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Planets are (very likely) in orbits of stars

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Abstract

The probability that a randomly and uniformly chosen point from the circumball of a tetrahedron lies outside of the inscribed ball of the tetrahedron can be bounded very sharply from below in terms of the edge lengths of the tetrahedron. One can imagine four stars in the Universe (vertices) with known mutual distances and a small (exo-) planet orbiting between them within the circumsphere. The least probability that the planet is outside of the insphere is given in terms of the distances of the stars. The least probability occurs for the regular tetrahedron and it is $0.962962\ldots$. Geometrically, this is a tricky corollary of (refinements of) the famous Euler inequality: circumradius is at least three times bigger than the inradius of a tetrahedron with equality for a regular tetrahedron. The Euler inequality can be extended to Euclidean simplices in all dimensions and to non-Euclidean planes. The most relevant cases of 3D and 4D being in accordance with the relativity theory are considered.

Keywords: Euler's inequality, refined Euler's inequality in 3D and 4D, exoplanet.

AMS subject classification (2020): 51M04, 51M16, 85A15.

1 Introduction

In this paper we shall geometrically explain why planets detected to be in vicinity of four stars must, in fact, with high probability, orbit around one of the four stars. The lower bound of the probability is given in terms of the distances of the stars. The essence of the argument is Euler's inequality $R \geq nr$ from 1765 between the circumradius R and the inradius r of an n-dimensional simplex and particularly its refinements as given in [1], [2] and [3].
A popular introductionary text on some mathematical and computational aspects in astronomy or astrophysics is [4] (in Croatian) and a textbook on the topic is e.g. [5]. A popular history book on math is [6]. Many Euler's contributions can be found there.

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We shall first consider the case of three stars and a planet moving in their plane ($n = 2$). Next is the main 3D-case ($n = 3$), because four (general) points form a space.
Finally, we shall consider 4D-case (i.e. when $n = 4$) which is important because the Universe is by Einstein's relativity theory four dimensional (hyperbolic) space-time object.

2 Triangle of stars

To begin with, we start with an easier question. Let T be a triangle of stars and a, b and c their mutual distances. Let X be a randomly and uniformly chosen planet (point) within the circumcircle of T. What is the minimal probability p that X lies outside of of the incircle of T? That is, what is the minimal probability that X is rather close to one of the stars? The probability that a randomly and uniformly chosen point within the circumcircle of T (of radius R) which is within the incircle of T (of radius r) is equal to the quotient of their areas $r^2\pi/R^2\pi = (r/R)^2$. Hence, $p = 1-(r/R)^2$. In [1] (see also [2], [3]) we proved a refined Euler's inequality $R \geq 2r$ in the form

$$\frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc} \geq 2. \tag{1}$$

with equalities if and only if the triangle T is equilateral. Recall Heron's formula for the area S of T in terms of side lengths a, b, c of T:

$$16S^2 = (2s)d_3(a, b, c),$$

where $2s = a + b + c$ is the perimeter of T and $d_3(a, b, c) := (a + b - c)(a - b + c)(-a + b + c)$. Then the probability q that X is within the incircle is equal to:

$$q = \left(\frac{r}{R}\right)^2 = \left(\frac{S}{s} : \frac{abc}{4S}\right)^2 = \left(\frac{4S^2}{sabc}\right)^2 = \left(\frac{16S^2}{4sabc}\right)^2 = \left(\frac{2sd_3(a, b, c)}{4sabc}\right)^2 = \left(\frac{d_3(a, b, c)}{2abc}\right)^2 \leq \frac{1}{4}.$$

Hence,

$$p = 1 - q = 1 - \left(\frac{d_3(a, b, c)}{2abc}\right)^2 \geq 0.75. \tag{2}$$

The last inequality follows from Euler's $R \geq 2r$ and above inequality.

3 Tetrahedron of stars

Now, the main topic in this paper is to show that with a high probability an (exo-) planet from the vicinity of four stars must, in fact, orbit around one of the stars. We quote Theorem 4.1. from [3]: the probability that a randomly and uniformly chosen point within the circumsphere of the tetrahedron T is within the insphere of T is at most equal to

$$\sqrt{\frac{d_3(aa', bb', cc')}{[3(aa' + bb' + cc')]^3}} \leq \frac{1}{27}. \tag{3}$$

Here a, b and c are side lengths forming the base of T, a' is opposite to a etc. Therefore, the probability p that a randomly and uniformly chosen point (planet) from the circumball of the tetrahedron T of four stars (vertices of T) is outside of the inscribed ball of T is at least equal to

$$p \geq 1 - \sqrt{\frac{d_3}{(3e_1)^3}}, \tag{4}$$

where $d_3 := d_3(aa', bb', cc')$ and $e_1 := aa' + bb' + cc'$. (Recall that aa', bb', cc' form a triangle, called Crelle's triangle of T.) By Euler's inequality $R \geq 3r$ (and [3]), it follows from (4) that the probability

$$p \geq 1 - 1/27 = 0.962962\ldots \tag{4'}$$

And this is rather close to 1. This can be considered as a geometric proof that a planet very likely must orbiting around a star and taken by gravity rules must stay in the orbit for good (or at least for billions of years). The equality in (4) in fact occurs if and only if $aa' = bb' = cc'$, i.e. when Crelle's triangle of T is equilateral. In particular, we have equality in (4) and (4') when T is a regular tetrahedron.
This geometric-probabilistic proof shows that with the chance of at least 96.2962% we can expect that a planet moving in vicinity of four stars (with known mutual distances of stars) is rather close to one of the stars instead of being somewhere in deep space formed by four stars. And then the gravity rule force the planet to orbit (eliptically) around one of the star. (A little quiz for the reader: four girls are bathing in a lake or sea; their mutual distances are 20 meters; three girls have on them red bikini; what has the fourth girl on? If you don't know, see [4].)
As an astronomy example consider four stars from the Ursa Major (the Great Bear) constellation: Dubhe, Merak, Phecda and Megrez. They are roughly 120 ly far from the Earth. With data available from the Internet, a little computation shows that the chance that a planet close to the constellation (within the circumball) but out of the inball of the four stars, that is, rather close to one of the stars, is roughly a bit more than 97%. In other words, it is quite likely that a planet is not wandering in deep space between four stars.
A geometric example is the tetrahedron whose vertices are the origin and three unit points on the coordinate axes. Its Crelle's triangle is regular (with side lengths $\sqrt{2}$), hence the probability in question is minimal and is (at least) 96.2962%. For the tetrahedron $ABCD$ with $AB = BC = CD = 1$ and $BC \perp AB$, $CD \perp AB$, BC, i.e. an "ortoscheme", the considered probability is at least 97%. In any concrete example, we can, of course, compute R and r exactly and hence the probability, but that is not the issue here.

4 4-simplex

Four dimensional geometry in astrophysics is also of interest since our space-time Universe is four-dimensional. It seems, the hyperbolic geometry is prevailing, according to relativity theory, but 3D and even more 4D hyperbolic geometry is still to some degree "tabula rasa". One possible inequality for edge lengths of a hyperbolic tetrahedron can be obtained by applying Theorem 2.3. from [3] to corresponding Crelle's triangle (if it exists). In any way, the best known approximation to our geometric-probabilistic approach is Euclidean 3D- and 4D-geometry.
Recall, our refinement of Euler's $R \geq 4r$ for a 4-dimensional simplex $T = A_0A_1A_2A_3A_4$ in terms of symmetric functions of edge lengths $a_{ij} = A_iA_j$, $i < j$, of T is (5.8) from [3]

$$\left(\frac{R}{r}\right)^2 \geq 8\frac{\sum a_{ij}^2}{5\prod(i,j,k,l)^{1/15}} \geq 4^2, \tag{5}$$

where $\sum a_{ij}^2$ is the sum of squared lengths of all ten edges of T and the symbol (i,j,k,l) is defined by $(i,j,k,l) := d_3(a_{ij}a_{kl}, a_{ik}a_{jl}, a_{il}a_{jk})$ for all $0 \leq i < j < k < l \leq 4$. The equality in (5) holds for a regular 4-simplex. From (5) we have a sharp lower bound for the probability p that a randomly and uniformly chosen point from the circumball of the 4-simplex T is out of the inball of T. It is given by

$$p \geq 1 - \left(\frac{r}{R}\right)^4 \geq \frac{25\prod(i,j,k,l)^{2/15}}{64(\sum a_{ij}^2)^2} \geq 1 - 4^{-4}. \tag{6}$$

In the case of a regular 4D-simplex (of stars) with all edge lengths $a_{ij} = 1$, we have $(i,j,k,l) = 1$ for all five choices of i,j,k,l, and the probability is $p = 1 - 2^{-8} = 99.609\ldots\%$.

5 Conclusion

We have proved here that planets quite likely orbit around stars. This fact was observed back in ancient times, as well as the fact that satellites must orbit around planets; the first examples, naturally, being Moon around Earth and Earth around Sun. It was firmly established only by Kepler and later explained by Newton. In fact, the Croatian mathematician Ruđer Bošković was the first to give a procedure to compute a planet's orbit from three observations of its position. (Bošković did not get the Grand Prix of the Academy in 1752 for his studies on Saturn and Jupiter, but the prize was given to Euler.)
In this paper we provided a pure geometric proof of these facts.
In searching for an exo-planet, where humans have to move once in the future, one can imagine the following experiment. Suppose we get a spectral signal that a certain planet has water, but we don't know the exact position of this planet. Then our method predicts its position with high probability within four nearest stars. This prediction then focus the search for the planet to much less space.
By the procedure (recursive algorithm) proposed in [3] for refined Euler's inequality, we can proceed further to higher dimensions and prove that the limit when dimension

n tends to infinity for the probability that a randomly and uniformly chosen point from the circumball that is out of the inball of simplex is equal to 1.
In a similar manner to astrophysics, we can apply our method to the micro world as well, by considering for instance electrons in atoms and other subatomic particles and predict with some high probability the behavior and position of particles, but this is already in the domain of quantum physics. Such "predictology" is also very much appreciated in biochemistry, molecular biology, information theory and other modern sciences.

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