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# Models for Modular Curves, Modular Forms and $\eta$-quotients

DOCTORAL THESIS

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# Extended summary

Modular group $\Gamma$ is a subgroup of $SL_2(\mathbb{Z})$ of finite index. A modular group acts on the complex upper half plane by linear fractional transfomations. Quotient of this action is a set $X(\Gamma)$ which is compactified by adding cusps. Cusps are real numbers or $\infty$ which are fixed by some parabolic element in $\Gamma$. On the set $X(\Gamma)$ we can define complex structure so it becomes a compact Riemann surface. The set $X(\Gamma)$ is called a modular curve and it consists of orbits of elements of the upper half plane and classes of nonequivalent cusps. If $\Gamma$ is the congruence subgroup $\Gamma_0(N)$, then this quotient is denoted $X_0(N)$.

We observe the following mapping - an element of the modular curve $X(\Gamma)$ is mapped into a point in the projective plane using three linearly independent modular forms of some even weight greater or equal to 2. This mapping is holomorphic. We prove a formula which connects the degree of the map and the degree of the image curve. If the degree of the map equals 1, then the image curve is a model of the modular curve $X(\Gamma)$.

From the formula we can check whether the map is birational equivalence by computing all values except the degree of the map. There are two directions in calculations. On one hand, we need to calculate the divisors of the modular forms. We describe the cusps of congruence subgroups and calculate the divisors of the Ramanujan delta function $\Delta$, Eisenstein series $E_4$ and various $\eta$-quotients with regard to the group $\Gamma_0(N)$. On the other hand, we calculate the degree of the image curve. We develop an algorithm which calculates this degree and the defining polynomial. The degree is connected to the rank of a matrix of the coefficients of a homogeneous system of equations. The system is obtained from the fact that a finite number of coefficient in the Fourier expansion of a modular form must be equal to zero if a form is zero. This finite number of coefficients is calculated from the Sturm bound. The algorithm is based on the Hilbert zero theorem, linear algebra and the theory of modular forms.

A class of modular forms called $\eta$-quotients are obtained by multiplication and scaling from the Dedekind $\eta$-function. Under certain assumptions, these functions are modular forms on $\Gamma_0(N)$. There is a formula for the divisor of an $\eta$-quotient and they have integral Fourier expansions. We search for $\eta$-quotients on $\Gamma_0(p)$ for a prime $p$ and create mappings with these functions. We observe the map from $X_0(N)$ defined with $\Delta$, $E_4^3$ and $\Delta_N$. We determine those $\eta$-quotients that have the zero of the maximal order at the cusp $\infty$ and use these functions to create models of $X_0(N)$.

**Keywords**: compact Riemann surface, modular group, modular curve, modular form, divisor of modular form, $\eta$-quotients