 + (-60 c6 + 88 c7 + 160 + (-6 c6 + c7 + 72) c5 + (-6 c5 + 4 c6 + 10 c7 + 60 - 20 c4) c4) c4
+ (28 c7 - 4 + (-3 c7 - 56 + 2 c6) c6 + (12 c6 + 2 c7 - 210 - 14 c5) c5 + (-30 c5 - 4 c6 - 5 c7 - 182 + 6 c4) c4 + (-2 c4 + 15 c5
- 18 c6 + 7 c7 + 44 - 6 c3) c3) c3 + (-60 + (-8 c7 + 8) c7 + (-28 + 8 c7 + 24 c6) c6 + (112 c6 + 28 c7 + 88 + (2 c6 - 3 c7 + 28

$+4\,c5)\,c5)\,c5+(-48+(-36+c7)\,c7+(-92+4\,c7+4\,c6)\,c6+(-12\,c6-14\,c7-112+14\,c5)\,c5+(28\,c5-8\,c6-4\,c7+68$

$+4\,c4)\,c4)\,c4+(44+(-80+4\,c7)\,c7+(-12\,c7-72-8\,c6)\,c6+(34\,c6+3\,c7-56-10\,c5)\,c5+(-8\,c5+48\,c6-8\,c7+12$

$+4\,c4)\,c4+(16\,c4+14\,c5-28\,c6+15\,c7+162-2\,c3)\,c3)\,c3+(4+(-4+11\,c7)\,c7+(-40-12\,c6)\,c6+(-4\,c6-2\,c7-52$

$-18\,c5)\,c5+(-16\,c5+16\,c6-20\,c7+60-4\,c4)\,c4+(-20\,c4-10\,c5-28\,c6+6\,c7-44+18\,c3)\,c3+(20\,c3-20\,c4+20\,c5$

$+24\,c6+16\,c7+24-12\,c2)\,c2)\,c2)\,c2)\,c2)\,c2)\,c2+\big(16+(-2+(-10+(10+c7)\,c7)\,c7)\,c7+(8+(-86+(-3\,c7-56)\,c7)\,c7$

$+(48+(82+3\,c7)\,c7+(2\,c7+8)\,c6)\,c6)\,c6+(-36+(76+(3+2\,c7)\,c7)\,c7+(22+(93+10\,c7)\,c7+(-34\,c7-102$

$+8\,c6)\,c6)\,c6+(12+(-53+7\,c7)\,c7+(-26\,c7-48+42\,c6)\,c6+(8\,c6-15\,c7+30-14\,c5)\,c5)\,c5)\,c5+(-4+(-70+(11$

$-5\,c7)\,c7)\,c7+(60+(10+10\,c7)\,c7+(-34\,c7+16\,c6)\,c6)\,c6+(26+(-39+3\,c7)\,c7+(-14\,c7-116+36\,c6)\,c6+(28\,c6$

$+47\,c7+36+(-c6+c7-60-c5)\,c5)\,c5)\,c5+(-12+(-6-5\,c7)\,c7+(28\,c7+16+4\,c6)\,c6+(-16\,c7+42+(-10-3\,c7$

$+2\,c6)\,c6+(4\,c6+2\,c7-50-c5)\,c5)\,c5+(-20+(16+3\,c7)\,c7+(-2\,c7-28)\,c6+(-4\,c6-7\,c7-6)\,c5+(10\,c5+8\,c6-2\,c7$

$+16-8\,c4)\,c4)\,c4)\,c4)\,c4+(-36+(18-2\,c7)\,c7^2+(24+(-76+8\,c7)\,c7+(-5\,c7-34+10\,c6)\,c6)\,c6+(68+(-8+17\,c7)\,c7$

$+(26\,c7-46-32\,c6)\,c6+(-18+(-27+c7)\,c7+(-2\,c7-59-2\,c6)\,c6+(6\,c6+2\,c7+26-6\,c5)\,c5)\,c5)\,c5+(-28+(116$

$-17\,c7)\,c7+(16+(76-2\,c7)\,c7+(3\,c7-52+6\,c6)\,c6)\,c6+(-182+(-32+5\,c7)\,c7+(-10\,c7-88-12\,c6)\,c6+(24\,c6-7\,c7$

$+69-12\,c5)\,c5)\,c5+(162+(-35+9\,c7)\,c7+(-16\,c7-26-18\,c6)\,c6+(58\,c6-16\,c7+100-6\,c5)\,c5+(-26\,c5+46\,c6+c7$

$+4-42\,c4)\,c4)\,c4)\,c4+(24+(81+(13+2\,c7)\,c7)\,c7+(-72+(-85-3\,c7)\,c7+(-5\,c7-2)\,c6)\,c6+(-12+(-31-5\,c7)\,c7$

$+(16\,c7+131+6\,c6)\,c6+(-23\,c6+5\,c7+45+c5)\,c5)\,c5+(-48+(-48-2\,c7)\,c7+(27\,c7+140+18\,c6)\,c6+(-30\,c6-31\,c7$

$+165+41\,c5)\,c5+(-4\,c5-56\,c6-16\,c7-214+66\,c4)\,c4)\,c4+(-6+(19-12\,c7)\,c7+(13+2\,c7-9\,c6)\,c6+(14\,c6+12\,c7$

$-26+13\,c5)\,c5+(-32\,c5-8\,c6+9\,c7+13+61\,c4)\,c4+(20\,c6+15\,c7+3+(-20+c7-3\,c5)\,c5+(-c5-c6-63+9\,c4)\,c4$

$+(4\,c4-4\,c5-2\,c6+3\,c7+9-6\,c3)\,c3)\,c3)\,c3)\,c3)\,c3+(-24+(-2+(5+8\,c7)\,c7)\,c7+(-28+(-44+35\,c7)\,c7+(-52+4\,c7$

$-48\,c6)\,c6)\,c6+(78+(-69-24\,c7)\,c7+(168+(76-2\,c7)\,c7+(6\,c7-64+4\,c6)\,c6)\,c6+(-120+(48-5\,c7)\,c7+(-4\,c7$

$-176-14\,c6)\,c6+(11\,c6+6\,c7+18+7\,c5)\,c5)\,c5)\,c5+(16+(214+(25+3\,c7)\,c7)\,c7+(-160+(-118-7\,c7)\,c7+(-60$

$-8\,c7-12\,c6)\,c6)\,c6+(-16+(-126-6\,c7)\,c7+(52\,c7+200+8\,c6)\,c6+(-24\,c6+2\,c7+176-8\,c5)\,c5)\,c5+(-52+(-128$

$-6\,c7)\,c7+(108+42\,c7+4\,c6)\,c6+(-84\,c6+21\,c7+86-18\,c5)\,c5+(-32\,c5-28\,c6-12\,c7-28+28\,c4)\,c4)\,c4)\,c4+(76$

$+(-62+(-96-c7)\,c7)\,c7+(62+(-49+10\,c7)\,c7+(21\,c7+102-22\,c6)\,c6)\,c6+(-14+(74-13\,c7)\,c7+(6\,c7+348)\,c6+($

$-29\,c6+c7-37+42\,c5)\,c5)\,c5+(-42+(27+6\,c7)\,c7+(-45\,c7+314+14\,c6)\,c6+(-128\,c6-20\,c7+68+107\,c5)\,c5$

$+(110\,c5-14\,c6+16\,c7-384+170\,c4)\,c4)\,c4+(-76+(32-16\,c7)\,c7+(-63\,c7-50+66\,c6)\,c6+(126\,c6+7\,c7-213+(c6$

$+18-c5)\,c5)\,c5+(233\,c7+286+(-54+c7)\,c6+(-2\,c6-9\,c7-302+11\,c5)\,c5+(18\,c5+16\,c6-3\,c7+48-48\,c4)\,c4)\,c4$

$+(-61+(35-4\,c7)\,c7+(2\,c7-74)\,c6+(16\,c7-120-10\,c5)\,c5+(18\,c5-10\,c6-10\,c7-226-18\,c4)\,c4+(32\,c4-9\,c5$

$-15\,c6+15\,c7+118-11\,c3)\,c3)\,c3)\,c3+((62+(-2-c7)\,c7)\,c7+(-28+(90+13\,c7)\,c7+(14\,c7+20-8\,c6)\,c6)\,c6+($

$-40+(9-3\,c7)\,c7+(-73\,c7+76-18\,c6)\,c6+(36\,c6-45\,c7+120+25\,c5)\,c5)\,c5+(-36+(88-26\,c7)\,c7+(-42\,c7+52$

$+112\,c6)\,c6+(100\,c6+15\,c7-360+(108+c7-3\,c5)\,c5)\,c5+(60\,c7+136+(-2\,c7-80-4\,c6)\,c6+(10\,c6+7\,c7+118$

$-8\,c5)\,c5+(-14\,c5+4\,c6-4)\,c4)\,c4)\,c4+(-116+(-18+27\,c7)\,c7+(-92\,c7-210+106\,c6)\,c6+(202+(26+c7)\,c7+($

$-4\,c7+246-4\,c6)\,c6+(10\,c6+6\,c7+44-6\,c5)\,c5)\,c5+(-98+(76+8\,c7)\,c7+(19\,c7-288+6\,c6)\,c6+(-44\,c6-32\,c7$

$-522-2\,c5)\,c5+(106\,c5-32\,c6-35\,c7-230+38\,c4)\,c4)\,c4+(60+(-27-16\,c7)\,c7+(2\,c7-308-12\,c6)\,c6+(-4\,c6$

$+51\,c7-145-57\,c5)\,c5+(66\,c5+46\,c6+7\,c7+344+30\,c4)\,c4+(-19\,c4-36\,c5-7\,c6-2\,c7+268+15\,c3)\,c3)\,c3)\,c3+(4$

$+(-126-34\,c7)\,c7+(128+(52-c7)\,c7+(4\,c7+100+12\,c6)\,c6)\,c6+(-60+(129-5\,c7)\,c7+(-26\,c7-164-16\,c6)\,c6$

$+(36\,c6+14\,c7-136+5\,c5)\,c5)\,c5+(108+(2-6\,c7)\,c7+(-14\,c7-160-12\,c6)\,c6+(52\,c6+12\,c7-104-12\,c5)\,c5+(6\,c5$

$+24\,c6+22\,c7+36-20\,c4)\,c4)\,c4+(64+(38+10\,c7)\,c7+(49\,c7-242-14\,c6)\,c6+(-32\,c6+52\,c7-322-121\,c5)\,c5+($

$-128\,c5-2\,c6-11\,c7+694-100\,c4)\,c4+(-118\,c4+96\,c5+26\,c6-108\,c7-96+111\,c3)\,c3)\,c3+(16+(-84+20\,c7)\,c7+($

$-12\,c7-76-72\,c6)\,c6+(-18\,c6-5\,c7+228-56\,c5)\,c5+(-62\,c5+56\,c6-42\,c7-120)\,c4+(122\,c4+362\,c5+198\,c6$

$-54\,c7+120-32\,c3)\,c3+(-324\,c3-12\,c4+46\,c5+44\,c6+44\,c7-44+32\,c2)\,c2)\,c2)\,c2)\,c2)\,c2+\big(8+(-40+(3\,c7$

$-14)\,c7)\,c7+(14+(25-20\,c7)\,c7+(50+(30+2\,c7)\,c7+(-c7-30-2\,c6)\,c6)\,c6)\,c6+(-20+(112+(-2\,c7+36)\,c7)\,c7+($

$-124+(-49-c7)\,c7+(-3\,c7-44+4\,c6)\,c6)\,c6+(-34+(-48+2\,c7)\,c7+(7\,c7+145+5\,c6)\,c6+(-13\,c6+32+7\,c5)\,c5)\,c5)$

$c5+(-2+(-55+(-12-5\,c7)\,c7)\,c7+(80+(-50+10\,c7)\,c7+(11\,c7+42-4\,c6)\,c6)\,c6+(4+(13-9\,c7)\,c7+(7\,c7+44$

$-32\,c6)\,c6+(-45\,c6+43\,c7+35+7\,c5)\,c5)\,c5+(-26+(52-2\,c7)\,c7+(-2\,c7+36+10\,c6)\,c6+(-44\,c6-16\,c7-36$

$+54\,c5)\,c5+(56\,c5-32\,c6+14\,c7-20+4\,c4)\,c4)\,c4)\,c4+(-28+(42+(8-7\,c7)\,c7)\,c7+(-10+(-27+5\,c7)\,c7+(10\,c7$

$+84-6\,c6)\,c6)\,c6+(-4+(29+15\,c7)\,c7+(95-36\,c6)\,c6+(4\,c6-13\,c7+12+28\,c5)\,c5)\,c5+(42+(79-17\,c7)\,c7+($

$-12\,c7-124+14\,c6)\,c6+(8\,c7-207+(61+c6)\,c6+(-c6-2\,c7+46+3\,c5)\,c5)\,c5+(36\,c7+156+(-c7+4)\,c6+(-2\,c6$

$+3\,c7-17+3\,c5)\,c5+(-11\,c5-8\,c6+9\,c7-60)\,c4)\,c4)\,c4+(8+(-50+5\,c7)\,c7+(-13\,c7-62+(c7+34-c6)\,c6)\,c6$

$+(118+(39-2\,c7)\,c7+(c7-1-c6)\,c6+(2\,c7-100+5\,c5)\,c5)\,c5+(-166+(-9+c7)\,c7+(6\,c7+2+3\,c6)\,c6+(-20\,c6$

$-2\,c7-57+8\,c5)\,c5+(3\,c5-37\,c6+11\,c7-112+35\,c4)\,c4)\,c4+(38+(18-3\,c7)\,c7+(-2\,c7-99-2\,c6)\,c6+(6\,c6+8\,c7$

$-61-7\,c5)\,c5+(-12\,c5+8\,c6+13\,c7+209-10\,c4)\,c4+(-35\,c4+25\,c5+8\,c6-18\,c7-4+(c4+c5+20-c3)\,c3)\,c3)\,c3)$

$$c3)\ c3 + (6 + (3 + (-16 + 6\ c7)\ c7)\ c7 + (4 + (50 - 16\ c7)\ c7 + (-17\ c7 + 22 + 12\ c6)\ c6)\ c6 + (84 + (43 - 42\ c7)\ c7 + (25\ c7 + 144$$
$$+ 64\ c6)\ c6 + (-155 + (c7 - 22 - c6)\ c6 + (-3\ c6 + c7 + 17 + c5)\ c5)\ c5 + ((-88 + 12\ c7)\ c7 + (-24\ c7 - 200 + (-2\ c7 + 60$$
$$- 4\ c6)\ c6)\ c6 + (208 + (-36 - c7)\ c7 + (13\ c7 + 248 + 12\ c6)\ c6 + (-18\ c6 - 2\ c7 - 45 - 3\ c5)\ c5)\ c5 + (-128 + (-6\ c7 + 16)\ c7$$
$$+ (12 + 12\ c6)\ c6 + (-24\ c6 - c7 - 228 + 15\ c5)\ c5 + (36\ c5 - 12\ c6 - 6\ c7 - 12 + 36\ c4)\ c4)\ c4)\ c4 + (30 + (-313 - 61\ c7)\ c7$$
$$+ (132 + (158 + c7)\ c7 + (6\ c7 + 14 + 4\ c6)\ c6)\ c6 + (148\ c7 + 41 + (-20\ c7 - 274 - 13\ c6)\ c6 + (29\ c6 - 9\ c7 - 328 - 3\ c5)\ c5)\ c5$$
$$+ (-4 + (116 + 7\ c7)\ c7 + (-44\ c7 - 336 - 12\ c6)\ c6 + (100\ c6 + 8\ c7 - 335 - 40\ c5)\ c5 + (-3\ c5 + 108\ c6 + c7 + 316$$
$$- 52\ c4)\ c4)\ c4 + (-32 + (-14 + 24\ c7)\ c7 + (-12\ c7 - 118 + 21\ c6)\ c6 + (20\ c6 - 14\ c7 - 29 - 87\ c5)\ c5 + (-122\ c5 + 20\ c6 + 9\ c7$$
$$+ 590 - 171\ c4)\ c4 + (-24\ c6 - 91\ c7 - 103 + (200 + c5)\ c5 + (-8\ c5 + c7 + 34 - 8\ c4)\ c4 + (4\ c4 + c5 + 2\ c6 - 4\ c7 + 51$$
$$+ 3\ c3)\ c3)\ c3)\ c3)\ c3 + (-72 + (73 + (67 + c7)\ c7)\ c7 + (-156 + (39 - 12\ c7)\ c7 + (-11\ c7 - 138 + 14\ c6)\ c6)\ c6 + (8 + (-97$$
$$+ 11\ c7)\ c7 + (6\ c7 - 196 + 12\ c6)\ c6 + (43\ c6 - 16\ c7 + 46 - 8\ c5)\ c5)\ c5 + (20 + (-9 + 19\ c7)\ c7 + (16\ c7 - 192 - 42\ c6)\ c6$$
$$+ (28\ c6 + 46\ c7 - 148 - 96\ c5)\ c5 + (-116\ c5 - 12\ c6 - 30\ c7 + 378 - 108\ c4)\ c4)\ c4 + (-26 + (-103 + 75\ c7)\ c7 + (-17\ c7 - 14$$
$$- 110\ c6)\ c6 + (-94\ c6 - 43\ c7 + 705 + c5)\ c5 + (92\ c6 - 235\ c7 - 394 + (c6 + 170 - c5)\ c5 + (7\ c5 - 4\ c6 - 3\ c7 - 64$$
$$+ 24\ c4)\ c4)\ c4 + (-71\ c7 + 204 + (c7 + 161 - c6)\ c6 + (-5\ c6 - 4\ c7 + 530 + 15\ c5)\ c5 + (-7\ c5 + 6\ c6 + 16\ c7 + 501 + 5\ c4)\ c4$$
$$+ (-20\ c4 - 8\ c5 + 22\ c6 - 13\ c7 - 299 + 12\ c3)\ c3)\ c3)\ c3 + (76 + (47 - 22\ c7)\ c7 + (26\ c7 + 208 - 22\ c6)\ c6 + (-252\ c6 + 13\ c7$$
$$- 244 + (-c6 + c7 - 8 - c5)\ c5)\ c5 + (-18\ c7 + 204 + (-2\ c7 + 228 - 4\ c6)\ c6 + (8\ c6 + 11\ c7 + 436 - 15\ c5)\ c5 + (-44\ c5 + 8\ c6$$
$$+ 10\ c7 + 44 - 20\ c4)\ c4)\ c4 + (-22 + (123 - c7)\ c7 + (10\ c7 + 372 + 12\ c6)\ c6 + (-29\ c6 - 28\ c7 + 11 + 39\ c5)\ c5 + (-18\ c5$$
$$- 68\ c6 + 4\ c7 - 284)\ c4 + (-18\ c4 + 5\ c5 + 26\ c6 - 6\ c7 - 835 - 4\ c3)\ c3)\ c3 + (16 + (-45 - 10\ c7)\ c7 + (-6\ c7 + 142 + 16\ c6)\ c6$$
$$+ (28\ c6 - 23\ c7 + 188 + 57\ c5)\ c5 + (68\ c5 + 8\ c6 + 12\ c7 - 478 + 56\ c4)\ c4 + (72\ c4 - 21\ c5 - 8\ c6 + 43\ c7 + 182 - 71\ c3)\ c3 + ($$
$$-16\ c3 - 20\ c4 - 160\ c5 - 128\ c6 + 10\ c7 - 122 + 160\ c2)\ c2)\ c2)\ c2)\ c2)\ c2 + (-13 + (-5 + (-3\ c7 - 4)\ c7)\ c7 + (46 + (17$$
$$- 4\ c7)\ c7 + (8\ c7 - 8 + 10\ c6)\ c6)\ c6 + (7 + (8 + 28\ c7)\ c7 + (-70\ c7 - 158 + (38 - c7)\ c6)\ c6 + (56 + (c7 - 30)\ c7 + (2\ c7 + 75$$
$$+ 2\ c6)\ c6 + (-5\ c6 - 2\ c7 - 50 + c5)\ c5)\ c5)\ c5 + (8 + (-68 - 16\ c7)\ c7 + (52 + (34 + 3\ c7)\ c7 + (-c7 + 30)\ c6)\ c6 + (-18 + (61$$
$$- 3\ c7)\ c7 + (-13\ c7 - 127 + 4\ c6)\ c6 + (10\ c6 + 6\ c7 - 77 - 5\ c5)\ c5)\ c5 + (11 + (37 + 6\ c7)\ c7 + (-11\ c7 - 60 + 11\ c6)\ c6$$
$$+ (23\ c6 - 21\ c7 - 50 + 5\ c5)\ c5 + (c5 + 6\ c6 + c7 + 28 - 17\ c4)\ c4)\ c4)\ c4 + (8 + (45 + 24\ c7)\ c7 + (-83 + (-c7 + 5)\ c7 + (-7\ c7$$
$$- 28 + 6\ c6)\ c6)\ c6 + (-56 + (-35 + 9\ c7)\ c7 + (-11\ c7 - 126 + 6\ c6)\ c6 + (26\ c6 - 9\ c7 + 40 - 22\ c5)\ c5)\ c5 + (-16\ c7 + 51 + ($$
$$-2\ c7 - 117 + c6)\ c6 + (60\ c6 - 9\ c7 + 68 - 53\ c5)\ c5 + (-72\ c5 + 30\ c6 - 7\ c7 + 127 - 51\ c4)\ c4)\ c4 + (-24 + (-27 - c7)\ c7$$
$$+ (20\ c7 + 84 - 15\ c6)\ c6 + (-56\ c6 + 13\ c7 + 161 + 3\ c5)\ c5 + (12\ c6 - 89\ c7 - 160 + (c7 + 140 - 3\ c5)\ c5 + (-c5 - c6 + 46$$
$$+ 9\ c4)\ c4)\ c4 + (71 + (5 + c7)\ c7 + (16 - c7)\ c6 + (2\ c6 - 4\ c7 + 38)\ c5 + (-4\ c5 + 10\ c6 + 80)\ c4 + (-5\ c4 + 3\ c5 + 3\ c6 - 6\ c7$$
$$- 73 + 3\ c3)\ c3)\ c3)\ c3)\ c3 + (41 + (-53 + (-18 + 7\ c7)\ c7)\ c7 + (4 + (2 - 5\ c7)\ c7 + (-11\ c7 - 76 + 4\ c6)\ c6)\ c6 + (-9 + (6$$
$$- 13\ c7)\ c7 + (37\ c7 - 29 + 10\ c6)\ c6 + (-31\ c6 + 19\ c7 - 79 + 5\ c5)\ c5)\ c5 + (-88 + (-128 + 25\ c7)\ c7 + (17\ c7 + 104 - 82\ c6)\ c6$$
$$+ (-46\ c6 - 20\ c7 + 428 - 79\ c5)\ c5 + (-26\ c7 - 133 + (16 + c6)\ c6 + (-c6 - 2\ c7 - 40 + 3\ c5)\ c5 + (5\ c5 + 2\ c6 - 3\ c7 + 62$$
$$- 3\ c4)\ c4)\ c4)\ c4 + (-9 + (50 - 38\ c7)\ c7 + (80\ c7 + 382 - 85\ c6)\ c6 + (22\ c7 - 234 + (c7 - 136)\ c6 + (-2\ c6 - 2\ c7 + 111$$
$$+ 2\ c5)\ c5)\ c5 + (271 + (-8 - 3\ c7)\ c7 + (-6\ c7 + 80)\ c6 + (20\ c6 + 15\ c7 + 404 - 6\ c5)\ c5 + (-32\ c5 + 16\ c6 - 3\ c7 + 313$$
$$- 50\ c4)\ c4)\ c4 + (-29 + (-28 + 7\ c7)\ c7 + (-3\ c7 + 327 + 8\ c6)\ c6 + (-4\ c6 - 20\ c7 + 37 + 19\ c5)\ c5 + (2\ c5 - 38\ c6 - 5\ c7 - 527$$
$$- 3\ c4)\ c4 + (50\ c4 - 20\ c6 + 30\ c7 - 280 - 36\ c3)\ c3)\ c3)\ c3 + (-48 + (220 + 53\ c7)\ c7 + (-87\ c7 - 134 + (-c7 - 43 - 6\ c6)\ c6)\ c6$$
$$+ (96 + (-147 + c7)\ c7 + (18\ c7 + 277 + 8\ c6)\ c6 + (-26\ c6 - 10\ c7 + 227 + 3\ c5)\ c5)\ c5 + (-16 + (-21 - c7)\ c7 + (13\ c7 + 214$$
$$- 4\ c6)\ c6 + (-50\ c6 + 15\ c7 + 138 + 4\ c5)\ c5 + (11\ c5 - 16\ c6 - 11\ c7 - 154 + 22\ c4)\ c4)\ c4 + (25 + (-78 - 10\ c7)\ c7 + (-21\ c7$$
$$+ 136 + 4\ c6)\ c6 + (4\ c6 - 25\ c7 + 298 + 108\ c5)\ c5 + (180\ c5 + 23\ c6 - 11\ c7 - 1314 + 187\ c4)\ c4 + (102\ c4 - 239\ c5 - 60\ c6$$
$$+ 155\ c7 + 252 - 146\ c3)\ c3)\ c3 + (50 + (129 - 43\ c7)\ c7 + (23\ c7 + 95\ c6)\ c6 + (-11\ c6 + 29\ c7 - 559 + 46\ c5)\ c5 + (17\ c5 - 54\ c6$$
$$+ 72\ c7 + 250 - 36\ c4)\ c4 + (-368\ c4 - 720\ c5 - 259\ c6 + 100\ c7 - 345 + 278\ c3)\ c3 + (1047\ c3 + 102\ c4 - 21\ c5 - 112\ c6 - 90\ c7$$
$$+ 43 - 109\ c2)\ c2)\ c2)\ c2)\ c2 + (-3 + (8 + (10 - 3\ c7)\ c7)\ c7 + (-3 + (-16 + 6\ c7)\ c7 + (2\ c7 + 5 - 4\ c6)\ c6)\ c6 + (-34 + (-12$$
$$+ 15\ c7)\ c7 + (-24\ c7 + 3 - 6\ c6)\ c6 + (7\ c6 + 8\ c7 + 82 - 12\ c5)\ c5)\ c5 + (-3 + (40 - 14\ c7)\ c7 + (23\ c7 + 74 + (-16 + c6)\ c6)\ c6$$
$$+ (37\ c7 - 75 + (-3\ c7 - 83 - c6)\ c6 + (4\ c6 + 2\ c7 + 58 - c5)\ c5)\ c5 + (50 + (-7 + 3\ c7)\ c7 + (-c7 - 9 - 3\ c6)\ c6 + (2\ c6 - 4\ c7$$
$$+ 79)\ c5 + (-5\ c5 + 3\ c6 + 3\ c7 + 13 - 9\ c4)\ c4)\ c4)\ c4 + (-3 + (87 + 14\ c7)\ c7 + (-31\ c7 - 42 + (-c7 - 6)\ c6)\ c6 + (-31\ c7 + 14$$
$$+ (2\ c7 + 71 + c6)\ c6 + (-3\ c6 + 3\ c7 + 86 - c5)\ c5)\ c5 + (8 + (-37 - 2\ c7)\ c7 + (11\ c7 + 113 - 4\ c6)\ c6 + (-26\ c6 + 2\ c7 + 82$$
$$+ 11\ c5)\ c5 + (4\ c5 - 23\ c6 + 7\ c7 - 146 + 23\ c4)\ c4)\ c4 + ((-3 - 8\ c7)\ c7 + (16 + 10\ c7 - 6\ c6)\ c6 + (-20\ c6 + 13\ c7 - 35$$
$$+ 27\ c5)\ c5 + (30\ c5 - 18\ c6 - c7 - 172 + 47\ c4)\ c4 + (-56\ c5 + 6\ c6 + 25\ c7 + 41 + (c5 - 32 + c4)\ c4 + (-c6 + c7 - 8 - c5$$
$$- c3)\ c3)\ c3)\ c3)\ c3 + (26 + (-55 - 35\ c7)\ c7 + (105 + (5 + 3\ c7)\ c7 + (5\ c7 + 55 - 7\ c6)\ c6)\ c6 + (11 + (36 - 12\ c7)\ c7 + (9\ c7$$
$$+ 122 - 3\ c6)\ c6 + (-26\ c6 + 10\ c7 - 64 + 15\ c5)\ c5)\ c5 + (-31 + (-29 - 5\ c7)\ c7 + (2\ c7 + 128 + 12\ c6)\ c6 + (-18\ c6 - 14\ c7 - 3$$
$$+ 32\ c5)\ c5 + (71\ c5 + 8\ c6 + 6\ c7 - 243 + 63\ c4)\ c4)\ c4 + (63 + (79 - 32\ c7)\ c7 + (8\ c7 - 99 + 50\ c6)\ c6 + (52\ c6 + 6\ c7 - 602$$
$$- c5)\ c5 + (-170\ c5 - 34\ c6 + 119\ c7 + 340 + (-c5 + c6 - 32 - 9\ c4)\ c4)\ c4 + (-34\ c6 + 2\ c7 - 189 + (c7 - 294 - 3\ c5)\ c5 + (8\ c5$$
$$- 4\ c6 - 4\ c7 - 330 - 2\ c4)\ c4 + (10\ c4 + c5 - 8\ c6 + 7\ c7 + 267 - 6\ c3)\ c3)\ c3)\ c3 + (-18 + (-51 + 35\ c7)\ c7 + (-58\ c7 - 298$$
$$+ 23\ c6)\ c6 + (190\ c6 - 50\ c7 + 241 - 51\ c5)\ c5 + (29\ c7 - 236 + (-143 + c6)\ c6 + (-c6 - 2\ c7 - 374 + 3\ c5)\ c5 + (21\ c5 - 2\ c6$$
$$- 6\ c7 - 155 + 21\ c4)\ c4)\ c4 + (-48\ c7 + 18 + (-2\ c7 - 385 - 6\ c6)\ c6 + (9\ c6 + 15\ c7 + 151 - 21\ c5)\ c5 + (c5 + 38\ c6 - c7 + 446$$
$$- c4)\ c4 + (-6\ c4 - 6\ c5 - 12\ c7 + 958 + 13\ c3)\ c3)\ c3 + (-37 + (90 + 2\ c7)\ c7 + (5\ c7 - 109 - 7\ c6)\ c6 + (-27\ c6 + 26\ c7 - 191$$
$$- 41\ c5)\ c5 + (-84\ c5 - 28\ c6 + 7\ c7 + 794 - 81\ c4)\ c4 + (-69\ c4 + 74\ c5 + 51\ c6 - 75\ c7 - 361 + 88\ c3)\ c3 + (-69\ c3 + 111\ c4$$
$$+ 279\ c5 + 166\ c6 - 47\ c7 + 213 - 427\ c2)\ c2)\ c2)\ c2)\ c2 + (-3 + (13 + (7 - 2\ c7)\ c7)\ c7 + (-16 + (-17 + 2\ c7)\ c7 + (21 + 2\ c7$$
$$- c6)\ c6)\ c6 + (-3 + (-8 + 3\ c7)\ c7 + (-8\ c7 + 5 + c6)\ c6 + (2\ c6 + c7 + 4 - 3\ c5)\ c5)\ c5 + (29 + (33 - 7\ c7)\ c7 + (2\ c7 - 51$$

$$
\begin{aligned}
&+14\,c6)\,c6+(-2\,c6-117+23\,c5)\,c5+(-3\,c5+5\,c6-2\,c7+33+(-c5-c6+c7-23+c4)\,c4)\,c4)\,c4+(2+(-19+10\,c7)\,c7\\
&+(-19\,c7-97+17\,c6)\,c6+(30\,c6-12\,c7+78-43\,c5)\,c5+(-86-8\,c7+(12+c7)\,c6+(-2\,c6-2\,c7-88+2\,c5)\,c5+(-4\,c6\\
&+3\,c7-80+12\,c4)\,c4)\,c4+(12+(14-c7)\,c7+(c7-76-c6)\,c6+(c6+c7+3)\,c5+(-7\,c5+8\,c6+163-3\,c4)\,c4+(-13\,c4\\
&+2\,c5+8\,c6-8\,c7+64+11\,c3)\,c3)\,c3+(8+(-102-23\,c7)\,c7+(40\,c7+75+(7+c6)\,c6)\,c6+(49\,c7-48+(-3\,c7-108\\
&-c6)\,c6+(4\,c6+2\,c7-93-c5)\,c5)\,c5+(12+(c7+17)\,c7+(-7\,c7-90+5\,c6)\,c6+(22\,c6-9\,c7-52-4\,c5)\,c5+(-4\,c5\\
&+6\,c6-c7+119-11\,c4)\,c4)\,c4+((59+6\,c7)\,c7+(-2\,c7-32)\,c6+(16\,c6-4\,c7-76-36\,c5)\,c5+(-74\,c5-2\,c6+5\,c7\\
&+670-85\,c4)\,c4+(-16\,c4+115\,c5+30\,c6-67\,c7-177+59\,c3)\,c3)\,c3+(-46+(-90+30\,c7)\,c7+(-18\,c7+53-51\,c6)\,c6\\
&+(20\,c6-15\,c7+478-24\,c5)\,c5+(27\,c5+10\,c6-34\,c7-250+42\,c4)\,c4+(319\,c4+488\,c5+93\,c6-39\,c7+283\\
&-303\,c3)\,c3+(-1184\,c3-141\,c4-58\,c5+113\,c6+64\,c7-43+189\,c2)\,c2)\,c2)\,c2+(-4+(6\,c7+12)\,c7+(-5\,c7-18+(-c7\\
&-3+c6)\,c6)\,c6+(2\,c7^2+6+(-3\,c7-21+c6)\,c6+(4\,c6-c7+10-3\,c5)\,c5)\,c5+(9+(11+c7)\,c7+(-c7-26-c6)\,c6\\
&+(4\,c6-c7+20-2\,c5)\,c5+(-14\,c5+c6+43-10\,c4)\,c4)\,c4+(-15+(-18+5\,c7)\,c7+(-3\,c7+29-7\,c6)\,c6+(-5\,c6+2\,c7\\
&+126-3\,c5)\,c5+(33\,c5+2\,c6-15\,c7-79+(17+c4)\,c4)\,c4+(50\,c5-7\,c6+6\,c7+40+(-c5+c6+62)\,c4+(-c4+c6-c7\\
&-59+c3)\,c3)\,c3)\,c3+(-1+(18-15\,c7)\,c7+(28\,c7+126-12\,c6)\,c6+(-62\,c6+23\,c7-110+33\,c5)\,c5+(118\,c5+28\,c6\\
&-7\,c7+100+(-3\,c5+c7+79-8\,c4)\,c4)\,c4+(-c7-24+(142+c6)\,c6+(-c6-2\,c7-84+3\,c5)\,c5+(5\,c5-10\,c6-222)\,c4\\
&+(6\,c4+c5-5\,c6+7\,c7-387-7\,c3)\,c3)\,c3+(-64\,c7+15+(-c7+43+c6)\,c6+(9\,c6-9\,c7+85+11\,c5)\,c5+(45\,c5+14\,c6\\
&-6\,c7-505+45\,c4)\,c4+(25\,c4-49\,c5-36\,c6+44\,c7+290-46\,c3)\,c3+(102\,c3-118\,c4-208\,c5-90\,c6+35\,c7-166\\
&+485\,c2)\,c2)\,c2)\,c2+((14+3\,c7)\,c7+(-6\,c7-13)\,c6+(11\,c6-4\,c7+8+12\,c5)\,c5+(-4\,c7-7+(12+c7-c6)\,c6+(-3\,c6\\
&+c7+10+c5)\,c5+(-c6+c7-24+2\,c4)\,c4)\,c4+(-2+(-12-c7)\,c7+(5+c7)\,c6+(-4\,c6+2\,c7+2+4\,c5)\,c5+(10\,c5\\
&-2\,c6-103+12\,c4)\,c4+(-2\,c4-15\,c5-5\,c6+9\,c7+35-8\,c3)\,c3)\,c3+(16+(28-9\,c7)\,c7+(7\,c7-24+12\,c6)\,c6+(-8\,c6\\
&+2\,c7-162+8\,c5)\,c5+(-15\,c5+4\,c6+5\,c7+100-16\,c4)\,c4+(-109\,c4-138\,c5-3\,c6+c7-94+114\,c3)\,c3+(605\,c3\\
&+81\,c4+47\,c5-54\,c6-19\,c7+29-157\,c2)\,c2)\,c2+(1+(-2+2\,c7)\,c7+(-4\,c7-17+2\,c6)\,c6+(7\,c6-3\,c7+16-5\,c5)\,c5\\
&+(-13\,c5-c6-14+(-12+c4)\,c4)\,c4+(11\,c5-17\,c6+2\,c7+6+(-c5+c6+34)\,c4+(-c4+c6-c7+52+c3)\,c3)\,c3+(\\
&-10\,c6+19\,c7-2+(-c6+c7-16-c5)\,c5+(-11\,c5-2\,c6+c7+139-11\,c4)\,c4+(-3\,c4+12\,c5+10\,c6-11\,c7-96\\
&+11\,c3)\,c3+(-53\,c3+55\,c4+77\,c5+22\,c6-10\,c7+66-285\,c2)\,c2)\,c2+(-2+(-3+c7)\,c7+(-c7+3-c6)\,c6+(c6+19\\
&-c5)\,c5+(2\,c5-c6-13+2\,c4)\,c4+(13\,c4+14\,c5-2\,c6+c7+11-14\,c3)\,c3+(-144\,c3-21\,c4-12\,c5+12\,c6+2\,c7-9\\
&+65\,c2)\,c2+(c5+c6-2\,c7+(c5-14+c4)\,c4+(-c5-c6+c7+11-c3)\,c3+(12\,c3-12\,c4-14\,c5-2\,c6+c7-13\\
&+91\,c2)\,c2+(-13\,c2+13\,c3+2\,c4+c5-c6+1+(-15\,c2-c3+c4+c5+1+(1+c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\,c1)\\
&c1)\,c1)\,c1
\end{aligned}
$$

\>

\> From (10) we have $b3 = b31 \cdot Y[7] + b30$, thus the sought relation among $c[1],\ldots,c[7]$ is given by

\> $$a7^2 \cdot Y[7]^2 - b7^2 = 0 \text{ or } (A7h)^2 \cdot (2 \cdot (1 + \mathrm{add}(c\|k, k = 1..6)) + c\|7) - (B7h)^2 = 0$$

\> This represents, compactly, an implicit heptagon's inverseradius-equation in terms of our c − substitutions

\> with remarkably small coefficients: because

\> [min([coeffs(expand(A7h))]), max([coeffs(expand(A7h))])], [min([coeffs(expand(B7h))]), max([coeffs(expand(B7h))])]

$$[-1614, 2180], [-1314, 1047] \tag{15}$$

\> In the following theorem we describe the inverse radius equation
in terms of elementary symmetric polynomials of squares of sides of a cyclic heptagon:

\>

\> **THE MAIN THEOREM**: Consider a cyclic heptagon with sides $a_1,\ldots, a_7$, $\rho = r^{-1}$ (inverse circumradius)

\> If $\epsilon_1,\ldots, \epsilon_7$ are the elementary symmetric polynomials in the squares of the sides,

\> $c[k] = \mathrm{sum}\left(\mathrm{binomial}(14 - 2\cdot j, k - j) \cdot (-1)^j \cdot \epsilon_j \cdot{}^{2\cdot j}, j = 0..k\right)$ and

\> $\Delta[7] = \mathrm{product}\left(4 - a[k]^2 \cdot \rho^2, k = 1..7\right) = \mathrm{sum}\left(4^{7-j} \cdot (-1)^j \cdot \epsilon_j\, \rho^{2\cdot j}, j = 0..7\right)$,

then (the square of the inverse circumradius) ρ^2 satisfies the following degree 38 equation

\>

\> $\rho_7^{el} = \rho^{-64} \cdot \mathrm{subs}\left(\mathrm{seq}\left(c\|k = c[k], k = 1..7, (A7h)^2 - \Delta_7 \cdot (B7h)^2\right)\right)$

\> The leading monomial of ρ_7^{el} is equal to $\epsilon_7^{10}\,\rho^{76}$.

\> REMARK: We have $\mathrm{length}(\rho_7^{el}) = 19649983$, $\mathrm{nops}(\rho_7^{el}) = 199695$

\> Moreover the coefficients of ρ_7^{el} have at most 22 digits:

\> $\left[\min\left(\left[\mathrm{coeffs}\left(\mathrm{expand}(\rho_7^{el})\right)\right]\right), \max\left(\left[\mathrm{coeffs}\left(\mathrm{expand}(\rho_7^{el})\right)\right]\right)\right]$

$$[-1143554017381916344320, 1208261736827975630848] \tag{16}$$

\> length(−1143554017381916344320), length(1208261736827975630848)

$$22, 22 \tag{17}$$

In conclusion, we can interpret the quantity $Q_7 := (A7h)^2 - \Delta 7 \cdot (B7h)^2$ as a kind of minimal condensed polynomial relation (among $c1, \ldots, c_7$), $Q_7 = 0$. It has only up to four-digit coefficients (between -1614 and 2180). Our formula for ρ_7^{el}, having 199695 monomial terms (with up to 22-digit coefficients), is expectedly large (as a 2200 pages book!). From this formula one can get other expressions by simple substitution (e.g., by side lengths - what might be unreasonable, instead one might rewrite it in monomial or Schur basis of symmetric functions, etc.). Similar explicit circumradius formulas we have obtained for cyclic octagons already in 2004 (see [9, 11, 14, 18]) (for partial results see [22]), but for heptagon area equation we need to compute resultant of two polynomials of degree 11 and 12- not yet achievable on our computer at hand.
Future research: One may expect, with more powerful computer system, to obtain circumradius equation for cyclic nonagon (cyclic 9-gon) which has degree 187 in circumradius squared.

References

[1] A. F. Möbius, *Ueber die Gleichungen, mittelst welcher aus der Seiten eines in einen Kreis zu beschriebenden Vielecks der Halbmesser des Kreises un die Flähe des Vielecks gefunden werden*, Crelle's Journal, 3:5–34. 1828.

[2] A. W. Richeson, *Extension of Brahmagupta's Theorem*, American Journal of Mathematics,Vol.52 No.2, (Apr.,1930),pp.4An

[3] D. P. Robbins, *Areas of polygons inscribed in a circle*, Discrete & Computational Geometry, 12:223–236, 1994.

[4] D. P. Robbins, *Areas of polygons inscribed in a circle*, Amer. Math. Monthly 102 (1995), 523–530

[5] V. V. Varfolomeev, *Inscribed polygons and Heron Polynomials*, Sbornik: Mathematics, 194 (3):311–331, 2003.

[6] V. V. Varfolomeev, *Galois groups of Heron-Sabitov polynomials for pentagons inscribed in a circle*, Sbornik: Mathematics, 195 (3):3–16, 2004.

[7] D. Svrtan, D. Veljan and V. Volenec, *Geometry of pentagons: from Gauss to Robbins*, submitted to arXiv on 29 Mar 2004., https://arxiv.org/abs/math/0403503

[8] F. Miller Maley, David P. Robbins, Julie Roskies, *On the areas of cyclic and semicyclic polygons*, submitted to arXiv on 16. Jul 2004., https://arxiv.org/abs/math/0407300
submitted to arXiv on 9. Aug 2004., https://arxiv.org/abs/math/0408104

[9] I. Pak, *The area of cyclic polygons: Recent progress on Robbins' conjectures*, Advances in Applied Mathematics, Volume 34, Issue 4, May 2005, Pages 690–696.

[10] D. Svrtan, *Generalized Chebyshev Symmetric Multivariable Polynomials Associated to Cyclic and Tangential Polygons*, invited talk, Conference on Difference Equations, Special Functions and Applications, München, Germany, 25.07.2005.–29.07.2005.

[11] D. Svrtan, *A new approach to rationalization of surds*, submitted

[12] D. Svrtan, *On the Robbins problem for cyclic polygons*, 2005., https://www.mathos.unios.hr/images/predavanja/abs_svrtan.pdf

[13] M. Fedorchuk and I. Pak, *Rigidity and polynomial invariants of convex polytopes*, Duke Math. J. 129 (2) 371–404, 15 August 2005., https://doi.org/10.1215/S0012-7094-05-12926-X

[14] Pech, P., *Computations of the Area and Radius of Cyclic Polygons Given by the Lengths of Sides*, ADG2004 (Hong, H. and Wang, D., eds.), LNAI, 3763, Gainesville, Springer, 2006, 44–58.

[15] D. Svrtan, *On circumradius equation for cyclic polygons, Rigidity and Flexibility*, Invited talk, Wienna, Austria, 23.04.2006.–06.05.2006.

[16] G. Panina, G. N. Khimshiashvili, *Cyclic polygons are critical points of area*, Zap. Nauchn. Sem. POMI, 2008, Volume 360, 238–245

[17] R. Connelly, *Comments on generalized Heron polynomials and Robbins' conjectures Author links open overlay panelRobert Connelly*, Discrete Mathematics Volume 309, Issue 12, 28 June 2009, Pages 4192–4196

[18] D. Svrtan, *Geometry of Cyclic Heptagons/Octagons via "New" Brahmagupta Formula*, International Congress of Mathematicians/Rajendra Bhatia(ed.).-Hyderabad:HindustanBookAgency,2010.140-141, Tuesday, August 24 18:20-18:35 D. Svrtan, University of Zagreb *Intrinsic geometry of cyclic heptagons/octagons via new Brahmagupta formula*

[19] Moritsugu S., *Radius Computation for an Inscribed Pentagon in Sanpou-Hakki (1690)*, ACM Communications in Computer Algebra, 44(3), 2010, 127–128.

[20] Moritsugu S., *Radius Computation for Inscribed Polygons and Its Solution in Sanpou-Hakki (1690)*, Kyoto University RIMS Kokyuroku, 2011. (in press; in Japanese).

[21] Moritsugu S., *Computing Explicit Formulae for the Radius of Cyclic Hexagons and Heptagons*, Bulletin of JSSAC(2011), Vol. 18, No. 1, pp. 3–9

[22] S. Moritsugu, *Computation and Analysis of Explicit Formulae for the Circumradius of Cyclic Polygons* Communications of JSSAC (2018), Vol. 3, pp. 1-–17

[23] Moritsugu S., *Computing the Integrated Circumradius and Area Formulae for Cyclic Heptagons by Numerical Interpolation*, Bulletin of the Japan Society for Symbolic and Algebraic Computation 28 (1), 3–13, 2022-04

[24] Moritsugu S., *Applying new Brahmagupta's formula by Svrtan to the problems on cyclic polygons (Computer Algebra : Foundations and Applications)*, 2022. Jun, Research Institute for Mathematical Sciences, Kyoto University, RIMS Kokyuroku, Vol. 2224, 103–113.

Proceedings of the 4th Croatian Combinatorial Days
CroCoDays 2022
September 22 – 23, 2022

ISBN: 978-953-8168-63-5
DOI: 10.5592/CO/CCD.2022.10

Planets are (very likely) in orbits of stars

Darko Veljan

Abstract

The probability that a randomly and uniformly chosen point from the circumball of a tetrahedron lies outside of the inscribed ball of the tetrahedron can be bounded very sharply from below in terms of the edge lengths of the tetrahedron. One can imagine four stars in the Universe (vertices) with known mutual distances and a small (exo-) planet orbiting between them within the circumsphere. The least probability that the planet is outside of the insphere is given in terms of the distances of the stars. The least probability occurs for the regular tetrahedron and it is 0.962962.... Geometrically, this is a tricky corollary of (refinements of) the famous Euler inequality: circumradius is at least three times bigger than the inradius of a tetrahedron with equality for a regular tetrahedron. The Euler inequality can be extended to Euclidean simplices in all dimensions and to non-Euclidean planes. The most relevant cases of 3D and 4D being in accordance with the relativity theory are considered.

Keywords: Euler's inequality, refined Euler's inequality in 3D and 4D, exoplanet.

AMS subject classification (2020): 51M04, 51M16, 85A15.

1 Introduction

In this paper we shall geometrically explain why planets detected to be in vicinity of four stars must, in fact, with high probability, orbit around one of the four stars. The lower bound of the probability is given in terms of the distances of the stars. The essence of the argument is Euler's inequality $R \geq nr$ from 1765 between the circumradius R and the inradius r of an n-dimensional simplex and particularly its refinements as given in [1], [2] and [3].
A popular introductionary text on some mathematical and computational aspects in astronomy or astrophysics is [4] (in Croatian) and a textbook on the topic is e.g. [5]. A popular history book on math is [6]. Many Euler's contributions can be found there.

(Darko Veljan) Department of Mathematics, University of Zagreb, Croatia, darko.veljan@gmail.com

We shall first consider the case of three stars and a planet moving in their plane ($n = 2$). Next is the main 3D-case ($n = 3$), because four (general) points form a space.
Finally, we shall consider 4D-case (i.e. when $n = 4$) which is important because the Universe is by Einstein's relativity theory four dimensional (hyperbolic) space-time object.

2 Triangle of stars

To begin with, we start with an easier question. Let T be a triangle of stars and a, b and c their mutual distances. Let X be a randomly and uniformly chosen planet (point) within the circumcircle of T. What is the minimal probability p that X lies outside of of the incircle of T? That is, what is the minimal probability that X is rather close to one of the stars? The probability that a randomly and uniformly chosen point within the circumcircle of T (of radius R) which is within the incircle of T (of radius r) is equal to the quotient of their areas $r^2\pi/R^2\pi = (r/R)^2$. Hence, $p = 1-(r/R)^2$. In [1] (see also [2], [3]) we proved a refined Euler's inequality $R \geq 2r$ in the form

$$\frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc} \geq 2. \tag{1}$$

with equalities if and only if the triangle T is equilateral. Recall Heron's formula for the area S of T in terms of side lengths a, b, c of T:

$$16S^2 = (2s)d_3(a, b, c),$$

where $2s = a + b + c$ is the perimeter of T and $d_3(a, b, c) := (a + b - c)(a - b + c)(-a + b + c)$. Then the probability q that X is within the incircle is equal to:

$$\begin{aligned} q = \left(\frac{r}{R}\right)^2 = \left(\frac{S}{s} : \frac{abc}{4S}\right)^2 = \left(\frac{4S^2}{sabc}\right)^2 = \left(\frac{16S^2}{4sabc}\right)^2 = \\ \left(\frac{2sd_3(a, b, c)}{4sabc}\right)^2 = \left(\frac{d_3(a, b, c)}{2abc}\right)^2 \leq \frac{1}{4}. \end{aligned}$$

Hence,

$$p = 1 - q = 1 - \left(\frac{d_3(a, b, c)}{2abc}\right)^2 \geq 0.75. \tag{2}$$

The last inequality follows from Euler's $R \geq 2r$ and above inequality.

3 Tetrahedron of stars

Now, the main topic in this paper is to show that with a high probability an (exo-) planet from the vicinity of four stars must, in fact, orbit around one of the stars. We quote Theorem 4.1. from [3]: the probability that a randomly and uniformly chosen point within the circumsphere of the tetrahedron T is within the insphere of T is at most equal to

$$\sqrt{\frac{d_3(aa', bb', cc')}{[3(aa' + bb' + cc')]^3}} \leq \frac{1}{27}. \tag{3}$$

Here a, b and c are side lengths forming the base of T, a' is opposite to a etc. Therefore, the probability p that a randomly and uniformly chosen point (planet) from the circumball of the tetrahedron T of four stars (vertices of T) is outside of the inscribed ball of T is at least equal to

$$p \geq 1 - \sqrt{\frac{d_3}{(3e_1)^3}}, \tag{4}$$

where $d_3 := d_3(aa', bb', cc')$ and $e_1 := aa' + bb' + cc'$. (Recall that aa', bb', cc' form a triangle, called Crelle's triangle of T.) By Euler's inequality $R \geq 3r$ (and [3]), it follows from (4) that the probability

$$p \geq 1 - 1/27 = 0.962962\ldots \tag{4'}$$

And this is rather close to 1. This can be considered as a geometric proof that a planet very likely must orbiting around a star and taken by gravity rules must stay in the orbit for good (or at least for billions of years). The equality in (4) in fact occurs if and only if $aa' = bb' = cc'$, i.e. when Crelle's triangle of T is equilateral. In particular, we have equality in (4) and (4') when T is a regular tetrahedron.
This geometric-probabilistic proof shows that with the chance of at least 96.2962% we can expect that a planet moving in vicinity of four stars (with known mutual distances of stars) is rather close to one of the stars instead of being somewhere in deep space formed by four stars. And then the gravity rule force the planet to orbit (eliptically) around one of the star. (A little quiz for the reader: four girls are bathing in a lake or sea; their mutual distances are 20 meters; three girls have on them red bikini; what has the fourth girl on? If you don't know, see [4].)
As an astronomy example consider four stars from the Ursa Major (the Great Bear) constellation: Dubhe, Merak, Phecda and Megrez. They are roughly 120 ly far from the Earth. With data available from the Internet, a little computation shows that the chance that a planet close to the constellation (within the circumball) but out of the inball of the four stars, that is, rather close to one of the stars, is roughly a bit more than 97%. In other words, it is quite likely that a planet is not wandering in deep space between four stars.
A geometric example is the tetrahedron whose vertices are the origin and three unit points on the coordinate axes. Its Crelle's triangle is regular (with side lengths $\sqrt{2}$), hence the probability in question is minimal and is (at least) 96.2962%. For the tetrahedron $ABCD$ with $AB = BC = CD = 1$ and $BC \perp AB$, $CD \perp AB$, BC, i.e. an "ortoscheme", the considered probability is at least 97%. In any concrete example, we can, of course, compute R and r exactly and hence the probability, but that is not the issue here.

4 4-simplex

Four dimensional geometry in astrophysics is also of interest since our space-time Universe is four-dimensional. It seems, the hyperbolic geometry is prevailing, according to relativity theory, but 3D and even more 4D hyperbolic geometry is still to some degree "tabula rasa". One possible inequality for edge lengths of a hyperbolic tetrahedron can be obtained by applying Theorem 2.3. from [3] to corresponding Crelle's triangle (if it exists). In any way, the best known approximation to our geometric-probabilistic approach is Euclidean 3D- and 4D-geometry.
Recall, our refinement of Euler's $R \geq 4r$ for a 4-dimensional simplex $T = A_0A_1A_2A_3A_4$ in terms of symmetric functions of edge lengths $a_{ij} = A_iA_j$, $i < j$, of T is (5.8) from [3]

$$\left(\frac{R}{r}\right)^2 \geq 8\frac{\sum a_{ij}^2}{5\prod(i,j,k,l)^{1/15}} \geq 4^2, \tag{5}$$

where $\sum a_{ij}^2$ is the sum of squared lengths of all ten edges of T and the symbol (i,j,k,l) is defined by $(i,j,k,l) := d_3(a_{ij}a_{kl}, a_{ik}a_{jl}, a_{il}a_{jk})$ for all $0 \leq i < j < k < l \leq 4$. The equality in (5) holds for a regular 4-simplex. From (5) we have a sharp lower bound for the probability p that a randomly and uniformly chosen point from the circumball of the 4-simplex T is out of the inball of T. It is given by

$$p \geq 1 - \left(\frac{r}{R}\right)^4 \geq \frac{25\prod(i,j,k,l)^{2/15}}{64(\sum a_{ij}^2)^2} \geq 1 - 4^{-4}. \tag{6}$$

In the case of a regular 4D-simplex (of stars) with all edge lengths $a_{ij} = 1$, we have $(i,j,k,l) = 1$ for all five choices of i,j,k,l, and the probability is $p = 1 - 2^{-8} = 99.609\ldots\%$.

5 Conclusion

We have proved here that planets quite likely orbit around stars. This fact was observed back in ancient times, as well as the fact that satellites must orbit around planets; the first examples, naturally, being Moon around Earth and Earth around Sun. It was firmly established only by Kepler and later explained by Newton. In fact, the Croatian mathematician Ruđer Bošković was the first to give a procedure to compute a planet's orbit from three observations of its position. (Bošković did not get the Grand Prix of the Academy in 1752 for his studies on Saturn and Jupiter, but the prize was given to Euler.)
In this paper we provided a pure geometric proof of these facts.
In searching for an exo-planet, where humans have to move once in the future, one can imagine the following experiment. Suppose we get a spectral signal that a certain planet has water, but we don't know the exact position of this planet. Then our method predicts its position with high probability within four nearest stars. This prediction then focus the search for the planet to much less space.
By the procedure (recursive algorithm) proposed in [3] for refined Euler's inequality, we can proceed further to higher dimensions and prove that the limit when dimension

n tends to infinity for the probability that a randomly and uniformly chosen point from the circumball that is out of the inball of simplex is equal to 1.
In a similar manner to astrophysics, we can apply our method to the micro world as well, by considering for instance electrons in atoms and other subatomic particles and predict with some high probability the behavior and position of particles, but this is already in the domain of quantum physics. Such "predictology" is also very much appreciated in biochemistry, molecular biology, information theory and other modern sciences.

References

[1] D. Veljan, *Improved Euler's inequality in plane and space*, Journal of Geometry, **112** (2021), art 31.

[2] D. Veljan, *Refinements of Euler's inequalities in plane, space and n-space*, Proc. 3th Croatian Combinatorial Days, Zagreb, September 21-22, 2020 (Eds. T. Došlić and S. Majstorović), 129-140.

[3] D. Veljan, *Refined Euler's inequalities in plane geometries and spaces*, Rad HAZU, Matem. zn., to appear.

[4] D. Klobučar, *Mathematics Our Everyday 3*, Element, Zagreb, 2020 (Universe, pp.211-334) (in Croatian).

[5] J. L. Lawrence, *Celestial Calculations: A Gentle Introduction to Computational Astronomy*, The MIT Press, Boston, 2019.

[6] C. Pickover, *The Math Book*, Sterling, New York, 2009. (Croatian translation with some additions appeared as "Veličanstvena matematika", Izvori, Zagreb, 2023.)