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Complexity function for a variant of Flory model on a ladder

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Abstract

In this article, we compare the famous Flory model with its variant that was recently introduced by the authors. Instead of looking at these models on a one-dimensional lattice, we consider a two-row ladder. For both models we compute the complexity function, and we analyze the differences of static and dynamic versions of these models.

Keywords: dynamic lattice systems, equilibrium lattice systems, complexity function, configurational entropy, jammed configuration.

AMS subject classification (2020): 82B20, 82C20, 05B40, 05A15, 05A16.

1 Introduction

Random sequential adsorption (RSA) refers to a process where particles are sequentially introduced in a system at random. Placement (adsorption) of a new particle is accepted if the particle does not fall into the exclusion region of another particle that has already been adsorbed. The process continues until a *jammed configuration* is reached, i.e. until no more new particles can be adsorbed. These jammed configurations for various models have been extensively studied in the literature (see [18, §7] for a comprehensive overview). The main question related to jammed configurations concerns the expected density of the adsorbed particles. There are two natural ways to interpret this question. One is through the explained RSA approach where particles are sequentially introduced in a system until a jammed configuration is reached. This is usually referred to as the *dynamic* model. The limit to which the expected density of particles converges in this case is called *jamming limit*. The other way to look at this problem is to consider the set of all the possible jammed configurations

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and to sample one such configuration at random. This is referred to as the *static* (or *equilibrium*) model. The static model is usually described with the so-called *complexity function* (also known as *configurational entropy*), and the expected density of particles in a jammed configuration converges to the argument of the maximum of the complexity function. The precise definitions of the jamming limit and the complexity function are given later. It is not clear, at first, that the two models are different. The assumption that both approaches lead to the same expected density of large jammed configurations is referred to as the *Edwards hypothesis*, see [2] for a recent review. In the models studied in the present paper, as is the case for most irreversible deposition models, Edwards hypothesis is violated.

The dynamic model (RSA) has been studied extensively because of its wide applicability to diverse aspects of physics, chemistry and biology. It was first studied in one dimension. In his pioneering paper [12], Paul Flory studied the attachment of pendant groups in a polymer chain. Closely related to his work is the Page-Rényi car-parking problem (see [24]), which is a discrete version of the original car-parking model introduced by Alfred Rényi in [27]. To determine the value of the jamming limit one can run experiments (see [22]), computer simulations (see [23, 30, 31]), or use mathematical analysis (see [1, 3, 10, 12, 13, 18, 27]). Even though various analytic solutions have been found, for both continuum and lattice versions of RSA, for most of the models studied in the literature, jamming limit is only known approximately and is obtained with the aid of computer simulations.

The static (equilibrium) model has also received attention in the literature (see [4, 6, 7, 15, 18–20]). The configurational entropy of jammed configurations is usually determined either by means of direct combinatorial reasoning [4, 7, 19], or by using the transfer-matrix approach [6, 9, 20]. Recently, a new method for determining complexity function has been developed in [17], inspired by the theory of renewal processes.

Most of the results in the literature (in both the static and the dynamic case) have been obtained for one-dimensional lattices. The problem gets more involved in higher dimensions. In this paper, we consider a slightly more general structure than the one-dimensional lattice, namely a two-row ladder (see Figure 1). We employ the new approach from [17] to develop the complexity function for both the original Flory model, and the modified version of Flory model that was introduced in [25] and subsequently studied in [8, 16, 26]. This modified version of the original Flory model was introduced in terms of a combinatorial settlement model where the impact of the architect would be as small as possible, and people would have a lot of freedom in the process of building the settlement. This minimal intervention from the side of the architect is given through the condition that houses are not allowed to be blocked from the sunlight, and that the tracts of land on which the settlements are built are of rectangular shapes. A house is blocked from the sunlight if there are other houses on the neighboring sites to the right (east), to the left (west), and below (to the south) of that house. Additionally, it is assumed that houses receive sunlight from the eastern, southern and western boundary of the tract of land on which the settlement is being built. Here, we consider the two-row ladder as the rectangular tract of land on which the houses are built. The described condition imposes that each particle (house) in the upper (northern) row of the two-row ladder needs to

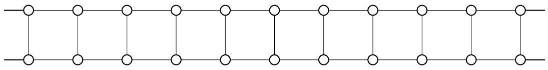

Figure 1: A two-row ladder graph.

have at least one of its neighboring sites free, while no restrictions are placed upon particles (houses) in the lower (southern) row as they are guaranteed to receive sunlight from the southern border. Compare this to the original Flory model, where each deposited particle enforces all of its neighboring sites to be vacant.
Flory model on a two-row ladder has already been studied in the literature in the dynamic case (see [1, 3, 10, 13]). The static case was considered in one dimension (see [18, §7.2]). On the other hand, the model introduced in [25] was originally studied on a two-dimensional lattice, and some observations have been made both in the static and the dynamic case (see [25,26]), but the complexity was addressed only in the one-dimensional version of that model, the so-called Riviera model (see [8,16]).

Notation. The length of the two-row ladder graph that we are considering is denoted by L. Notice that such a graph of length L has $2L$ sites that can be occupied by deposited particles (atoms, houses), or vacant. Each configuration of length L can be represented with two binary 0/1 sequences (both of length L, one representing the upper row, and the other one representing the lower row of a two-row ladder graph) where 1 is interpreted as an occupied site, and 0 as vacant site.
The total number of different jammed configurations on the two-row ladder of length L is denoted by J_L, and the total number of jammed configurations on the two-row ladder of length L that have precisely N deposited particles is denoted by $J_{N,L}$. The *density (saturation, coverage)* of any such configuration of length L with N particles is defined as N/L. Notice that, using the convention that the length of the two-row ladder graph with $2L$ sites is denoted by L, the density will always attain values in $[0, 2]$.

2 Flory model on a ladder

Flory model on a one-dimensional lattice is the most famous model within the theory of RSA. Each site on the lattice can be occupied by an atom, or vacant. Each atom blocks its nearest neighbor sites. This model (and the related Page-Rényi car-parking process) has been extensively studied in the literature, in both dynamic and static setting (see [12, 14, 24] for the dynamic version and [16, 18] for both versions). As announced in the introduction, we are dealing with Flory model on a two-row ladder graph. Most of the analytic treatments of such models are restricted to one-dimensional lattice. It would be very interesting to see how all these results translate to the truly two-dimensional lattice. This, however, seems to be an extremely technical task (if one wants to obtain some analytical result, and not just run simulations), so we provide here a very modest entrée into two-dimensional space.
To get a better feeling about the Flory model on a two-row ladder graph, and about the structure of jammed configurations of this model, we display in Figure 2 all the

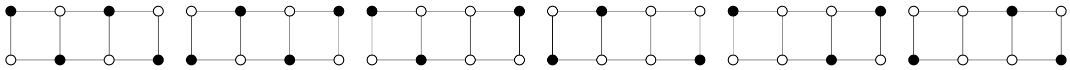

Figure 2: All the jammed configurations of Flory model on a two-row ladder graph of length 4.

possible jammed configurations on the two-row ladder graph of length 4.
Based on Figure 2, we can easily compare static and dynamic model in the case of the two-row ladder graph of length 4. Notice that $J_4^{\text{Flory}} = 6$. More precisely, $J_{3,4}^{\text{Flory}} = 4$, and $J_{4,4}^{\text{Flory}} = 2$. Denoting the random variable counting the number of occupied sites in the static model on a two-row ladder graph of length L by $X_s^{\text{Flory}}(L)$, and random variable counting the number of occupied sites in the dynamic model by $X_d^{\text{Flory}}(L)$, we read from Figure 2 that

$$X_s^{\text{Flory}}(4) \sim \begin{pmatrix} 3 & 4 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Inspecting in which order particles have to be introduced in the system to end up in each of the jammed configurations shown in Figure 2, one easily obtains

$$X_d^{\text{Flory}}(4) \sim \begin{pmatrix} 3 & 4 \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}.$$

Using the sequence of random variables $(X_d(L))_L$, that count the number of occupied sites in a dynamic model of length L, we can give a precise definition of the jamming limit of a certain model.

Definition 1. *For a fixed length $L \in \mathbb{N}$, let $X_d(L)$ denote the random variable which models the number of occupied sites in a jammed configuration of length L, reached through the RSA procedure. The* jamming limit *is then defined as*

$$\rho_\infty = \lim_{L\to\infty} \frac{\mathbb{E}[X_d(L)]}{L}, \tag{1}$$

if this limit exists, where $\mathbb{E}[X]$ denotes the expected value of a random variable X.

Remark 1. The convergence of expected values in (1) is the most common notion of convergence considered in these kind of problems. However, in some instances it was possible to prove convergence in probability for these random variables, in addition to convergence of expectations (see e.g. [14, 24] for discrete Rényi car-parking problem). Note that here, due to boundedness of random variables, the convergence in probability is strictly stronger than the convergence of expected values.

As mentioned in the introduction, the dynamic version of the Flory model on a two-row ladder has already been studied in the literature. It was independently shown in [10] and [1] that the jamming limit of this model is

$$\rho_\infty^{\text{Flory}} = 1 - \frac{1}{2e}, \tag{2}$$

and this result was later reconstructed in [3, 13].

Remark 2. Note that in [1, 3, 10, 13] the density is defined as $\frac{N}{2L}$, so the constant above is divided by two.

We now turn to the static model and to determining the complexity function. We first recall the definition of complexity function of a certain model.

Definition 2. *For a fixed density $\rho \geq 0$, take $((N_i, L_i))_i$ to be any sequence of pairs of non-negative integers such that $\lim_{i\to\infty} L_i = +\infty$ and $\lim_{i\to\infty} \frac{N_i}{L_i} = \rho$. Note that J_{N_i,L_i} is the number of configurations of length L_i with density $\frac{N_i}{L_i}$. We are interested in the quantity*

$$\limsup_{i\to\infty} \frac{\ln J_{N_i,L_i}}{L_i}$$

which is the exponential rate of growth of these configurations. If we now take the supremum over all such sequences, we arrive at the definition of complexity function $S(\rho)\colon [0,\infty) \to [0,\infty)$

$$S(\rho) = \sup_{(N_i,L_i)} \limsup_{i\to\infty} \frac{\ln J_{N_i,L_i}}{L_i} \tag{3}$$

where the supremum runs over all the sequences such that $N_i/L_i \to \rho$.

Remark 3. Whenever we encounter $J_{N,L} = 0$ for some (N, L), we will redefine it as $J_{N,L} = 1$ so that $\ln J_{N,L} = 0$ can be computed. Consequently, if there are no configurations with densities approaching a certain ρ, we get $S(\rho) = 0$. Also note that the $\limsup$ can be replaced with $\lim$ since we can, if needed, pass to a subsequence.

Remark 4. This definition implies that the number of configurations with density $N/L \approx \rho$ grows as $e^{LS(\rho)}$ for large L. The density $\rho_\star$ at which the complexity function $S(\rho)$ attains its maximum, i.e. the density corresponding to the largest rate of growth, is called the *equilibrium density* and is a static setting analogue of jamming limit as it can be shown that

$$\rho_\star = \lim_{L\to\infty} \frac{\mathbb{E}[X_s(L)]}{L}.$$

where $X_s(L)$ is the number of occupied sites in the static (equilibrium) model in which each jammed configuration of length L is equally likely. Furthermore, the random variables $X_s(L)/L$ can be shown to converge in distribution to delta distribution concentrated at $\rho_\star$.

Remark 5. In most commonly encountered models the sup in the definition is superfluous, as any choice of the sequence (say $((N_L, L))_L$ where $N_L = \lfloor \rho L \rfloor$) will produce the same limit.

Remark 6. Lastly, note that the complexity function is bounded

$$S(\rho) \leq 2 \ln 2.$$

This follows from the trivial bound $J_{N,L} \leq 2^{2L}$. A more precise bound

$$S(\rho) \leq 2H(\rho/2)$$

follows from the inequality $J_{N,L} \leq \binom{2L}{N} \sim \exp\left(2L \cdot H(N/(2L))\right)$, where $H(p)$ is the Shannon's entropy function $H(p) = -p \ln p - (1-p)\ln(1-p)$, see [5].

Figure 3: Vertices in the graph encoding the Flory model on a ladder.

To be able to apply the method from [17], we first need to determine the bivariate generating function for this model. To this end, we apply the so-called transfer matrix method (see [29, §4.7], [11, §V], and [21, §2–4]). This is a well known method for counting words of a regular language. Notice that every jammed configuration on a ladder is composed of the nodes shown in Figure 3. The transfer matrix is then of the shape

$$A(x) = \begin{bmatrix} 0 & 0 & 0 & x & 0 & 0 \\ 1 & 0 & 0 & 0 & x & 0 \\ 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 & x \\ 1 & 0 & 0 & 0 & x & 0 \end{bmatrix},$$

where x is a formal variable associated with the number of atoms. Taking into consideration what the possible starting and ending nodes of a jammed configuration in the Flory model on a two-row ladder graph are, we define vectors

$$a(x) = (x, 0, x, 0, x^2, x^2)^T \qquad \text{and} \qquad b = (0, 1, 0, 1, 1, 1)^T.$$

Now the bivariate generating function enumerating all the jammed configurations in the Flory model on a two-row ladder of length L with precisely N atoms is given by

$$\begin{aligned} F_{\text{Flory}}(x, y) &= 1 + 2xy + \sum_{n=2}^{\infty} a(x)^T \cdot [A(x)]^{n-2} \cdot b \cdot y^n \\ &= 1 + 2xy + a(x)^T \sum_{n=0}^{\infty} [A(x) \cdot y]^n \cdot b \cdot y^2 \\ &= \frac{1 + xy - xy^2}{1 - xy - xy^2}, \end{aligned} \tag{4}$$

where y is a formal variable associated with the length of the configuration. Hence, we have just proved the following lemma.

Lemma 1. *The bivariate generating function enumerating the total number of jammed configurations in the Flory model on a two-row ladder of length L, where precisely N sites are occupied with atoms, is*

$$F_{Flory}(x, y) = \frac{1 + xy - xy^2}{1 - xy - xy^2}.$$

We will now briefly describe the method from [17], based on the classical Legendre transform, which enables one to derive the complexity function directly from the bivariate (rational) generating function for the sequence $J_{N,L}$.

Remark 7. In order to follow the procedure described here, the generating function one starts with does not actually need to be rational. It suffices for the function to be analytic in y on some disk around the origin for each $x > 0$. But in order to actually calculate the complexity, one has to be able to express the radius of convergence (or the modulus of the non-removable singularity closest to the origin) of that function in terms of x. This is why rational generating functions are particularly amenable to this approach.

Let

$$F(x,y) = \sum_L \left(\sum_N J_{N,L}\, x^N \right) y^L = \frac{p(x,y)}{q(x,y)}$$

be the bivariate generating function of the sequence $(J_{N,L})$, where p and q are coprime polynomials. For each fixed $x > 0$, let $y_0(x)$ be the root of the polynomial $q(x,y)$ with the smallest modulus. Then, by the Cauchy–Hadamard theorem from complex analysis (see e.g. [32, Theorem 2.4.3]), it follows

$$\limsup_{L\to\infty} \sqrt[L]{\sum_N J_{N,L}\, x^N} = |y_0(x)|^{-1}.$$

As $x > 0$ and the sum above runs over $0 \le N \le 2L$ we obtain the bounds

$$\sqrt[L]{\max_N \left(J_{N,L}\, x^N\right)} \le \sqrt[L]{\sum_N J_{N,L}\, x^N} \le \sqrt[L]{(2L+1) \max_N \left(J_{N,L}\, x^N\right)}$$

and consequently

$$\limsup_{L\to\infty} \sqrt[L]{\max_N \left(J_{N,L}\, x^N\right)} = \limsup_{L\to\infty} \sqrt[L]{\sum_N J_{N,L}\, x^N} = |y_0(x)|^{-1}.$$

After taking logarithms

$$\limsup_{L\to\infty} \frac{\ln\left(\max_N \left(J_{N,L}\, x^N\right)\right)}{L} = -\ln|y_0(x)|. \tag{5}$$

Next, note

$$\ln\left(\max_N \left(J_{N,L}\, x^N\right)\right) = \max_N \left(\ln J_{N,L} + N \ln x\right) = L \max_N \left(\frac{\ln J_{N,L}}{L} + \frac{N}{L} \ln x\right)$$

and so (5) reads

$$\limsup_{L\to\infty} \max_N \left(\frac{\ln J_{N,L}}{L} + \frac{N}{L} \ln x\right) = -\ln|y_0(x)|. \tag{6}$$

We next wish to show the following claim

Claim 1. *The complexity function $S(\rho)$ associated to the sequence $J_{N,L}$ satisfies the equation*

$$\sup_\rho \left\{S(\rho) + \rho \ln x\right\} = -\ln|y_0(x)|. \tag{7}$$

Proof. Fix $\rho \geq 0$ and let (N_i, L_i) be an arbitrary sequence such that $N_i/L_i \to \rho$. Clearly

$$\frac{\ln J_{N_i,L_i}}{L_i} + \frac{N_i}{L_i} \ln x \leq \max_N \left(\frac{\ln J_{N,L_i}}{L_i} + \frac{N}{L_i} \ln x \right)$$

and thus

$$\limsup_{i\to\infty} \left(\frac{\ln J_{N_i,L_i}}{L_i} \right) + \rho \ln x = \limsup_{i\to\infty} \left(\frac{\ln J_{N_i,L_i}}{L_i} + \frac{N_i}{L_i} \ln x \right) \leq$$
$$\leq \limsup_{i\to\infty} \max_N \left(\frac{J_{N,L_i}}{L_i} + \frac{N}{L_i} \ln x \right) \leq \limsup_{L\to\infty} \max_N \left(\frac{\ln J_{N,L}}{L} + \frac{N}{L} \ln x \right).$$

Since this holds for any sequence (N_i, L_i) with $N_i/L_i \to \rho$ we get

$$S(\rho) + \rho \ln x = \sup_{(N_i,L_i)} \limsup_{i\to\infty} \left(\frac{\ln J_{N_i,L_i}}{L_i} \right) + \rho \ln x \leq \limsup_{L\to\infty} \max_N \left(\frac{\ln J_{N,L}}{L} + \frac{N}{L} \ln x \right).$$

Since the latter inequality holds for any ρ we conclude

$$\sup_\rho \{S(\rho) + \rho \ln x\} \leq \limsup_{L\to\infty} \max_N \left(\frac{\ln J_{N,L}}{L} + \frac{N}{L} \ln x \right). \tag{8}$$

For the opposite inequality, let us take a sequence $((N_L, L))_L$ where for each L, $0 \leq N_L \leq 2L$ is the number at which the maximum in the right hand side of (8) is attained. Passing to a subsequence $((N_{L_i}, L_i))_i$, if needed, we may assume that the $\limsup$ on the right hand of (8) is actually lim. After passing to a subsequence once more, if needed, we may assume that $N_{L_i}/L_i \to \tilde{\rho}$ for some $\tilde{\rho} \in [0, 2]$. Therefore

$$\limsup_{L\to\infty} \max_N \left(\frac{\ln J_{N,L}}{L} + \frac{N}{L} \ln x \right) = \lim_{i\to\infty} \left(\frac{\ln J_{N_{L_i},L_i}}{L_i} + \frac{N_i}{L_i} \ln x \right) =$$
$$= \lim_{i\to\infty} \left(\frac{\ln J_{N_{L_i},L_i}}{L_i} \right) + \tilde{\rho} \ln x \leq S(\tilde{\rho}) + \tilde{\rho} \ln x$$

where in the last step we used the fact that $((N_{L_i}, L_i))_i$ is just one of the sequences over which the supremum is taken in the definition of $S(\tilde{\rho})$. Finally

$$\limsup_{L\to\infty} \max_N \left(\frac{\ln J_{N,L}}{L} + \frac{N}{L} \ln x \right) \leq S(\tilde{\rho}) + \tilde{\rho} \ln x \leq \sup_\rho \{S(\rho) + \rho \ln x\} \tag{9}$$

and putting the statements (6), (8) and (9) together gives the claim (7). □

If we set $x = e^t > 0$ then the relation (7) becomes

$$\sup_\rho \{t\rho - (-S(\rho))\} = -\ln |y_0(e^t)|$$

and it expresses the fact that the function $-\ln |y_0(e^t)|$ is the convex conjugate (or *Legendre-Fenchel transform*) of the function $-S(\rho)$, see [28, §12].

Since we know the function $-\ln|y_0(e^t)|$, and seek to find $-S(\rho)$, we actually need to invert the transformation. Luckily, the Legendre transformation is an involution and therefore

$$\sup_t \left\{t\rho - (-\ln|y_0(e^t)|)\right\} = -S(\rho),$$

which we can express in terms of x as

$$\sup_{x>0} \{\rho \ln x + \ln|y_0(x)|\} = -S(\rho). \tag{10}$$

Remark 8. The Legendre transformation is actually an involution only on the set of closed convex functions, and if applied twice on any function it returns the closed convex envelope of that function [28, Corollary 12.1.1 and Theorem 12.2]. As the complexity functions for similar models in the literature are known to be closed concave ($-S$ is closed convex), it is reasonable to think that the complexity function in our model must be as well. With this in mind, a more precise restatement of (10) would be

$$\sup_{x>0} \{\rho \ln x + \ln|y_0(x)|\} = \mathrm{cl}\left(\mathrm{conv}\left(-S(\rho)\right)\right).$$

Using the first derivative test, we see that x_0 at which maximum on the left hand side of (10) is attained must satisfy

$$\rho = -x_0 \frac{y_0'(x_0)}{y_0(x_0)} = \left[\frac{x}{y}\frac{\partial_x q}{\partial_y q}\right]_{x=x_0, y=y_0(x_0)} \tag{11}$$

where we used the implicit function theorem and the fact that $y_0(x)$ is defined by the relation $q(x, y_0(x)) = 0$. If we, now, for each $0 \le \rho \le 2$ solve (11) and $q(x_0, y_0) = 0$ for $x_0 = x_0(\rho)$ and $y_0 = y_0(x_0(\rho))$, then we can write the complexity function as

$$S(\rho) = -\rho \ln x_0 - \ln|y_0|. \tag{12}$$

In order to find the equilibrium density $\rho_\star$ of the model we need to solve the optimization problem

$$\rho_\star = \arg\max_\rho \{-\rho \ln x_0 - \ln|y_0|\}.$$

Note that the complexity function $S(\rho)$ can be parameterized using x as

$$S(\rho(x)) = x\frac{y_0'(x)}{y_0(x)} \ln x - \ln|y_0(x)|,$$

and it is easier to solve the optimization problem in this form. The solution $x = x_\star$ at which the maximum is attained must satisfy the equation

$$\frac{y_0'(x)}{y_0(x)} \ln x + x\frac{y_0''(x)y_0(x) - (y_0'(x))^2}{(y_0(x))^2} \ln x + x\frac{y_0'(x)}{y_0(x)}\frac{1}{x} - \frac{y_0'(x)}{y_0(x)} = 0.$$

or equivalently

$$\left(\frac{y_0'(x)}{y_0(x)} + x\frac{y_0''(x)y_0(x) - (y_0'(x))^2}{(y_0(x))^2}\right) \ln x = 0.$$

(a) $\frac{N}{L} = \frac{1}{2} + \frac{1}{18}$ (b) $\frac{N}{L} = 1$

Figure 4: Examples of two jammed configurations with extreme densities for the Flory model with $L = 9$.

From here, it is easy to see that $x_\star = 1$, and the corresponding $y_\star = y_0(1)$ is the solution of the equation $q(1, y_\star) = 0$. The equilibrium density $\rho_\star$ and the maximum value of the complexity function can then be calculated using (11) and (12) as

$$\begin{aligned} \rho_\star &= \frac{1}{y_\star} \frac{\partial_x q(1, y_\star)}{\partial_y q(1, y_\star)}, \\ S(\rho_\star) &= -\ln |y_\star|. \end{aligned} \tag{13}$$

Remark 9. Before we proceed to apply this method in order to compute the complexity of the Flory model on a two-row ladder, let us briefly discuss the support of S. As each occupied site in a maximal configuration must have all of its three neighbors unoccupied, and as each unoccupied site has at least one occupied neighbor, if follows $E \leq 3N$, where N and E stand for the number of occupied and unoccupied sites in a configuration, respectively. From here, $2L = E + N \leq 4N$ and therefore $\frac{N}{L} \geq \frac{1}{2}$. Also, to each occupied site corresponds the unique unoccupied site (above or below) which is on the same rung of the ladder. Hence, $N \leq E$ or, equivalently, $2N \leq N + E = 2L$ and therefore $\frac{N}{L} \leq 1$. One can actually show, more precisely, that $\left\lfloor \frac{L}{2} \right\rfloor + 1 \leq N \leq L$. Examples of the most and the least saturated maximal configurations for $L = 9$ are depicted in Figure 4.
The previous discussion implies that the complexity $S(\rho)$ can be non-zero only on the interval $\frac{1}{2} \leq \rho \leq 1$.

It is clear from the method described above that complexity is determined by the denominator of the bivariate generating function. In our case, the denominator of the expression (4) is

$$q(x, y) = 1 - xy - xy^2,$$

which coincides with the one that appears in [17, §4.2 *Isolated empty sites* (4.21)]. The name *Isolated empty sites* of the model discussed in [17, §4.2] already suggests that this is equivalent to the Flory model on a two-row ladder, because in the Flory model on the two-row ladder we are not allowed to have two empty columns in a row, and in non-empty columns we always have exactly one atom. Following the method described above, Krapivsky and Luck in [17, equation (4.28)] obtain the complexity function

$$S_{\text{Flory}}(\rho) = \rho \ln(\rho) - (1 - \rho) \ln(1 - \rho) - (2\rho - 1) \ln(2\rho - 1), \tag{14}$$

for $1/2 < \rho < 1$ (see Figure 5). The density for which the complexity function is maximized is

$$\rho_\star^{\text{Flory}} = \frac{5 + \sqrt{5}}{10}, \tag{15}$$

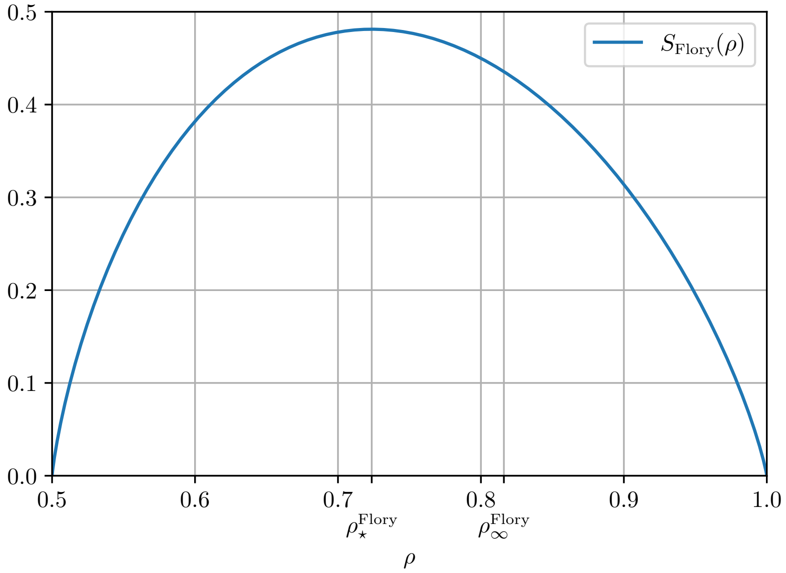

Figure 5: The complexity of the Flory model on the ladder. Also depicted on the plot are the jamming limit $\rho_\infty^{\mathrm{Flory}} = 1 - (2e)^{-1}$, and the equilibrium density $\rho_\star^{\mathrm{Flory}} = (5+\sqrt{5})/10$.

and the value of the maximum is

$$S(\rho_\star^{\mathrm{Flory}}) = \ln\left(\frac{1+\sqrt{5}}{2}\right). \tag{16}$$

This can be obtained from (13) by setting $x_\star = 1$ and $y_\star = \frac{\sqrt{5}-1}{2}$. The interpretation of the value calculated in (16) is that on the two-row ladder of length L, there are roughly

$$\left(\frac{1+\sqrt{5}}{2}\right)^L$$

jammed configurations. Comparing the jamming limit $\rho_\infty^{\mathrm{Flory}}$ from equation (2), which summarizes the dynamic model, and the value of $\rho_\star^{\mathrm{Flory}}$ (i.e. the argument of the maximum of the complexity function) given in (15), which summarizes the equilibrium model, we see that Edwards hypothesis does not hold (see Figure 5).

Remark 10. Notice that the constant appearing in relation (16) is the famous golden ratio. It is in fact very easy to see that the sequence J_L counting the number of all jammed configurations of length L satisfies the well-known Fibonacci recursion

$$J_L^{\mathrm{Flory}} = J_{L-1}^{\mathrm{Flory}} + J_{L-2}^{\mathrm{Flory}}.$$

3 Settlement model on a ladder

Recently, a combinatorial settlement planning model was introduced in [25] and the same model has been further studied in [8,16,26]. As in the case of Flory model, most of the results have been developed for the one-dimensional version of this model, the

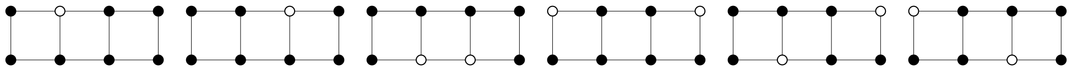

Figure 6: All the jammed configurations of the settlement model from [25] on a two-row ladder graph of length 4 ($J_4 = 6$, $J_{6,4} = 4$, $J_{7,4} = 2$).

so-called Riviera model. The complexity function for the Riviera model has been derived in [16], and in the same paper the authors discuss the dynamic model, and they obtain the jamming limit using simulations. In this paper, we are interested in the model from [25] on a ladder. As in the previous section, we first show some concrete examples of jammed configurations of this model. In Figure 6 are shown all the jammed configurations of combinatorial settlement model, introduced in [25], on a two-row ladder graph of length 4. Recall that there are no restrictions on the houses placed in the bottom row, while each house in the top row (which is not on the boundary) must have at least one of its immediate neighbors unoccupied. It is clear from Figure 6 that

$$X_s(4) \sim \begin{pmatrix} 6 & 7 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix},$$

where $X_s(4)$ is a random variable that counts the number of occupied sites in the static combinatorial settlement model on a two-row ladder graph of length 4. Denoting by $X_d(4)$ the corresponding random variable in the dynamic version of the model, and carefully analyzing in which order the houses had to be built to reach a particular jammed configurations, we have

$$X_d(4) \sim \begin{pmatrix} 6 & 7 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

This particular model has been considered in the static case in [8], but only bivariate generating function was developed. However, in light of the new method (introduced in [17]) for computing complexity function, we know that this result about the bivariate generating function can be used to obtain the desired complexity function. From [8, §4.1 and relation (A.4)], we have

$$F(x,y) = \frac{1 - xy + x^2y - 2x^3y^2 + x^4y^2 + x^4y^3 + x^5y^3 - x^8y^5}{1 - xy - x^3y^2 + x^4y^3 - x^5y^3 - x^6y^4 + x^9y^6}. \tag{17}$$

It is not hard to see what the extreme configurations look like for this model (see also [25, Theorem 4.5]) and to conclude

$$L + 2 \le N \le 2L - \lfloor L/3 \rfloor = \frac{5}{3}L + (L/3 - \lfloor L/3 \rfloor) \le \frac{5}{3}L + \frac{2}{3},$$

and from there

$$1 + \frac{2}{L} \le \frac{N}{L} \le \frac{5}{3} + \frac{2}{3L}.$$

In Figure 7 are given examples of the most and the least saturated maximal configurations for $L = 9$. The inequality above immediately implies that the support of the complexity function of this model will be contained in the interval $1 \le \rho \le 5/3$.

(a) $\frac{N}{L} = 1 + \frac{2}{9}$ (b) $\frac{N}{L} = \frac{5}{3}$

Figure 7: Examples of two jammed configurations with extreme densities for the settlement model with $L = 9$.

Using the same notation as in the previous section, we have

$$q(x, y) = 1 - xy - x^3y^2 + x^4y^3 - x^5y^3 - x^6y^4 + x^9y^6. \tag{18}$$

As it is not possible to explicitly solve for $y_0(x)$ in the equation $q(x, y) = 0$, the first step is to find a parametrization of this equation which will enable us to express its roots in terms of the new variables. To this end, we employ the parametrization

$$\begin{cases} t = y^{\frac{1}{3}}, \\ u = xy^{\frac{2}{3}}, \end{cases} \quad \text{i.e.} \quad \begin{cases} x = \dfrac{u}{t^2}, \\ y = t^3. \end{cases}$$

Plugging this into the equation $q(x, y) = 0$ (where $q(x, y)$ is given in (18)) gives us

$$(u^4 - u)t + (u^9 - u^6 - u^3 + 1) - u^5t^{-1} = 0.$$

Multiplying by t, we get

$$(u^4 - u)t^2 + (u^9 - u^6 - u^3 + 1)t - u^5 = 0.$$

From here it follows that

$$t_0^{\pm} = \frac{-(u^9 - u^6 - u^3 + 1) \pm \sqrt{u^{18} - 2u^{15} - u^{12} + 8u^9 - 5u^6 - 2u^3 + 1}}{2(u^4 - u)}. \tag{19}$$

These two expression for t_0 in terms of u enable us to express the roots $y_0(x)$ of the polynomial $q(x, y_0(x))$ for each $x > 0$ in terms of the parameter u. In fact, all the quantities of interest can now be expressed in terms of u in the following way

$$y_0 = [t_0(u)]^3, x_0 = \frac{u}{[t_0(u)]^2}, \quad \rho(u) = \left[\frac{x}{y}\frac{\partial_x q}{\partial_y q}\right]_{x=x_0, y=y_0}, \quad S(\rho) = -\rho \ln x_0 - \ln |y_0|.$$

We have, thus, proved the following theorem which provides a parameterized expression for the complexity function of the combinatorial settlement planning model on a two-row ladder.

Theorem 1. *The complexity function of the combinatorial settlement planning model on a two-row ladder is given as*

$$S(\rho) = -\rho \ln x_0 - \ln |y_0|,$$

where

$$\begin{cases} \rho(u) = \left.\frac{-9x^8y^5+6x^5y^3+5x^4y^2-4x^3y^2+3x^2y+1}{-6x^8y^5+4x^5y^3+3x^4y^2-3x^3y^2+2x^2y+1}\right|_{x=x_0, y=y_0}, \\ x_0 = u\,[t_0(u)]^{-2}, \\ y_0 = [t_0(u)]^3, \end{cases}$$

and $t_0(u)$ is given by (19).

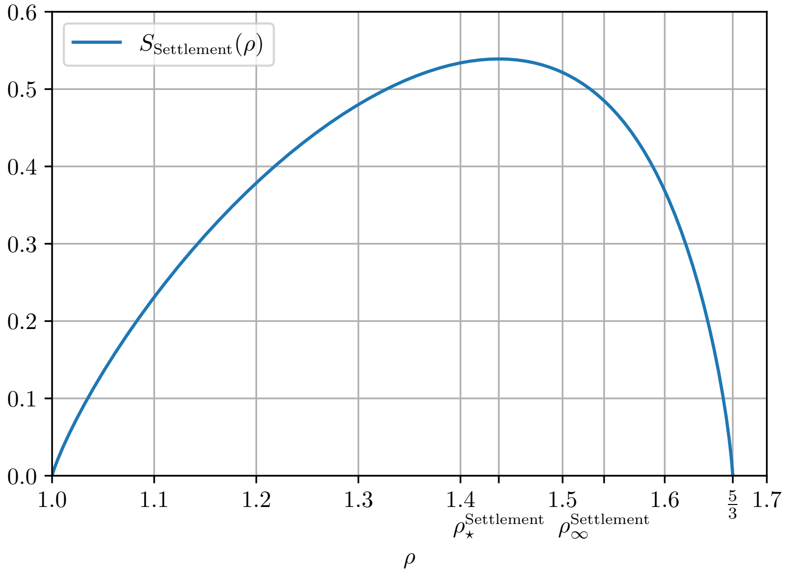

Figure 8: The complexity of the combinatorial settlement model on the ladder. Also depicted on the plot are (a Monte Carlo estimation of) the jamming limit $\rho_\infty^{\text{Settlement}} \approx 1.54064$, and the equilibrium density $\rho_\star^{\text{Settlement}} \approx 1.4374959$.

Remark 11. The equilibrium density $\rho_\star^{\text{Settlement}}$ for this model can be calculated using (13). More precisely,

$$\rho_\star^{\text{Settlement}} = \frac{1 + y_\star + y_\star^2 + 6y_\star^3 - 9y_\star^5}{1 + 2y_\star + 4y_\star^3 - 6y_\star^5} \approx 1.4374959$$

where $y_\star$ is the root of the polynomial $1 - y - y^2 - y^4 + y^6$ with the smallest modulus.

Remark 12. A closer inspection of the parametrization given in the theorem above and the equation (19) reveals that, in order to obtain the whole range of values $x > 0$, it suffices to take $0 < u \leq u_0$, where $u_0 \approx 0.70633685$ is the positive root of the polynomial appearing under the square root in (19). For each such a value u, both t_0^+ and t_0^-, have to be used in order to produce the whole range $x > 0$.

Theorem 1 enables us to produce the graph in Figure 8 showing the complexity function of the settlement model. Note that, as previously discussed, the complexity is supported on $[1, \frac{5}{3}]$.
When it comes to the dynamic version of the combinatorial settlement planning model, there are no analytical results for the jamming limit. Even in the simplest case of Riviera model (which lives on a one-dimensional lattice), the exact solution is not known, and it is not clear whether it is even possible to derive one. One-dimensional RSA, and similar models, usually enjoy a peculiar property that is commonly referred to as the shielding property. The deposition of any elementary object splits the line into two half-lines which evolve independently from each other. This ensures the exact solvability of this class of models, by means of a common analytical approach based on tracking empty intervals (see [18, §7]). Clearly, the original Flory model has the shielding property. The Riviera model, on the other hand, does not enjoy the shielding property. On the contrary, the rule that at least

one of the two neighboring lots of each house should remain forever unbuilt couples both neighboring sites of any occupied one. The ensuing lack of exact solvability has observable consequences (see [16, §3]). The authors of [16] even write that this model (in all likelihood) cannot be solved by analytical means. In the case of the two-row ladder, the situation is even more involved, so we only provide Monte Carlo estimation of the jamming limit. The value is approximately $\rho_\infty^{\text{Settlement}} \approx 1.54064$, and comparing this value with $\rho_\star^{\text{Settlement}} \approx 1.4374959$, which is the argument of the maximum of the complexity function from Theorem 1 (see Figure 8) again shows the violation of Edwards hypothesis.

4 Conclusions

In this paper we use the novel approach developed in [17] to compute the complexity function (configurational entropy) of two similar, but essentially different, models on a two-row ladder graph. One of them is the famous Flory model, and the other one is the combinatorial settlement model introduced in [25]. Using the obtained configurational entropies, we compare the dynamic and static versions of these models and conclude weak violation of Edwards hypothesis.

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