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Extended abstract on Local Irregularity Conjecture and cactus graphs

Jelena Sedlar and Riste Škrekovski

Abstract

A graph is locally irregular if the degrees of the end-vertices of every edge are distinct. An edge coloring of a graph G is locally irregular if every color induces a locally irregular subgraph of G. A colorable graph G is any graph which admits a locally irregular edge coloring. The locally irregular chromatic index $\chi'_{\rm irr}(G)$ of a colorable graph G is the smallest number of colors required by a locally irregular edge coloring of G. The Local Irregularity Conjecture claims that all colorable graphs require at most 3 colors for a locally irregular edge coloring.

In this extended abstract, we summarize our results on the mentioned conjecture for the class of cactus graphs. First, we established that the conjecture holds for unicyclic graphs and cacti with vertex disjoint cycles. Then we observed that there exists a cactus graph, the so called bow-tie graph B, which is colorable and requires at least 4 colors for a locally irregular edge coloring [8]. As B is a cactus graph and all non-colorable graphs are also cacti, the class of cactus graphs seems to be a relevant for the Local Irregularity Conjecture. By further researching this class, we established that 4 colors are the maximum number of colors required by a locally irregular edge coloring of any colorable cactus graph [9]. Using the same approach, but with the more elaborated argument, we established that the bow-tie graph B is the only connected colorable cactus graph which requires 4 colors for a locally irregular edge coloring [10]. Our last result indicates that B is possibly the only connected graph with $\chi'_{\rm irr}(B) = 4$, and consequently it could be the only graph which contradicts the present form of the Local Irregularity Conjecture.

Keywords: locally irregular edge coloring; Local Irregularity Conjecture; cactus graphs.

AMS subject classification (2020): 05C15.

(Jelena Sedlar) University of Split, Faculty of civil engineering, architecture and geodesy, Croatia, jelena.sedlar@gradst.hr

(Riste Škrekovski) University of Ljubljana, FMF, Ljubljana, Slovenia and Faculty of Information Studies, Novo Mesto, Slovenia, riste.skrekovski@fmf.uni-lj.si

1 Introduction

For all graphs mentioned in this paper it is tacitly assumed they are simple, finite and connected. A *locally irregular* graph is any graph in which the two end-vertices of every edge have distinct degrees. A *locally irregular k-edge coloring* (also called *k-liec* for short) is any k-edge coloring of G such that every color induces locally irregular subgraph of G. We say that a graph G is *colorable* if it admits a locally irregular edge coloring. For a colorable graph G, the smallest number k such that G admits a k-liec is called the *locally irregular chromatic index* of G and denoted by $\chi'_{\mathrm{irr}}(G)$. Since isolated vertices of G do not influence the local irregularity of an edge coloring, if such vertices arise in a graph by edge deletion we will ignore them. We first need to establish which graphs are colorable. For that purpose, any graph with edge disjoint cycles is called a *cactus* graph. We define a class $\mathfrak{T}$ of cacti as follows:

- the triangle K_3 belongs to $\mathfrak{T}$,
- for every graph G in $\mathfrak{T}$, a graph H which is also from $\mathfrak{T}$ is obtained as: let v be a vertex of G of degree 2 from a triangle in G, and let w be a vertex of degree one in a path of even length or in a graph consisting of a triangle and an odd-length path pending on one vertex of the triangle, then identifying v and w yields H.

Obviously, every graph G in $\mathfrak{T}$ is a special kind of cactus graph in which every cycle is a triangle, all triangles are vertex disjoint, and all vertices in G are of degree ≤ 3. Also, every vertex of degree 3 belongs to a triangle, a pair of triangles is connected by a path of odd length and there may be paths of even length pending at a vertex of a triangle. It is known [1] that paths of odd length, cycles of odd length and cactus graphs from $\mathfrak{T}$ are the only non-colorable connected graphs. As for colorable graphs, the following conjecture was proposed [1].

Conjecture 1 (Local Irregularity Conjecture)**.** *For every colorable connected graph G, $\chi'_{\mathrm{irr}}(G) \leq 3$.*

Many partial results support this conjecture, the conjecture is true for trees [2], graphs with vertex disjoint cycles [8], graphs in which the minimum degree is at least 10^{10} [7], k-regular graphs for $k \geq 10^7$ [1]. For general graphs, the first constant upper bound on $\chi'_{\mathrm{irr}}(G)$ that was found is 328 [4], and then it was decreased to 220 [6].

2 Preliminaries

Let us introduce some basic notions and notation related to colorings, which are already introduced in literature, mainly in [2].
The number of a-colored edges incident to a vertex u is called the *a-degree* of u and it is denoted by $d^a_G(u)$. Similarly we have b-degree, c-degree, etc. The *a-sequence* of a vertex $u \in V(G)$ is defined as $d^a_G(v_1), \ldots, d^a_G(v_k)$ where $v_1, \ldots, v_k$ are all neighbors

of u. We usually assume that neighbors v_i of u are denoted so that the a-sequence is non-increasing.
For a graph G and a subgraph G_0 of G, a k-liec ϕ of G is said to be an *extension* of k-liec ϕ^0 of G_0 if $\phi(e) = \phi^0(e)$ for every $e \in E(G_0)$.
Let G be a colorable graph which admits a k-liec for $k \geq 3$. If G contains a leaf u, then a graph G' obtained from G by appending an even length path P at u also admits a k-liec. An *ear* of a graph G is any subgraph P of G such that P is a path and for any internal vertex $u \in V(P)$ it holds that $d_G(u) = 2$. If G contains an ear $P_q = u_0u_1 \dots u_q$ of length $q \geq 3$, then at least one internal vertex u_i of P_q is bichromatic by a k-liec of G, so if G' is obtained from G by replacing an ear P_q by an ear P_{q+2r} of the length $q + 2r$, then G' also admits a k-liec. In both situations we say G is obtained by *trimming* G'.

Remark 1. Let G' be a colorable graph which admits a k-liec for $k \geq 3$. If G is obtained by trimming G', then G also admits a k-liec.

A graph G is *totally trimmed* if it does not contain a pending path of length ≥ 3 and an ear of length ≥ 5. Because of Observation 1, in the rest of the paper we will tacitly assume every graph G is totally trimmed.
A *shrub* is any tree rooted at its leaf. An *almost locally irregular k-edge coloring* of a shrub G, or *k-aliec* for short, is any k-liec of G or any edge coloring of G such that only the edge incident to the root is locally regular. The following are useful results for trees from [2].

Theorem 1. *Every shrub admits a* 2*-aliec.*

Theorem 2. *Every colorable tree T satisfies $\chi'_{\text{irr}}(T) \leq 3$. Moreover, if $\Delta(T) \geq 5$, then $\chi'_{\text{irr}}(T) \leq 2$.*

Let T be a tree with $\chi'_{\text{irr}}(T) = 3$ and ϕ a 3-liec of T. If there is precisely one vertex u in T which is incident to edges of all three colors, and every shrub rooted at u is colored by at most two colors, than ϕ is called a *special* coloring and u is called the *rainbow root* of ϕ. The proof of Theorems 1 and 2 from [2] implies the following observation.

Remark 2. Let T be a colorable tree with $\chi'_{\text{irr}}(T) = 3$. Every vertex of maximum degree in T is a neighbor of another vertex of maximum degree in T. Let u be a vertex of maximum degree in T, then there exists a special 3-liec of T for which u is the rainbow root. Also, for any shrub T_i of T rooted at u, there exists a 3-liec of T which uses color c only in T_i.

3 Our results

In this section we will present our results on unicyclic graphs and cacti from [8], [9] and [10]. Since the Local Irregularity Conjecture holds for trees, we first investigated unicyclic graphs given that they are obtained from trees by introducing a single edge. In [8] the following result is established.

Theorem 3. *Every colorable unicyclic graph G sattisfies $\chi'_{\text{irr}}(G) \leq 3$.*

A natural question that arises from this result is whether the bound $\chi'_{\mathrm{irr}}(G) \leq 3$ is tight, i.e. are there colorable unicyclic graphs which are not 2-colorable. The family of cycles of length $4k + 2$ are such graphs, but this family is not an isolated case, there exist other unicyclic graphs which require three colors, for example the graph from Figure 1. One can assure infinitely many such graphs for example by taking longer threads of suitable parity in the given graph.

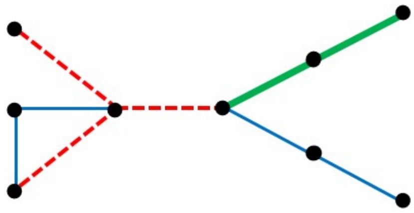

Figure 1: A colorable unicyclic graph distinct from cycle which requires 3 colors for locally irregular edge coloring.

The approach for unicyclic graphs usually extends to cactus graphs, so we further investigated that class of graphs. Let G be a cactus graph with at least two cycles, let C be a cycle in G and let u be a vertex from C. We say that u is a *root vertex* of C if the connected component of $G - E(C)$ which contains u is a cyclic graph. A cycle C of G is a *proper end-cycle* if $G - V(C)$ contains at most one cyclic connected component. Every cactus graph with vertex disjoint cycles contains at least two proper end-cycles, given it is not a unicyclic graph. Focusing the attention to a proper end-cycle of a cactus graph with vertex disjoint cycles, we obtained the following result [8].

Theorem 4. *Every colorable cactus graph G with vertex disjoint cycles satisfies $\chi'_{\mathrm{irr}}(G) \leq 3$.*

The attempt to extend the approach to cacti in which cycles may share a vertex yielded the so called bow-tie graph B, illustrated by Figure 2, and the following result.

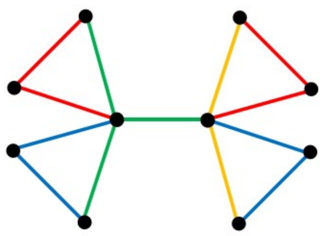

Figure 2: The bow-tie graph B and a locally irregular 4-edge coloring of it.

Proposition 1. *For the bow-tie graph B, it holds that $\chi'_{\mathrm{irr}}(B) = 4$.*

Proof. The edge coloring of the graph B shown in Figure 2 is locally irregular and it uses 4 colors, thus $\chi'_{\rm irr}(G) \leq 4$. It remains to establish that B does not admit a locally irregular coloring with less than 4 colors. Let u and v denote two vertices of degree 5 in B. Let $u_1, \ldots, u_4$ (resp. $v_1, \ldots, v_4$) be the four vertices neighboring u (resp. v) distinct from v (resp. u), so that u_1 and u_2 (resp. v_1 and v_2) belong to a same triangle of B. Let ϕ be a locally irregular edge coloring of B which uses less than 4 colors. Notice that $\phi(u_1u_2)$ must be equal to precisely one of $\phi(u_1u)$ and $\phi(u_2u)$, we may assume $\phi(u_1u_2) = \phi(u_1u) \neq \phi(u_2u)$. Similarly, we may assume $\phi(u_3u_4) = \phi(u_3u) \neq \phi(u_4u)$. Let us denote $a = \phi(uv)$.
If $\phi(u_1u_2) = \phi(u_3u_4)$, then it must hold $a = \phi(u_1u_2) = \phi(u_3u_4)$, otherwise ϕ would not be locally irregular. Thus, the a-degree of u by ϕ equals 3.
If $\phi(u_1u_2) \neq \phi(u_3u_4)$, then it must hold $\phi(u_2u) = \phi(u_4u)$, otherwise ϕ would use at least 4 colors. Notice that $\phi(u_1u_2) = a$ or $\phi(u_3u_4) = a$ would imply ϕ is not a liec, thus it must be $a = \phi(u_2u) = \phi(u_4u)$, so again we obtain the a-degree of u by ϕ equals 3.
The analogous analisys of the edges in the two triangles of B containing v yields that a-degree of v also equals 3. Since uv is an edge in B colored by a and a-degree of both u and v equals 3, this is a contradiction with ϕ being locally irregular. $\square$

The above result implies that B contradicts the Local Irregularity Conjecture. This gives rise to the following questions: are 4 colors now enough for a liec of all colorable graphs, and are there other graphs beside B which require 4 colors. Since the class of cactus graphs yielded the first known counterexample, we attempted to answer these two questions for that class.
To establish that all colorable cacti admit a 4-liec, we need the following definitions. A *grape* G is any cactus graph with at least one cycle in which all cycles share a vertex u, and the vertex u is called the *root* of G. A *berry* G_i of a grape G is any subgraph of G induced by $V(G'_i) \cup \{u\}$, where u is the root of G and G'_i a connected component of $G - u$. Notice that a berry G_i can be either a unicyclic graph in which u is of degree 2 or a tree in which u is a leaf, so such berries will be called *unicyclic berries* and *acyclic berries*, respectively. A unicyclic berry G_i is said to be *triangular* if its cycle is the triangle.
An *end-grape* G_u of a cactus graph G is any subgraph of G such that:

- G_u is a grape rooted at u where u is the only vertex of G_u incident to edges from $G - E(G_u)$, and
- u is incident to either one edge from $G - E(G_u)$ or two such edges which then must belong to a same cycle of $G - E(G_u)$, and such edges are called the *exit edges* of G_u.

This notion is illustrated by Figure 3. Also, for an end-grape G_u rooted at u, the graph $G_0 = G - E(G_u)$ will be called the *root component* of G_u. In [9], we established the following theorem.

Theorem 5. *Every colorable cactus graph G satisfies $\chi'_{\rm irr}(G) \leq 4$.*

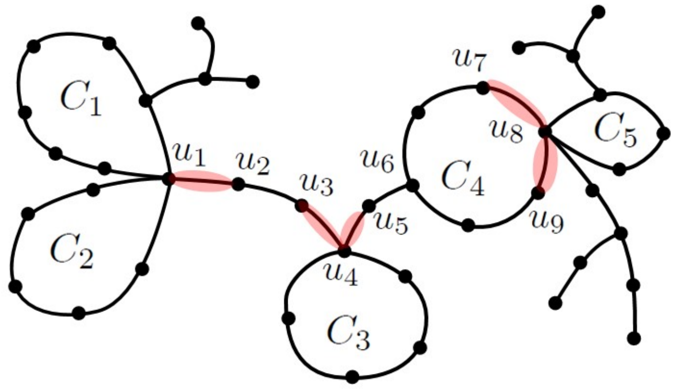

Figure 3: A cactus graph G with five cycles which contains two end-grapes, G_{u_1} and G_{u_8}. The end-grape G_{u_1} has two unicyclic berries and one exit edge u_1u_2. The end-grape G_{u_8} consists of one unicyclic and one acyclic berry and it has two exit edges u_8u_7 and u_8u_9. Notice that the cycle C_3 is not an end-grape of G since it has two exit edges u_4u_3 and u_4u_5 and they do not belong to the same cycle.

The result of Theorem 5 is obtained by induction on the number of cycles in G, using Theorem 3 as the basis of the induction. This approach enables one to assume that the root component G_0 of an end-grape G_u is already colored by a locally irregular edge coloring and then one has to extend such a coloring to the end-grape G_u by using at most 4 colors in all. This is done by removing precisely one edge incident to the root vertex u from every unicyclic berry of G_u, denote such set by E_u. Notice that $G_u - E_u$ becomes a tree and thus admits a 3-liec in colors a, b and c, the edges of E_u can then be colored by the fourth color d. The problem with local irregularity can thus arise only on edges incident to u in G. This is avoided by carefully chosing the coloring of the tree $T = G_u - E_u$ using Observation 2.
This result can be further extended to a claim that every colorable cactus graph distinct from B requires at most three colors for the locally irregular edge coloring. Our argument of this claim is lengthy but uses the same approach as Theorem 5. The main difference is that in proving Theorem 5, no special attention needs to be given to the a- and b-degrees of the neighbors of u in T since there is fourth color d to use it for at least one of the two edges incident to u in G_0. When the fourth color must not be used, then a great care has to be taken of these a- and b-degrees in T because the same colors must be used for both edges incident to u in G_0. So, one has to avoid colors a and b for edges incident to u in G_u to spare them for G_0. That is not always possible, so special berries and alternative colorings for them need to be introduced, which is done in [10].

Primary coloring. A berry coloring by which all edges are incident to u have a same color, say c, is a desirable coloring since it is very convenient when gathering berries in a grape. For that purpose, it is also desirable that the c-degree of the neighbors of u is ≤ 2.

Definition 1. *Let G be a graph rooted at a vertex u with $d_G(u) \in \{1, 2\}$. Let v be* *a neighbor of u and w the other neighbor of u if $d_G(u) = 2$. A* standard primary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G with the following properties:*

- *every edge non-incident to u is locally irregular by $\phi_{a,b,c}$;*
- *u is monochromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(u) = \{c\}$;*
- *$d^c(v) \leq 2$ and $d^c(w) \leq 2$.*

It would be convenient to use a standard primary coloring for all berries, but berries $B_1, \ldots, B_7$ from Figure 4 (dashed edges are not included in a berry) do not admit such a coloring, so these berries are called *alternative berries*. For such berries, we introduce an alternative primary coloring as follows.

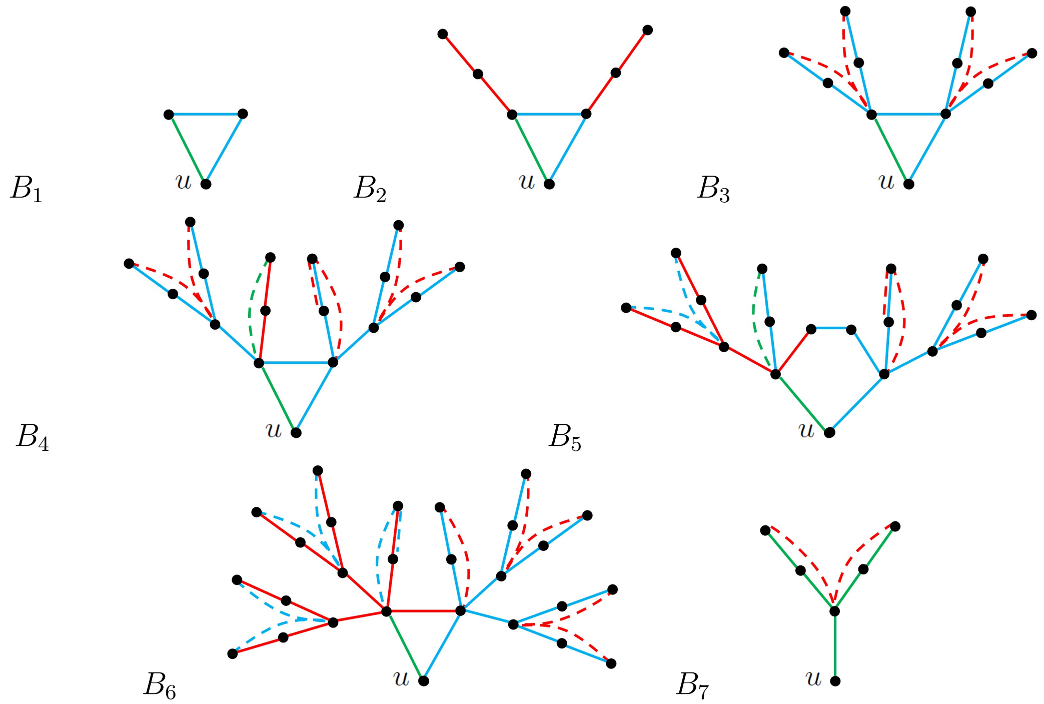

Figure 4: Alternative berries $B_1, \ldots, B_7$ rooted at u. Dashed edges are not included in the berry.

Definition 2. *Let G be a graph rooted at a vertex u with $d_G(u) \in \{1, 2\}$. Let v be a neighbor of u and w the other neighbor of u if $d_G(u) = 2$.*

- *If $d_G(u) = 1$, an* alternative primary coloring *of G is any 3-liec $\phi_{a,b,c}$ of G such that v is monochromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(v) = \{c\}$.*
- *If $d_G(u) = 2$, an* alternative primary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G with the following properties:*
  - *every edge $e \neq uv$ is locally irregular by $\phi_{a,b,c}$;*

- *u is bichromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(uv) = c$ and $\phi_{a,b,c}(uw) = a$;*
- *$d^c(v) \leq 2$ and $d^a(w) \in \{2, 4\}$;*
- *if $d^c(v) = 2$ then $d^a(w) = 4$.*

Alternative primary coloring of alternative berries is illustrated by Figure 4, all other berries admit a standard primary coloring [10]. In the sequel we will say shortly 'primary coloring', assuming standard primary coloring for standard berries and alternative primary coloring for alternative berries.

Secondary and tertiary coloring. So far we have one kind of coloring for every berry - a primary coloring. When we gather berries in an end-grape, if the problem with local irregularity arises for edges incident to the root u, it is convenient to have another kind of coloring for every berry, so that the change of coloring for some berries changes the color degrees of u and its neighbors. For standard berries, another kind of coloring is a so called secondary coloring, and for alternative berries it is tertiary coloring.

Definition 3. *Let $G \neq B_7$ be a graph rooted at a vertex u with $d_G(u) \in \{1, 2\}$. Let v be a neighbor of u and w the other neighbor of u if $d_G(u) = 2$.*

- *If $d_G(u) = 1$, a* secondary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G such that all edges not incident to u are locally irregular and $d^c(v) \neq 2$ where c is the color of uv by $\phi_{a,b,c}$.*
- *If $d_G(u) = 2$, a* secondary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G with the following properties:*
  - *every edge $e \neq uv$ is locally irregular by $\phi_{a,b,c}$;*
  - *u is bichromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(uv) = c$ and $\phi_{a,b,c}(uw) = a$;*
  - *$d^c(v) \leq 2$;*
  - *if $d^c(v) = 2$ then $d^a(w) \geq 3$.*

We exclude B_7 from the above definition, since we always want an alternative primary coloring of it.

Definition 4. *Let G be a graph rooted at a vertex u with $d_G(u) = 2$. Let v and w be the neighbors of u. A* tertiary coloring *of G is any 3-edge coloring $\phi_{a,b,c}$ of G with the following properties:*

- *every edge $e \neq uv$ is locally irregular by $\phi_{a,b,c}$;*
- *u is monochromatic by $\phi_{a,b,c}$, say $\phi_{a,b,c}(u) = \{c\}$;*
- *$d^c(v) \leq 2$ and $d^c(w) \in \{3, 4\}$.*

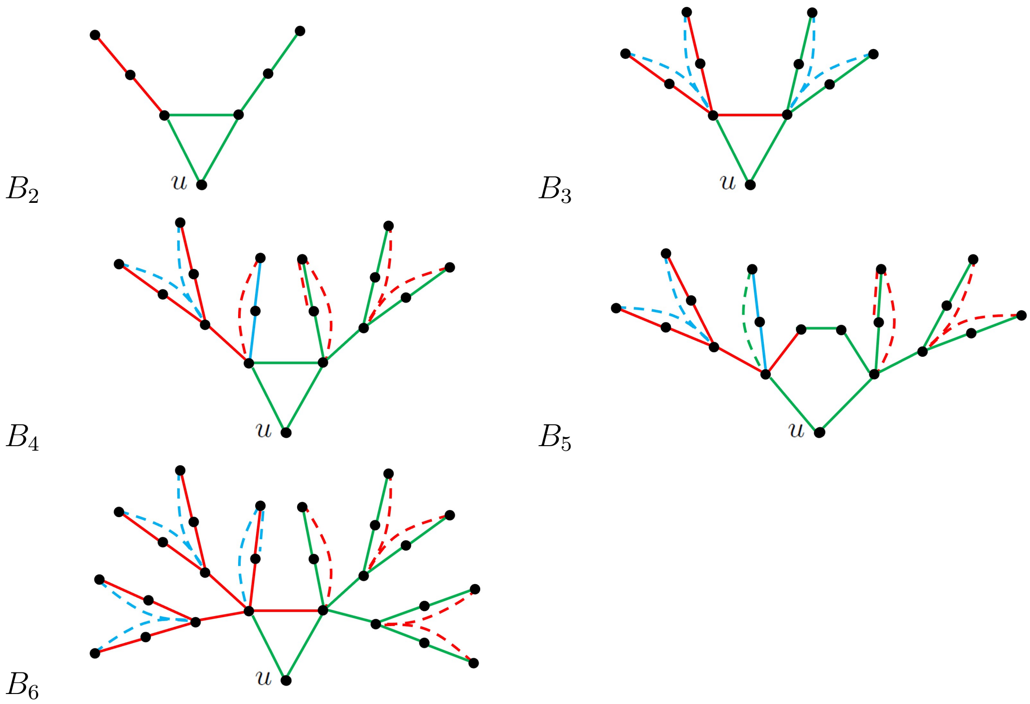

Figure 5: A tertiary coloring of berries $B_2, \ldots, B_6$ and its closures.

A tertiary coloring of $B_2, \ldots, B_6$ is shown in Figure 5 (without dashed lines). Notice that B_1 is the only berry which admits only one coloring.

Even with all these kinds of colorings, it turns out that the coloring of the root component G_0 cannot be extended to all possible end-grapes. To be more precise, there are six end-grapes to which it is not always possible to extend a coloring of G_0, we denote them by $A_1, \ldots, A_6$ and they are shown in Figure 6. For all other end-grapes, the following proposition [10] holds (here G_0' denotes a graph obtained from G_0 by adding a leaf neighbor to u).

Proposition 2. *Let G be a cactus graph with $\mathfrak{c} \geq 2$ cycles which is not a grape and which does not contain end-grapes $A_1, \ldots, A_6$. For an end-grape G_u of G with a root component G_0, there exists a colorable graph $\tilde{G}_0 \in \{G_0, G_0'\}$ such that every k-liec of $\tilde{G}_0$, for $k \geq 3$, can be extended to k-liec of G in which every berry of G_u is colored by a primary, secondary or tertiary coloring.*

If a cactus graph G contains only end-grapes $A_1, \ldots, A_6$, then by deleting an edge in every triangle of such end-grapes one can obtain a tree, a unicyclic graph or a cactus graph G^{op} which contains an end-grape distinct from $A_1, \ldots, A_6$. A tree and a unicyclic graph admit a 3-liec, and for G^{op} the induction hypothesis and Proposition 2 assure the existance of a 3-liec in which every berry is colored by a primary, secondary od tertiary coloring. Thus, it only remains to establish that a 3-liec of G^{op} can be extended to a 3-liec of G, i.e. to the removed edges of triangles in end-grapes of G. Since every berry of G^{op} is colored by a primary, secondary

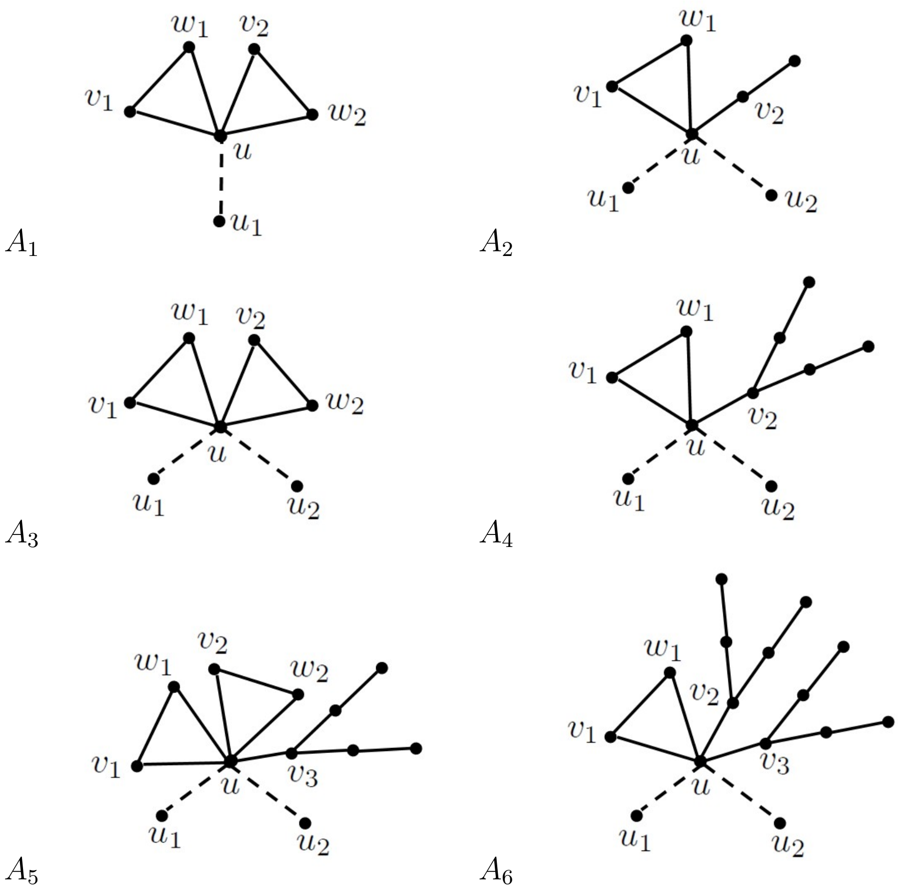

Figure 6: The singular end-grapes $A_1, \ldots, A_6$.

or tertiary coloring, the question reduces to whether such berry colorings can be extended to the removed edges of G. The extension of alternative primary and tertiary coloring is shown in Figures 4 and 5, and the existence of the extension for a standard primary and a secondary coloring of standard berries is established in [10]. This finally yields the following result [10].

Theorem 6. *Let $G \neq B$ be a colorable cactus graph. Then $\chi'_{\mathrm{irr}}(G) \leq 3$.*

Our investigation of the Local Irregularity Conjecture makes us believe that B is the only graph with $\chi'_{\mathrm{irr}}(G) \geq 4$, so the following conjecture is proposed in [10].

Conjecture 2. *The bow-tie graph B is the only colorable connected graph with $\chi'_{\mathrm{irr}}(B) > 3$.*

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