Osculating Circles of Conics in Cayley-Klein Planes

ABSTRACT

In the Euclidean plane there are several well-known methods of constructing an osculating (Euclidean) circle to a conic. We show that at least one of these methods can be “translated” into a construction scheme of finding the osculating non-Euclidean circle to a given conic in a hyperbolic or elliptic plane. As an example we will deal with the Klein-model of these non-Euclidean planes, as the projective geometric point of view is common to the Euclidean as well as to the non-Euclidean cases.

Key words: Cayley-Klein plane, elation, pencil of conics, osculating circle, curvature centre

MSC 2010: 51N30, 51M15, 53A35

1 Preliminary Remark

Although the problem of constructing an osculating circle at a point of a conic seems to be anachronistic in times of numerical approximation tools, knowledge about exact constructive methods is not at all obsolete, particularly since these methods are uniformly applicable. Beyond that, with the following projective geometric constructions of osculating circles, we place particular emphasis on synthetic argumentation, which is typically for geometry. Unfortunately Projective Geometry and Non-Euclidean Geometry in the sense of F. Klein does not have much space in nowadays Mathematic education such that valuable Geometry culture is in danger of vanishing. Our article might perhaps help to counteract these facts. The paper is also to be posed into the series of articles of classical Projective and Non-Euclidean Geometry initiated by the second author, see [5] - [8].

It is hard to say, how “well-known” the presented considerations are; they could be for example exercise material to lectures on classical Projective Geometry and not considered to be valuable enough to be published. To our knowledge lectures with related content still exist in Vienna (H. Stachel, H. Havlicek) and Graz (J. Wallner, O. Röschel), where they still belong to the syllabus in teachers education in Descriptive Geometry.

2 Euclidean Osculating Circles of Conics

We start with “permissible standard givens” of a conic in the Euclidean plane, i.e. from pair of conjugate diameters \( AC, BD \) of an ellipse, from a pair of line elements \((A,t_A),(B,t_B)\) of a parabola and from a pair of asymptotes \((r,s)\) and a point \( A \) of a hyperbola. The problem is to find the osculating circle at the given point \( A \).

In geometry courses for engineers one usually presents the construction recipe for the hyperosculating circles at the vertex of the conic; in lectures on differential geometry for Mathematicians this recipe is also presented and the analytic equality of 4th order is proved by calculation in each of the three cases. But for all three cases there is a uniform projective geometric idea for the solution of this (more general) problem. This idea uses properties of osculating resp. hyperosculating pencils of conics. This unified explanation of elementary construction of hyperosculating
circles at vertices of a conic might be not new, but it is, in our opinion, not at all so well-known as it is worthy to note.

In addition, the construction principle deduced from it can be used for all Cayley-Klein geometries, as it is shown in the following chapter only for the hyperbolic and the quasi-hyperbolic geometry (which is the dual geometry to the pseudo-Euclidean geometry) as an example.

Specifically, the construction follows two steps:

**Step 1:** Transform the given conic \( c \) by the standard shear transformation \( \sigma_1 : c \rightarrow c' \) into another conic \( c' \), which osculates \( c \) at the point \( A \), and has \( A \) as a vertex. The axis of the shear therefore is the tangent \( t_A \) in \( A \). As all elations with the center \( N \) on \( t_A \), this shear \( \sigma_1 \) maps the given conic \( c \) into \( c' \) which osculates the conic \( c \) at \( A \). The osculating circle of one is also osculating circle for the other one.

**Step 2:** Construct the hyperosculating circle \( k_A \) of the conic \( c' \) at \( A' \). For this one can use an additional elation \( \sigma_2 \) which also has \( t_A \) as its axis, but the center \( A \) and it should map \( c' \) into a hyperosculating conic \( c'' \). By demanding \( c' \rightarrow k_A \) the transformation \( \sigma_2 \) is uniquely defined; \( \sigma_2 \) transforms the point \( B' \) of the osculating conic \( c' \) (in the case of an ellipse this is a “neighbouring” vertex, in case of a hyperbola one of the asymptote’s ideal point “at infinity”, in the case of a parabola it is the additionally given point) into the point \( B'' \) of \( k_A \). The normal from the (fixed) point \( T = T' = T'' \in t_A \) (\( T = t_A \cap t_B \)) to the chord \( AB'' = AB' \) of circle \( k_A \) therewith passes through the center \( M_A \) of \( k_A \).

The following figures (Figures 1-3) show the construction costs, which in each of the three cases needs only a few lines.

**Elliptic case (Figure 1)**
The point \( A \) is transformed into the vertex of an ellipse \( c' \) by the shear \( \sigma_1 : c \rightarrow c' \) (tangent \( t_A \) at \( A \) is the axis of \( \sigma_1 \)). Furthermore, \( \sigma_2 : c' \rightarrow k_A \) with the same axis, but with center \( A \), transforms a “neighbouring” vertex \( B' \) of \( c' \) into the point \( B'' \) of \( k_A \). The line through the fixed point \( T = T' = T'' \) on \( t_A \) perpendicular to the chord \( AB'' = AB' \) of the osculating circle \( k_A \) intersects the normal \( n \) of the conic \( c \) at the point \( A \) in the center \( M_A \) of \( k_A \).

**Parabolic case (Figure 2)**
The shear \( \sigma_1 : c \rightarrow c' \) (axis is the tangent \( t_A \) at \( A \)) transforms the point \( A \) into the vertex of a parabola \( c' \), which osculates \( c \). The midpoint \( H \) of the chord \( AB \), together with the point \( T := t_A \cap t_B \), defines the diameter direction of the conic \( c \), and therefore is mapped into the point \( H' \) on the line normal to \( t_A \) at \( T \). For \( \sigma_2 : c' \rightarrow k_A \) (the center is \( A \)) the line \( AH' \) is a fixed line; it contains the point \( B'' = \sigma_2(B') \) of the hyperosculating conic \( k_A \). The line through the fixed point \( T = T' = T'' \) perpendicular to the chord \( AB'' = AH' \) of \( k_A \) passes through the sought-after curvature center \( M_A \).

**Hyperbolic case (Figure 3)**
At first we construct the tangent \( t_A \) at the point \( A \), \( (A \) is the midpoint of the tangent segment between the asymptotes). By a suitable shear \( \sigma_1 \) one can transform the point \( A \) into the vertex of the hyperbola \( c' \) whose asymptotes are \( r' \) and \( s' \) and the center is \( M' \). The elation \( \sigma_2 : c' \rightarrow k_A \) with center \( A \) transforms the ideal point \( R' = B' \) of the asymptote \( r' \) of the conic \( c' \) into the point \( B'' \) of the hyperosculating conic \( k_A \). Therefore, the line through the fixed point \( T \) on the tangent \( t_A \) perpendicular to \( AB'' = AB' \) passes through the common curvature center \( M_A \) of the conics \( k_A, c' \) and \( c \). As \( AB' \) is parallel to \( r' \), one only needs to draw the perpendicular line to the asymptote \( r' = M'T \) at the point \( T \).
Remark 1 Of course, there are other nice constructions of the Euclidean curvature circle at the point of a conic, too. We want to mention the one found in [1], which is based on the differential geometric investigations and goes as follows:

"Choose two points $P, Q$ on tangent $t_A$ symmetric to $A$, draw the tangents from $P$ and $Q$ to the given conic $c$, receive the contact points $P'$ and $Q'$. The normals $p$ and $q$ to the cords $AP'$ and $AQ'$ intersect the normal $n$ in $A$ in the points $M_P$ and $M_Q$. Then $M_A$ is the midpoint of the segment $(M_P, M_Q)$. Also this construction could be transformed to CK-planes."

3 Curvature Centres of Conics which are Given by General Data in CK-planes

In the preceding chapter we started with affine-special given data of a conic. Now we choose a projective-geometric system of data defining a conic:

Let a conic $c$ be given by two line elements $(A, a), (B, b)$ and a point $C$ in a (real) projective plane $\pi$ equipped with an "absolute (regular or degenerated) polarity" $\pi^\perp$. (The plane $\pi$ together with $\pi^\perp$ is therewith a Cayley-Klein plane, in short a CK-plane.)

These specifications of the conic $c$ can easily be derived from all other given data, which define conic uniquely, by applying the theorems of Pappus-Pascal and Brianchon, see [2]. These givens are also appropriate for the analytic treatment of the conic, as they can be interpreted as a projective coordinate system.

Via $\pi^\perp$ the place of action is a projective plane with an orthogonality structure $\perp$ and a concept of circles in the sense of Cayley-Klein; it is therefore a "projectively extended" non-Euclidean plane", a CK-plane, see e.g. [3].

Among these CK-planes we want to exclude the so-called isotropic planes (see e.g. [4]) from further considerations, because their orthogonality structure $\perp$ is too degenerated. These CK-planes are treated separately in [7]. It turns out that for degenerated absolute polarity $\pi^\perp$ one could "swap" the two steps described in chapter 2: At first one constructs the hyperosculating parabola $c'$ at the point $A$ of the given conic $c$ in the isotropic plane $\pi$ with absolute point $I$ at the ideal line $a$. This can be done with an elation with center $A$ and axis $a$. Next, the parabola $c'$ is transformed into the isotropic circle $k$ with ideal point $I$ using the shear with axis $a$.

In the last chapter, it will be shown that in some cases of CK-planes with degenerate absolute polarity one can easily find simplier constructions using one single elation alone.

The problem is to construct the curvature center $M_A$ at the point $A$ of the given conic $c$. (For the sake of simplicity we visualise again the Euclidean case in Figure 4, i.e. "⊥" and "circles" allow an elementary geometric interpretation.)

Step 1: determine the normal $n$ of the conic $c$ at the point $A$ with respect to $\perp$; i.e. one needs to construct the "absolute conjugate line" $n$ to the given tangent $a$ of the conic $c$ at $A$ with respect to $\pi^\perp$. The absolute pole $N$ of the normal $n$ is a point of $a$ and it has to be used as the center of an elation $\sigma_1 : c \rightarrow c'$ with axis $a$. One still needs a related pair of points to define elation $\sigma_1$ uniquely.

Step 2: the construction of a conic point $D$ on the (fixed) collineation ray on $BN$ using the theorem of Pappus-Pascal. (Naturally, it would be the same, if we constructed the point $E \in c$ on the collineation ray $CN$.) In Figure 4 the necessary lines are shown:

$D := BN \cap AP$, whereby, $P := BC \cap ((a \cap b) \vee (BN \cap AC))$.

Step 3: determination of the $4^{th}$ harmonic point $H$ to $N$ with respect to the pair $(B, D)$ on line $BN$. This point $H$ shall be related to $H' := n \cap BN$ in $\sigma_1$ and now $\sigma_1 : c \rightarrow c'$ is uniquely determined.

Step 4: corresponds to step 2 in chapter 2 to get $\sigma_2 : c' \rightarrow k_A$. Explicitly, one only needs the center $A$ of elation $\sigma_2$ and the fact that the point $B'' := \sigma_2(B')$ on $k_A$ belongs to the chord $AB'' = AB'$ of $k_A$, which is fixed under $\sigma_2$ where $B' := \sigma_1(B)$. Therewith the wanted curvature center $M_A$ is the intersection point of the normal $n$ with the line $s$, which is the absolute-conjugate line to $AB'$ through $a \cap b$.

Figure 4: Construction of the curvature center $M_A$ in a point $A$ of a conic $c$, which is given by two line elements $(A, a), (B, b)$ and an additional point $C$ in the (projective extended) elementary geometric plane. (Explanation of the construction see above.)
4  \( h \)-Curvature Circles of a Conic in the Projective Model of a Hyperbolic Plane

As an example for the construction of the osculating circle in a non-Euclidean plane with regular absolute polarity \( \pi^\perp \) the complete solution will be given in a hyperbolic plane, see Figure 5.

According to the projective geometric background of the idea of the construction, it is somehow natural to use the classical projective model of such a \( h \)-plane, (c.f. also [2]). This means that the place of action is essentially the inner domain of a (real) “absolute conic” \( u \), which can be taken as an ordinary circle in elementary geometric sense. Given a conic \( c \), the problem is to construct the \( h \)-curvature center \( M_A \) to the arbitrarily given point \( A \) (with tangent \( a \)).

It is expedient and practical to use a perspective collineation \( \kappa_1 : u \rightarrow c \) to construct the conic \( c \) as collinearly related image to the absolute conic \( u \). (In Figure 5 collineation \( \kappa_1 \) is defined with the center \( S \) and the axis \( s \), and the related pair of points \( (A_1, A_2) \).) Note, that if \( s \) is absolute polar of \( S \), the obtained conic \( c \) would be a circle.

Step 1: The osculating \( h \)-circle \( k \) of \( c \) at \( A \) has its center \( M \) on the \( h \)-normal \( n \) to \( a \) through \( A \). So as a first step one needs to construct this \( n \).

Step 2: Make \( A \) to a vertex of the conic \( c' \), which osculates \( c \) at \( A \). For this we use a “projective shear”, i.e. an elation \( \sigma_1 \) with axis \( a \) and centre \( N \in a \), which is the absolute pole of \( n \).

Step 3: Construct the hyperosculating circle of \( c' \) according to the description to Figure 4. In Figure 5 we used the points \( Q \) and \( Q' = : \sigma_1(Q) \) to get \( c' \) from \( c \) and the special point \( B' \) and its tangent \( b \). Finally we connect \( A \) with \( B' \) and erect the \( h \)-normal line to \( AB' \) through \( T := a \cap b \), it intersects \( n \) in the \( h \)-curvature center \( M \).

Step 4: If we do not use a graphics software like “Cinderella”, where we can directly draw \( h \)-circles in a \( h \)-plane, we still have to construct the \( h \)-osculating circle \( k \). This again can be done using a perspective collineation \( \kappa_2 : u \rightarrow k \); it has the center \( M \) and the axis \( m := \pi^\perp(M) \) and the related pair of points \( (A \in c, A_2 \in u) \). Because of

\[
\kappa_1 \circ \sigma_1 \circ \sigma_2 \circ \kappa_2 : u \rightarrow c \rightarrow c' \rightarrow k \rightarrow u
\]

The product of these perspective collineations must act as a projectivity \( \beta \) on \( u \). So the mapping \( \beta : c \rightarrow u \rightarrow k \) is determined by three pairs of points on \( u \), among them \( A_1, A_2 \) and \( Q_1, Q_2 \).

Figure 5: Osculating \( h \)-circle at the point \( A \) of the conic \( c \). (Construction in the classical projective model of a \( h \)-plane.)
5 Curvature Circles of a Conic in the CK-Planes with Singular Absolute Polarity

To start with, we give an overview over those CK-planes, c.f. [3]:

<table>
<thead>
<tr>
<th>$\pi^\pm$</th>
<th>acting in</th>
<th>“Absolute figure”</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>elliptic involutoric projectivity</td>
<td>(ideal) line $u$</td>
<td>pair of imaginary</td>
<td>Euclidean plane, $e - \text{plane}$</td>
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<tr>
<td></td>
<td></td>
<td>points $I,J \in u$</td>
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</tr>
<tr>
<td>hyperb. involutoric projectivity</td>
<td>(ideal) line $u$</td>
<td>pair of real</td>
<td>pseudo-Eucl. plane, $pe - \text{plane}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>points $I,J \in u$</td>
<td></td>
</tr>
<tr>
<td>elliptic involutoric projectivity</td>
<td>a pencil of lines $u$</td>
<td>pair of imaginary</td>
<td>dual Eucl. plane, quasi-elliptic plane</td>
</tr>
<tr>
<td></td>
<td>through (ideal) point $U$</td>
<td>lines $i,j$ through $U$</td>
<td>$qe - \text{plane}$</td>
</tr>
<tr>
<td>hyperb. involutoric projectivity</td>
<td>a pencil of lines $u$</td>
<td>pair of real</td>
<td>dual quasi-Pseudo-Eucl. plane, quasi-hyperbolic plane, $qh - \text{plane}$</td>
</tr>
<tr>
<td></td>
<td>through (ideal) point $U$</td>
<td>lines $i,j$ through $U$</td>
<td></td>
</tr>
<tr>
<td>degenerate inv. projectivity</td>
<td>(ideal) line $u$</td>
<td>point $U$ and line $u$</td>
<td>isotropic plane, Galilean plane, $i - \text{plane}$</td>
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<td></td>
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<td>(self-dual figure)</td>
<td></td>
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We treated the Euclidean case in Chapter 2 aiming at a unifying interpretation of the classical and well-known elementary constructions. We will now present constructions of osculating circles by using one single elation alone. Let us begin with

1) Pseudo-Euclidean case

Figure 6 shows the projective model of a pe-plane and the construction of a pe-circle osculating a conic $c$ at the point $A$.

Explanation to Figure 6:
The conic $c$ and its line element $(A,a)$ in the pe-plane with absolute points $I,J \in u$ are given. Now we choose $A$ as the center of an elation $\kappa$ and construct its axis $z$: (Also this type of elations is osculation preserving!) With center $A$ project $I,J$ onto $c$ getting $I', J'$. Intersect $u$ with the line $u' = I'J'$, get a fixed point $F$ and $AF = z$. Now $\kappa$ is well-defined and $\kappa(c) = z$: $k$ is the desired osculating pe-circle.

Remark 2 The same construction principle can be performed also in the Euclidean case. The imaginary rays $AI$ and $AJ$ are defined by the orthogonal-involution in the pencil with vertex $A$ and this orthogonal involution induces in $c$ an elliptic involutoric projectivity $\rho$ with involution center $R$. Now we had to construct the polar line $r$ to $R$ with respect to $c$; $r$ connects the imaginary points $I'$ and $J'$ and therefore is parallel to the elation axis $z$.

2) Quasi-hyperbolic case

Without loss of generality, let the absolute figure of the quasi-hyperbolic plane be a pair of parallel lines $i,j$. The conic $c$ and its line element $(A,a)$ are given. A $qh$-circle $k$ is a conic $k$ touching both absolute lines $i$ and $j$. The construction is now dual to the one of Figure 6, see Figure 7.

Figure 6: Osculating pe-circle at the point $A$ of the conic $c$. (Construction in a projective model of a pe-plane.)

Figure 7: Osculating $qh$-circle at the point $A$ of the conic $c$. (Construction in a $qh$-plane with an absolute figure $\{U;i,j\}$.)
Explanation to Figure 7:
Intersect the absolute lines $i$ and $j$ with the tangent $a$ of
the conic $c$ at $A$. The tangents $i', j'$ from these intersec-
tion points to $c$ intersect at the point $U'$ which corresponds
to the absolute point $U$ in the desired elation $\kappa$ with axis
$a$. Line $UU'$ intersects $a$ at the center $Z$ of $\kappa$ such that
$\kappa$ is well-defined by $\{Z, a, (U, U')\}$ and $k := \kappa(c)$ is the
desired osculating $qh$-circle. Note that $UA$ represents the
$qh$-normal $n$ of the $c$ at $A$.

Remark 3 The same construction principle can be per-
formed also in the quasi-elliptic case. The construction
“dualises” that of the Euclidean case. In the isotropic
case, because of self-duality, one can use both, the prin-
ciple of the “qh-construction” as well as that of the “pe-
construction” to find the needed elation. As the construc-
tion is obvious, it can be left to the reader to practice.

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