3D Graphic Statics via Grassmann Algebra

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Methods of graphic statics are used for solving problems of statics, evaluation of equilibrium and determination of internal forces as well as forces in supports of structural systems by applying only geometrical operations. They are based on construction of two reciprocal diagrams, the form diagram which shows the geometry of structure (location of internal nodes and supports, external loads and lengths of structural elements) and the force diagram where polygons of forces, assembled of vectors, represent equilibrium of internal forces in structural elements, forces in supports and external loads. Relation of the two reciprocal diagrams allows simultaneous control over the shape of the structure and internal forces in structural elements and thus finding an efficient geometry of the structure at an early stage of structural design process.
Developed in 19th century, methods of graphic statics were limited only to planar and simple spatial problems of statics. Owing to today's advanced tools for computer-aided design (CAD), the development and application of the methods of three-dimensional (3D) graphic statics, such as 3D algebraic polyhedral graphic statics and 3D vector-based graphic statics, both based on ideas from 19th century, are available.


Hermann Grassmann (1809. - 1877.) and Julius Plücker (1801. - 1868.)

Hermann Grassmann's great contribution to mathematics and mechanics was his concept of coordinatization of higher dimensional subsets (subspaces) of geometrical sets, and Julius Plücker is responsible for application of these ideas to the set of lines in the Extended Euclidean space.

Extended Euclidean Space

To the Euclidean space, whose basic elements are points, lines and planes, we add one plane at infinity, ideal plane, which contains lines at infinity and points at infinity, ideal lines and ideal points, so that every other plane in space contains exactly one ideal line, it is extended with this one line, which is then its intersection with all other planes parallel to it, and every line in space contains, or is extended by, one ideal point so that this point is its intersection with all other lines parallel to it. This kind of space is called the extended Euclidean space $\mathbf{P^3(\mathbb{R})}$, and it is a 3-dimensional projective space. The points of the extended Euclidean space are either the points of the Euclidean space E^3 or the ideal points, and the entire E^3 is canonically embedded in $\mathbf{P^3(\mathbb{R})}$ as the complement of the ideal plane. It is also embedded in the vector space $\mathbb{R}^4$or equivalently, we can think of it as an image of projection of $\mathbb{R}^4$ into three dimensions which can be done in infinitely many ways.

In the usual Cartesian coordinate system of $\mathbb{R}^4$, with four mutually orthogonal axes, a vector $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ is given by its coordinates x_0, x_1, x_2, x_3 which represent projections of this vector to the four axes. we denote the one-dimensional subspace, the span of this vector, by $\langle \boldsymbol{x} \rangle$ and it is a line of $\mathbb{R}^4$ passing through the origin point.

Then the **homogenous coordinates** of the corresponding point in $\mathbf{P^3(\mathbb{R})}$ are denoted by $(x_0 : x_1 : x_2 : x_3)$ and the homogeneity property states that

$$\lambda(x_0 : x_1 : x_2 : x_3) = (\lambda x_0 : \lambda x_1 : \lambda x_2 : \lambda x_3) = (x_0 : x_1 : x_2 : x_3), \quad \lambda \neq 0, \quad \lambda \in \mathbb{R}$$

must be true for every vector in $\mathbb{R}^4$.
Having in mind the first paragraph, we usually choose the first variable x_0, thus the set of ideal points becomes the plane of $\mathbf{P^3(\mathbb{R})}$ having the equation $x_0 = 0$, and the Euclidean space E^3 is isomorphic to its complement, a set given with the equation $x_0 \neq 0$, and is the image of the projection

$$(x_0 : x_1 : x_2 : x_3) = \left(1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \frac{x_3}{x_0}\right) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right)$$

from that set.

The homogenous coordinates of a plane $(\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3)$ are interpreted in E^3 as the plane passing through the points $(\alpha_1, 0, 0)$, $(0, \alpha_2, 0)$ and $(0, 0, \alpha_3)$, its intersections with the coordinate axes, unless the first coordinate $\alpha_0 = 0$ in which case the plane contains the origin point. The ideal plane is given by $(\alpha_0 : 0 : 0 : 0)$ with $\alpha_0 \neq 0$.

Incidence Relation

In the projective space, on the set of basic elements – points, lines and planes we have the incidence relation. This space can be axiomatically described with axioms of incidence if we want to employ synthetic geometry, or we can explore it analytically, which we will, using homogenous coordinates.
The incidence relation has threefold interpretations, three aspects. First is relation itself in a passive sense, to be incident meaning to lie in or to pass through, noting the relation between elements of the space. The other two are active, used to define an element with other lower or higher dimensional elements. First is the **meet** or intersection, for instance two lines meet or intersect at a point, and the other is the **join** or span, for instance the span of two points is a line, or the line is given as a join of two points.

Grassmann's outer product

Grassmann defined an operation, which he named outer product, that takes two elements of the vector space and attaches to this pair one element of another, higher dimensional, vector space. The adjective "outer" thus expressing that the result of the operation in not contained in the same vector space as the operands.
He then prescribes two properties that this operation must satisfy so that the resulting vector space is uniquely determined - this operation is *anti-commutative* and linear in both arguments, *bilinear.*

We present his construction for $\mathbb{R}^4$ and $k = 2$. Let e_i be the elements of canonical basis for $\mathbb{R}^4$. We denote the outer product of two elements e_i and e_j by $e_i \wedge e_j$ and for all elements $\boldsymbol{x} = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$ and $\boldsymbol{y} = y_0 e_0 + y_1 e_1 + y_2 e_2 + y_3 e_3$ in $\mathbb{R}^4$ we have

$$\boldsymbol{x} \wedge \boldsymbol{y} = (x_0 y_1 - x_1 y_0)\, e_0 \wedge e_1 + (x_0 y_2 - x_2 y_0)\, e_0 \wedge e_2 + (x_0 y_3 - x_3 y_0)\, e_0 \wedge e_3 + (x_2 y_3 - x_3 y_2)\, e_2 \wedge e_3 + (x_3 y_1 - x_1 y_3)\, e_3 \wedge e_1 + (x_1 y_2 - x_2 y_1)\, e_1 \wedge e_2$$

We relate the incidence relation to the concept of the outer product, the bilinear operation of Grassmann algebra. First, the meet operation, which defines elements of $\mathbf{P^3(\mathbb{R})}$ using lower dimensional elements is correspondent, in Grassmann's terminology, to the progressive product. For instance, a line, in Grassmann geometrical algebra, can be defined as outer product of two points. The adjective progressive emphasizing that this definition starts with lower and results in higher dimensional elements, or subspaces.
Analogously, given the duality principle, a line geometrically defined as intersecting line of two planes can be defined as outer product of two planes. This kind of outer product, where we start with higher and end up with lower dimensional objects is, in Grassmann's terminology, noted as regressive product.

A line L interpreted as the progressive product of two points $L = \boldsymbol{x} \wedge \boldsymbol{y}$ has Grassmann coordinates $L = (l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12}) \in \Lambda^2 \mathbb{R}^4$ where

$$l_{ij} = x_i y_j - x_j y_i$$

for $(i, j) \in \{(0,1), (0,2), (0,3), (2,3), (3,1), (1,2)\}$ and the following relation, **Plücker relation** must hold: $l_{01} l_{23} + l_{02} l_{31} + l_{03} l_{12} = 0$.

The **Plücker coordinates** of a line are the homogenous coordinates $(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12}) = (\boldsymbol{l}, \bar{\boldsymbol{l}})$, where $\boldsymbol{l} = (l_{01}, l_{02}, l_{03})$ and $\bar{\boldsymbol{l}} = (l_{23}, l_{31}, l_{12})$ and the Plücker relation now reads $\boldsymbol{l} \cdot \bar{\boldsymbol{l}} = 0$.

In the Table on the right we present the formulas in homogenous coordinates for the following incidence relations:

- a point incident with (lying in) a plane,
- a point incident with (lying on) a line,
- a line as join of two points,
- a line as meet of two planes,
- a point as meet of two intersecting lines,
- a plane as join of two intersecting lines,
- a plane as join of a point and non-incident line and
- a point as meet of a line and non-incident plane

Table Computing with homogenous coordinates

$\boldsymbol{x} \in \boldsymbol{\alpha} \iff$ $x_0 \alpha_0 + \bar{x}\bar{\alpha} = 0$	$\boldsymbol{x} \in L \iff$ $\boldsymbol{x} \cdot \bar{\boldsymbol{l}} = 0$ and $-x_0 \bar{\boldsymbol{l}} + \boldsymbol{x} \times \boldsymbol{l} = 0$
$L = \langle \boldsymbol{x}, \boldsymbol{y} \rangle \iff$ $L = (x_0 \bar{y} - y_0 \bar{x}, \boldsymbol{x} \times \boldsymbol{y})$	$L = \boldsymbol{\alpha} \cap \boldsymbol{\beta} \iff$ $L = (\boldsymbol{\alpha} \times \boldsymbol{\beta}, \alpha_0 \bar{\beta} - \beta_0 \bar{\alpha})$
$\boldsymbol{\alpha} = \langle L_1, L_2 \rangle \iff$ $\boldsymbol{\alpha} = (\bar{\boldsymbol{l}}_1 \cdot \boldsymbol{l}_2, -\boldsymbol{l}_1 \times \boldsymbol{l}_2)$	$\boldsymbol{x} = L_1 \cap L_2 \iff$ $\boldsymbol{x} = (\boldsymbol{l}_1 \cdot \bar{\boldsymbol{l}}_2, -\bar{\boldsymbol{l}}_1 \times \bar{\boldsymbol{l}}_2)$
$\boldsymbol{x} = \boldsymbol{\alpha} \cap L \iff$ $\boldsymbol{x} = (\boldsymbol{\alpha} \cdot \boldsymbol{l}, -\alpha_0 \boldsymbol{l} + \boldsymbol{x} \times \bar{\boldsymbol{l}})$	$\boldsymbol{\alpha} = \langle \boldsymbol{x}, L \rangle \iff$ $\boldsymbol{\alpha} = (\boldsymbol{x} \cdot \bar{\boldsymbol{l}}, -x_0 \bar{\boldsymbol{l}} + \boldsymbol{x} \times \boldsymbol{l})$

These formulas can be easily verified by direct computation using vector calculus. We write the homogenous coordinates of points and planes as

$$\boldsymbol{x} = (x_0 : x_1 : x_2 : x_3) = (x_0, \bar{x})$$

and

$$\boldsymbol{\alpha} = (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3) = (\alpha_0, \bar{\alpha}),$$

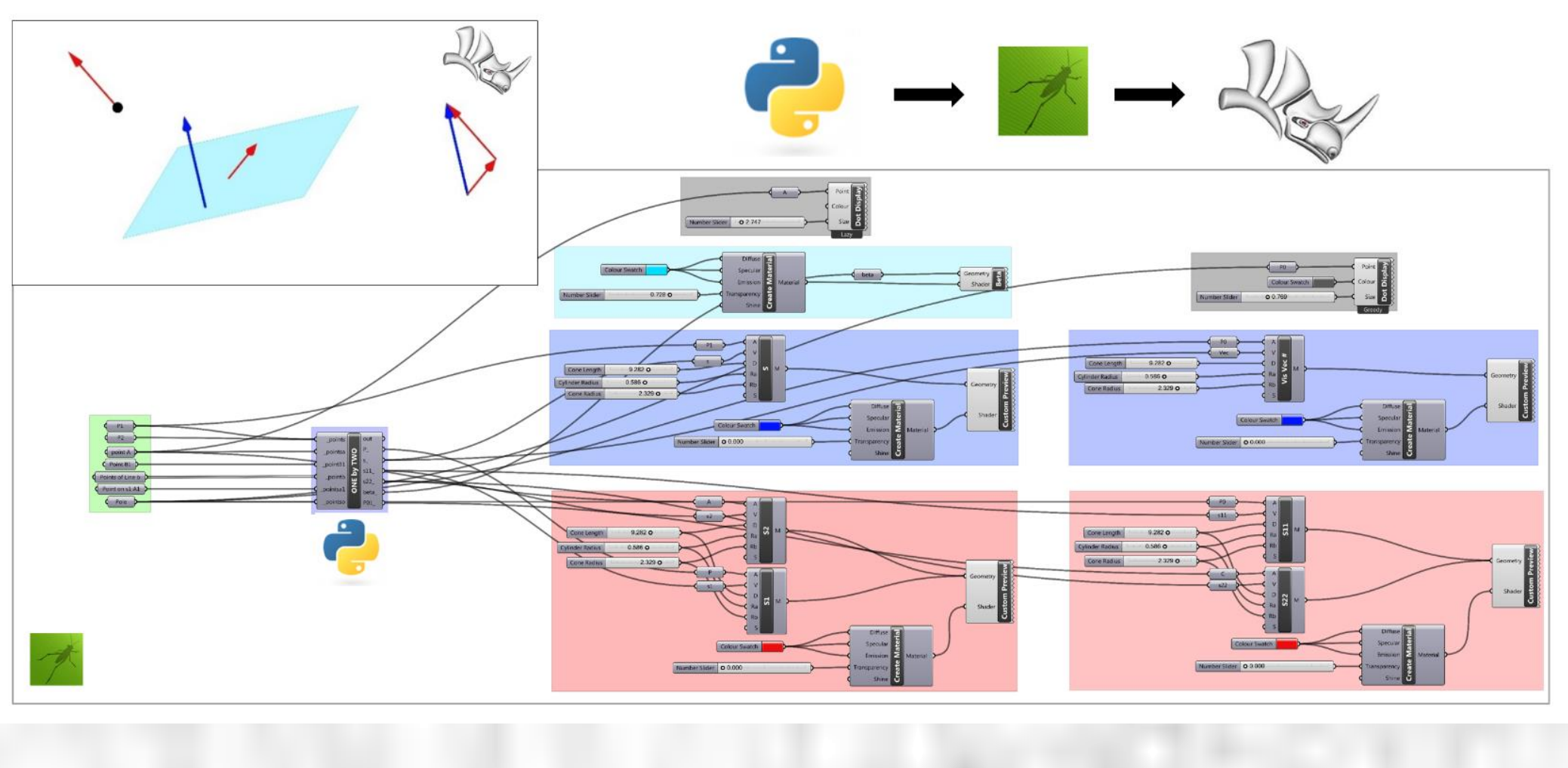
thus, emphasizing standard Cartesian coordinates of points in E^3, i.e. normal vectors of planes.

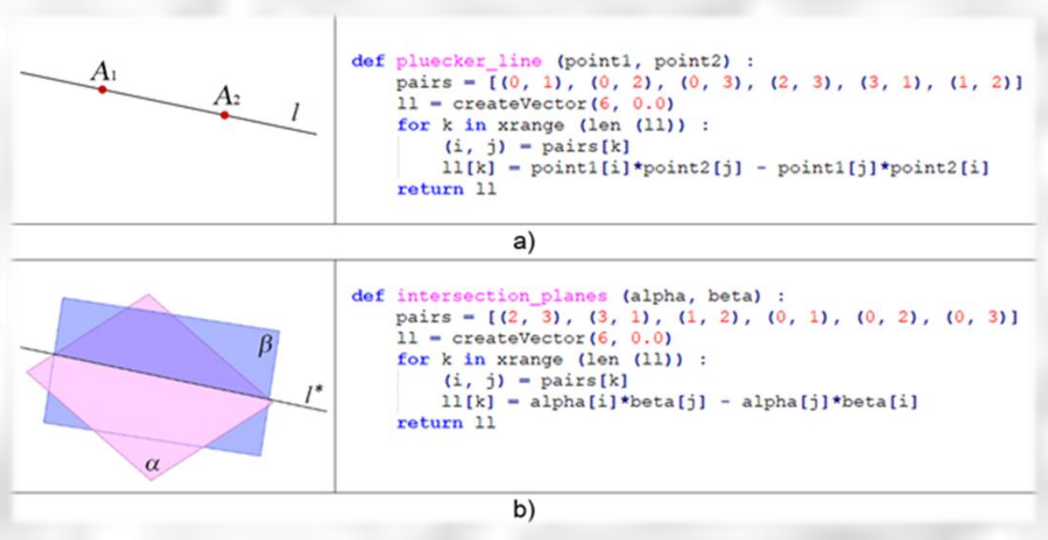
Homogenous coordinates of lines are as in (1).

Force coordinates

Since a force acting at a point in the space is lying on a line and due to the principle of transmissibility, we can use the line Plücker coordinates to derive the coordinates for a force. This is done by the further remark on the geometrical interpretation of the coordinates; the vector $\bar{\boldsymbol{l}}$ is orthogonal to the vector $\boldsymbol{l}$ and can be interpreted as the moment vector about the origin point of the coordinate system of a force F lying on the line.
Furthermore, the ideal point of this line is the point $(0 : l_{01} : l_{02} : l_{03})$, in other words the vector $\boldsymbol{l}$ represents the direction of the line, and if $\boldsymbol{x} = (1 : x_1 : x_2 : x_3)$ is any other (non-ideal) point on the line, then we have $\bar{\boldsymbol{l}} = x \times \boldsymbol{l}$, with $x = (x_1, x_2, x_3)$ being the Cartesian coordinates of the point in E^3.
Therefore, a force F can be given by its coordinates $F = (\boldsymbol{f}, \bar{\boldsymbol{f}})$, with $\boldsymbol{f}$ being is force vector and $\bar{\boldsymbol{f}}$ its moment vector about the origin point of the coordinate system. These coordinates are not homogenous since the quantity $\|\boldsymbol{f}\|$ represents its intensity.

For performance and visualisation of the examples of static equivalence and the examples of equilibrium finding that we present, we have developed a computer program based on algebraic translations of incidence operations using CAD tool Rhinoceros. Steps of the graphical procedures are carried out using basic operations of incidence geometry, which can be easily expressed in algebraic form using Grassmann algebra, thus enabling their conversion into a programme code.
The programme code is written in GhPython (Python interpreter and plug-in for Grasshopper), and the results are visualized in Rhinoceros (process is described in the figure bellow). All procedures of static equivalence and geometrical constructions are graphically performed in form diagram and then followed by force polygons in the force diagram.





```
def pluecker_line (point1, point2) :
    pairs = [(0, 1), (0, 2), (0, 3), (2, 3), (3, 1), (1, 2)]
    l1 = createVector(6, 0.0)
    for k in xrange (len (l1)) :
        (i, j) = pairs[k]
        l1[k] = point1[i]*point2[j] - point1[j]*point2[i]
    return l1
```

a)

```
def intersection_planes (alpha, beta) :
    pairs = [(2, 3), (3, 1), (1, 2), (0, 1), (0, 2), (0, 3)]
    l1 = createVector(6, 0.0)
    for k in xrange (len (l1)) :
        (i, j) = pairs[k]
        l1[k] = alpha[i]*beta[j] - alpha[j]*beta[i]
    return l1
```

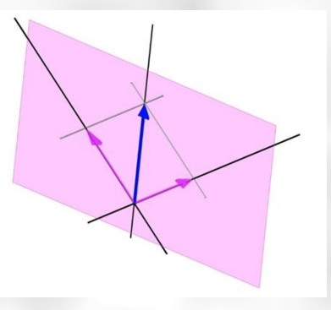
b)

Python codes: a) a line as join of two points
b) a line as meet of two planes

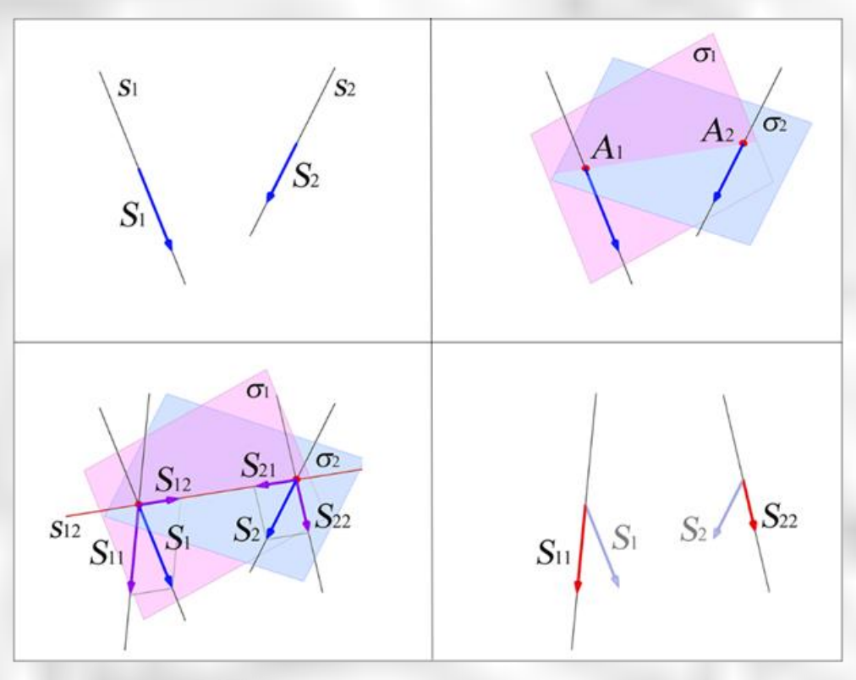
Examples

The procedures for replacing given force system with some other force system and procedures for finding equilibrating forces to the given force system described here are carried out using geometric constructions which can be considered as a partial three-dimensional extension of funicular polygon construction. The extension of funicular polygon is based on two principles :

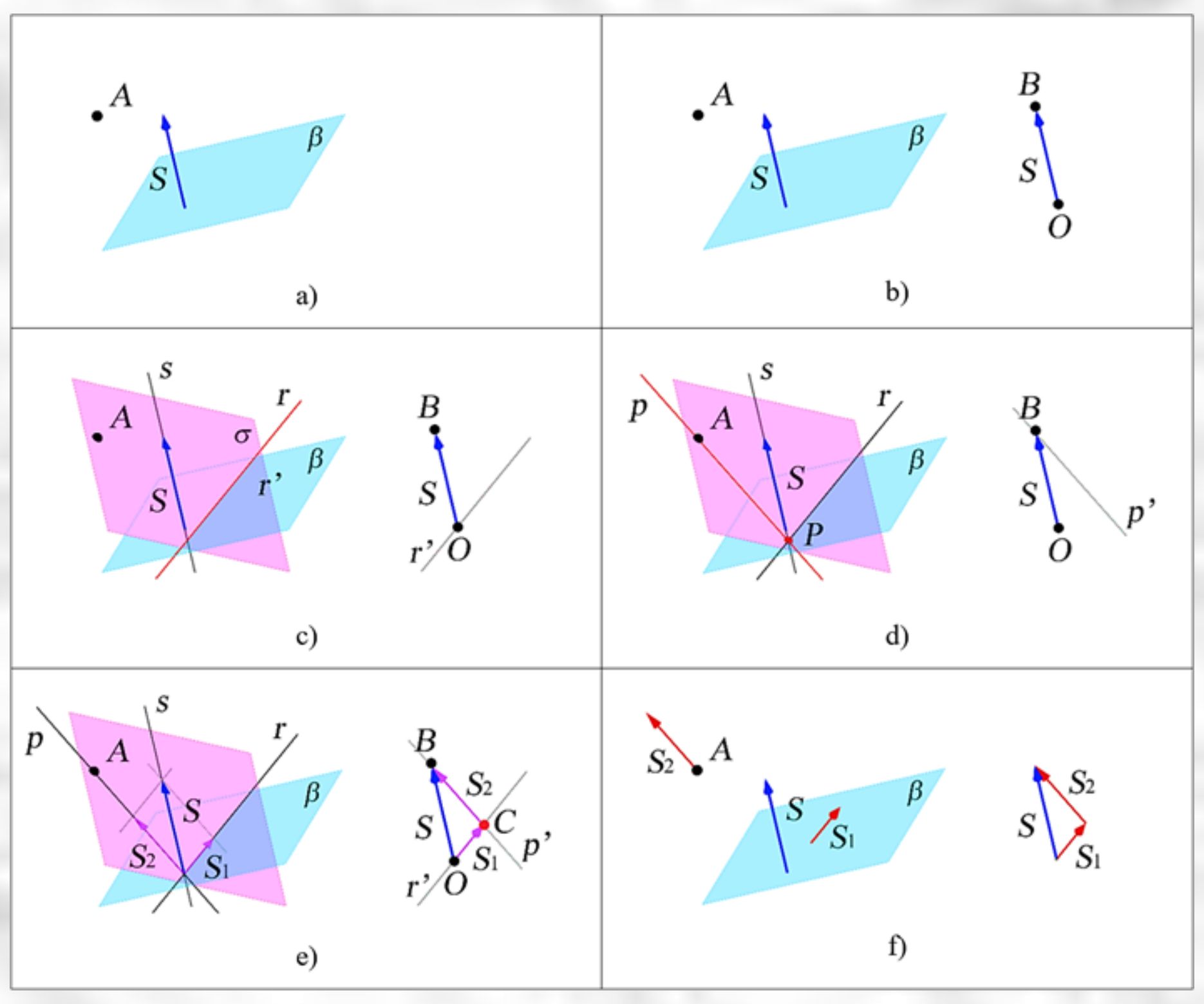
1) single force can be resolved into two force components along two given lines if and only if its line of action and two given lines are concurrent and coplanar



2) generally, when constructing funicular polygon, using first condition, each of two given forces is resolved into two components in such a way that one component of the first force and one component of the second force lie on the same line, they are opposite in sense and equal in magnitude, namely they cancel each other.



Replacing single force with a force acting at a given point and a force lying in a given plane



In this example, we will also show translations of geometric operations into Grassmann algebra expressions.

A plane σ is a join of the given point A and the line of action s of the given force S (in algebraic terms: join σ = [A ∧ s]) (Figure c).
First component of the force S acts along the line r, which is the intersection line of the plane σ and a given plane β (meet r = [σ ∧ β]).
Second component acts along the connecting line p of the point A and the intersection point P of the line s and the plane β (meet and then join: P = [s ∧ β], p = [A ∧ P]) (Figure d).
Previous steps were performed in the form diagram while the following ones will be performed in the force diagram.
From arbitrarily chosen point O vector s of the force S is drawn; head of s is the point B (B = O + s).
Lines r' and p' are drawn through O and B parallel to the lines r and p (r' = [O ∧ r], p' = [B ∧ p], where r and p are some vectors on lines r and p). Lines r' and p' intersect in the point C (C = [r' ∧ p']) (Figure e).

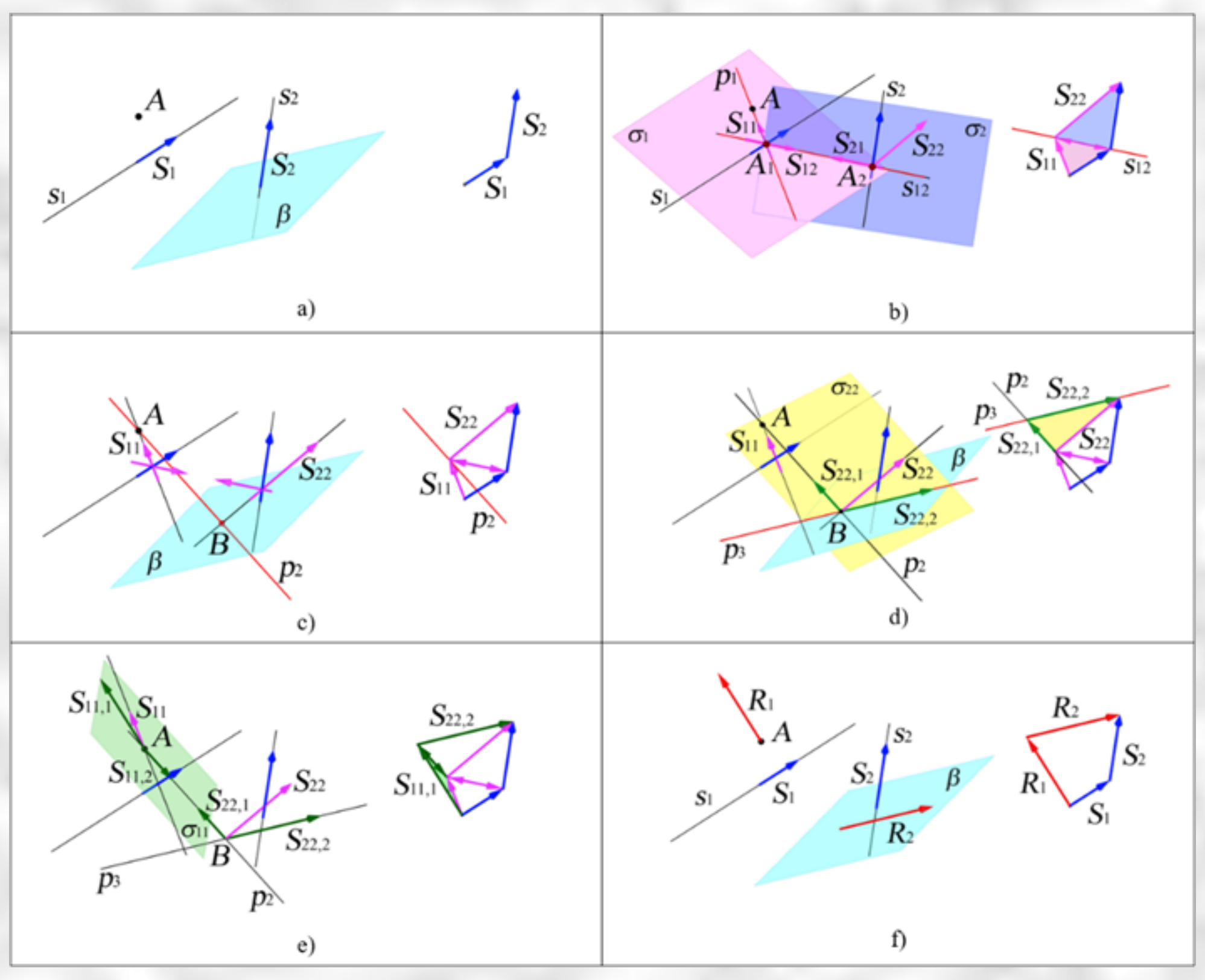
Vectors s_1 and s_2 of force components S_1 and S_2 on lines r and p are s_1 = C – O and s_2 = B – C.

Replacing two forces with a force acting at a given point and a force lying in a given plane

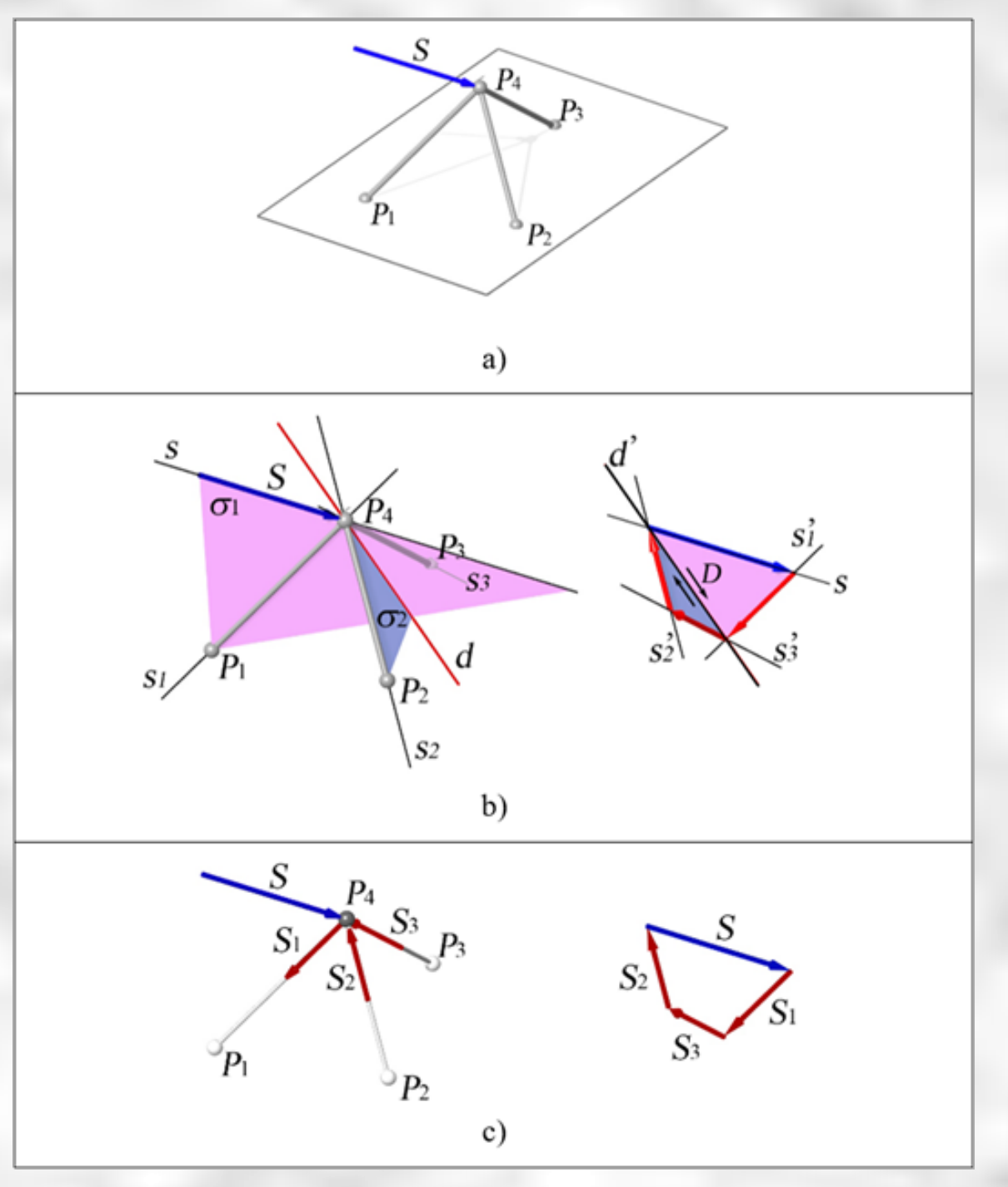
Theorem that "any system of forces can always be represented by two forces one of which lies in a given plane, and the other passes through a given point not lying in the plane" was also proved by Whitehead [5].

The first procedure for replacing two forces with a force acting at a given point and a force lying in a given plane is to resolve each of two given forces S_1 and S_2 into a component acting at the given point A and a component lying in a given plane β. The resultant R_1 of the two components acts at the point A and the resultant R_2 of the other two components lies in the plane β. Two obtained forces are statically equivalent to the given forces S_1 and S_2.

Another procedure, similar to the funicular polygon construction, is as follows. S_1 and S_2 are given forces acting on lines s_1 and s_2 (Figure a). The plane $σ_1$ is defined by the given point A and the line s_1, and the plane $σ_2$ by the line s_2 and a point A_1 arbitrarily chosen on the line s_1. The line s_{12} is the intersection of the planes $σ_1$ and $σ_2$. (The line s_{12} can also be considered as a connecting line between two arbitrarily chosen points, A_1 on the line s_1 and A_2 on the line s_2.) The component S_{12} of the force S_1 and the component S_{21} of the force S_2 act along the same line s_{12} and cancel each other. Second component S_{11} of the force S_1 acts along the line p_1 connecting the points A and A_1. Now, the force S_2 can be resolved in the plane $σ_2$. In that way given forces S_1 and S_2 are replaced with the forces S_{11} and S_{22} (Figure b).
The point B is the intersection of the line of action of the force S_{22}, line s_{22}, and the given plane β. Span (or join) of two points A and B is the line p_2 (Figure c). The plane $σ_{22}$ is defined as a join of the lines s_{22} and p_2. Planes $σ_{22}$ and β intersect in the line p_3.
Now, we resolve the force S_{22} into two components $S_{22,1}$ and $S_{22,2}$ along the lines p_2 and p_3 (Figure d), and the force S_{11} into components $S_{11,1}$ and $S_{11,2} = -S_{22,1}$ (Figure e). Remained components $S_{11,1} = R_1$, a force which acts at the given point A, and $S_{22,2} = R_2$, a force which lies in the given plane β (Figure f), represent equivalent force system to the system of forces S_1 and S_2. Reversion of the obtained forces R_1 and R_2 gives equilibrating forces to the given two-force system.



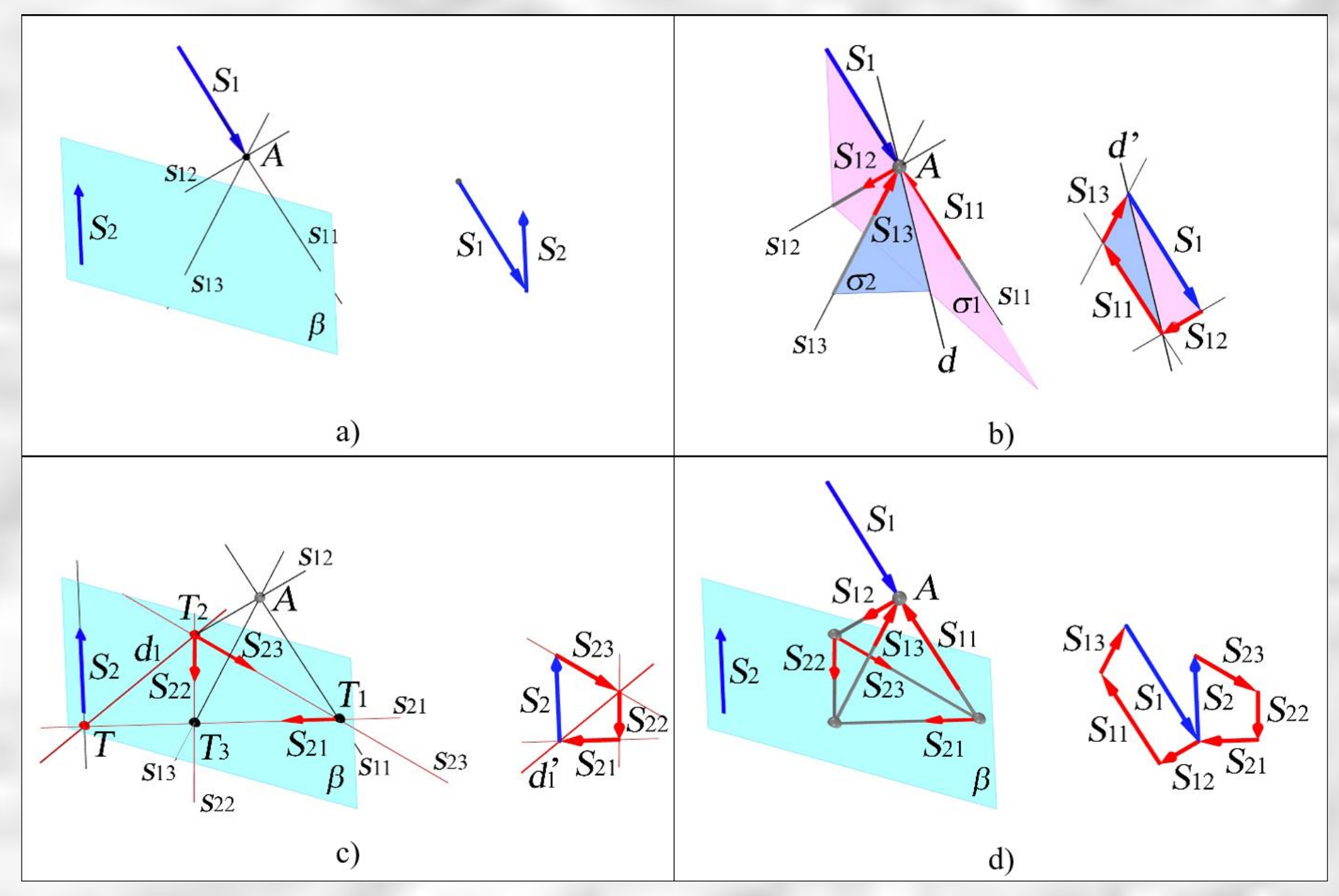
Equilibrium of a spatial node



Here, we describe an example of finding equilibrium of a given spatial node P_4 supported by three bars P_1P_4 (line s_1), P_2P_4 (line s_2) and P_3P_4 (line s_3). The bars are connected to the ground with spherical supports P_1, P_2 and P_3. Also, the node P_4 is a spherical node at which given force S acts along its line of action s (Figure a).
The procedure for equilibrium finding of a spatial node is similar to the procedure for replacing a single force with three forces acting along bars P_1P_4, P_2P_4 and P_3P_4. In the form diagram (Figure b), we define plane $σ_1$ containing the line of action s and the line s_2, and plane $σ_2$ containing the lines s_1 and s_3. The intersection line of the planes $σ_1$ and $σ_2$ is the line d. Now, all the lines in the form diagram are known, thus the procedure is carried out in the force diagram (Figure b, on the right side).
First, we place the line s_2' parallel to the line s_2 at the tail of the force S, and at the head of the force S, we place the line d' parallel to the line d. Since the force D = S+S_1 is the sum of the forces S and S_1, and since all of them lie in the plane parallel to the plane $σ_1$, the intersection of two lines gives the point which defines magnitudes of the forces D and S_1. The component S_1 acts along the line s_1, and the force D acts along the line d'. In that way S_1 and D are determined. Now, resultant force of the forces S_2 and S_3 must be in equilibrium with the force D. It is well known that two forces are in equilibrium if they act along the same line, they are opposite in sense and equal in magnitude. Thus, the force D and the force −D=S_2+S_3 act along the line d' and cancel each other out. Again, in the force diagram, we place a line s_2' at the tail of the force −D parallel to the line s_2 and at the head of the force −D we place a line s_3' parallel to the line s_3. Since all the forces −D, S_2 and S_3 lie in the same plane parallel to the plane $σ_2$, the intersection point of the lines s_2' and s_3' determines forces S_2 and S_3. In that way forces S, S_1, S_2 and S_3 form a closed polygon in the force diagram, i.e. the spatial node P_4 is in equilibrium (Figure c).

Equilibrating two-force system with forces on edges of given tetrahedron

Forces S_1 and S_2 are two given forces. The force S_1 acts at the given point A and the force S_2 lies in the given plane β (Figure a). Three lines s_{11}, s_{12} and s_{13} are given through the same point A, and lines s_{21}, s_{22} and s_{23} are lying in the plane β.
In our case, the six lines are given in a special position, that is they are edges of a tetrahedron. For each of the given forces, three equilibrating forces will be determined.
For the force S_1, using the procedure for finding equilibrium of a spatial node described in the previous subsection, three equilibrating forces S_{11}, S_{12} and S_{13}, acting along the given lines s_{11}, s_{12} and s_{13} at the same point A, which is also a vertex of the tetrahedron, will be determined (Figure b).
For the force S_2 three forces act in the same plane β along the given lines s_{21}, s_{22} and s_{23}. In this special position, points T_1, T_2 and T_3 are vertices of the given tetrahedron and also intersection points of the lines s_{11}, s_{12} and s_{13} with the plane β in general case. Since all lines, along which the three equilibrating forces S_{21}, S_{22} and S_{23} to the force S_2 act, are known, they can be determined in the force diagram. The line d_1, connecting the intersection point T_2 of the lines of action of the forces S_{22} and S_{23} and the intersection point T of the lines of action of the forces S_2 and S_{21}, represents Culmann's line (Figure c). Using well-known methods of planar (2D) graphic statics, we obtain equilibrating forces S_{21}, S_{22} and S_{23}.



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